Vector Differential Identities.

Peeter Joot

Jan 05, 2009. Last Revision: $Date : 2009/02/1605 : 41 : 36$

1 Some identities.

[Feynman et al.(1963)Feynman, Leighton, and Sands] electrodynamics chapter II lists a number of differential vector identities.

1. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) = \boldsymbol{\nabla}^2 T =$ a scalar field
2. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = 0$
3. $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{h}) =$ a vector field
4. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{h}) = 0$
5. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{h}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{h}) - \boldsymbol{\nabla}^2 \mathbf{h}$
6. $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{h} =$ a vector field

Let's see how all these translate to GA form.

1.1 Divergence of a gradient.

This one has the same form, but expanding it can be evaluated by grade selection

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) &= \langle \boldsymbol{\nabla} \boldsymbol{\nabla} T \rangle \\
&= (\boldsymbol{\nabla}^2) T
\end{aligned}
$$

A less sneaky expansion would be by coordinates

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) &= \sum_{k,j} (\sigma_k \partial_k) \cdot (\sigma_j \partial_j T) \\
&= \left\langle \sum_{k,j} (\sigma_k \partial_k)(\sigma_j \partial_j T) \right\rangle \\
&= \left\langle \left(\sum_{k,j} \sigma_k \partial_k \sigma_j \partial_j \right) T \right\rangle \\
&= \left\langle \boldsymbol{\nabla}^2 T \right\rangle \\
&= \boldsymbol{\nabla}^2 T
\end{aligned}
$$

1.2 Curl of a gradient is zero.

The duality analogue of this is

$$
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = -i(\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} T))
$$

Let's verify that this bivector curl is zero. This can also be done by grade selection

$$
\begin{aligned}
\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} T) &= \langle \boldsymbol{\nabla} (\boldsymbol{\nabla} T) \rangle_2 \\
&= \langle (\boldsymbol{\nabla} \boldsymbol{\nabla}) T \rangle_2 \\
&= (\boldsymbol{\nabla} \wedge \boldsymbol{\nabla}) T \\
&= 0
\end{aligned}
$$

Again, this is sneaky and side steps the continuity requirement for mixed partial equality. Again by coordinates is better

$$
\begin{aligned}
\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} T) &= \left\langle \sum_{k,j} \sigma_k \partial_k (\sigma_j \partial_j T) \right\rangle_2 \\
&= \left\langle \sum_{k<j} \sigma_k \sigma_j (\partial_k \partial_j - \partial_j \partial_k) T \right\rangle_2 \\
&= \sum_{k<j} \sigma_k \wedge \sigma_j (\partial_k \partial_j - \partial_j \partial_k) T
\end{aligned}
$$

So provided the mixed partials are zero the curl of a gradient is zero.

1.3 Gradient of a divergence.

Nothing more to say about this one.

1.4 Divergence of curl.

This one looks like it will have a dual form using bivectors.

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{h}) &= \boldsymbol{\nabla} \cdot (-i(\boldsymbol{\nabla} \wedge \mathbf{h})) \\
&= \langle \boldsymbol{\nabla}(-i(\boldsymbol{\nabla} \wedge \mathbf{h})) \rangle \\
&= \langle -i\boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{h}) \rangle \\
&= -(i\boldsymbol{\nabla}) \cdot (\boldsymbol{\nabla} \wedge \mathbf{h})
\end{aligned}
$$

Is this any better than the cross product relationship?

I don't really think so. They both say the same thing, and only possible value to this duality form is if more than three dimensions are required (in which case the sign of the pseudoscalar i has to be dealt with more carefully). Geometrically one has the dual of the gradient (a plane normal to the vector itself) dotted with the plane formed by the gradient and the vector operated on. The corresponding statement for the cross product form is that we have a dot product of a vector with a vector normal to it, so also intuitively expect a zero. In either case, because we are talking about operators here just saying this is zero because of geometrical arguments isn't neccessarily convicing. Let's evaluate this explicitly in coordinates to verify

$$
\begin{aligned}
(i\boldsymbol{\nabla}) \cdot (\boldsymbol{\nabla} \wedge \mathbf{h}) &= \langle i\boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{h}) \rangle \\
&= \left\langle i \sum_{k,j,l} \sigma_k \partial_k \left((\sigma_j \wedge \sigma_l) \partial_j h^l \right) \right\rangle \\
&= -i \sum_l \sigma_l \wedge \left(\sum_{k<j} (\sigma_k \wedge \sigma_j)(\partial_k \partial_j - \partial_j \partial_k) h^l \right)
\end{aligned}
$$

This inner quantity is zero, again by equality of mixed partials. While the dual form of this identity wasn't really any better than the cross product form, there is nothing in this zero equality proof that was tied to the dimension of the vectors involved, so we do have a more general form than can be expressed by the cross product, which could be of value in Minkowski space later.

1.5 Curl of a curl.

This will also have a dual form. That is

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{h}) &= -i(\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \times \mathbf{h})) \\
&= -i(\boldsymbol{\nabla} \wedge (-i(\boldsymbol{\nabla} \wedge \mathbf{h}))) \\
&= -i\langle \boldsymbol{\nabla}(-i(\boldsymbol{\nabla} \wedge \mathbf{h}))\rangle_2 \\
&= i\langle i\boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{h})\rangle_2 \\
&= i^2\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{h}) \\
&= -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{h})
\end{aligned}$$

Now, let's expand this quanity

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{h})$$

If the gradient could be treated as a plain old vector we could just do

$$\mathbf{a} \cdot (\mathbf{a} \wedge \mathbf{h}) = \mathbf{a}^2\mathbf{h} - \mathbf{a}(\mathbf{a} \cdot \mathbf{h})$$

With the gradient substituted this is exactly the desired identity (with the expected sign difference)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{h}) = \boldsymbol{\nabla}^2\mathbf{h} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{h})$$

A coordinate expansion to truely verify that this is valid is logically still required, but having done the others above, it is clear how this would work out.

1.6 Laplacian of a vector.

This one isn't interesting seeming.

2 Two more theorems.

Two theorems without proof are mentioned in the text.

2.1 Zero curl implies gradient solution.

Theorem was

$$\begin{array}{lll}
\text{If} & \boldsymbol{\nabla} \times \mathbf{A} & = 0 \\
\text{there is a} & \psi & \\
\text{such that} & \mathbf{A} & = \boldsymbol{\nabla}\psi
\end{array}$$

This appears to be half of an if and only if theorem. The unstated part is if one has a gradient then the curl is zero

$$\begin{aligned}
\mathbf{A} &= \boldsymbol{\nabla}\psi \\
&\Longrightarrow \\
\boldsymbol{\nabla} \times \mathbf{A} &= \boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0
\end{aligned}$$

This last was proven above, and follows from the assumed mixed partial equality. Now, the real problem here is to find ψ given $\mathbf{A}$. First note that we can remove the three dimensionality of the theorem by duality writing $\boldsymbol{\nabla} \times \mathbf{A} = -i(\boldsymbol{\nabla} \wedge \mathbf{A})$. In one sense changing the theorem to use the wedge instead of cross makes the problem harder since the wedge product is defined not just for $\mathbb{R}^3$. However, this also allows starting with the simpler $\mathbb{R}^2$ case, so let's do that one first.

Write

$$\mathbf{A} = \sigma^1 A_1 + \sigma^2 A_2 = \sigma^1(\partial_1\psi) + \sigma^2(\partial_2\psi) \tag{1}$$

The gradient is

$$\boldsymbol{\nabla} = \sigma^1\partial_1 + \sigma^2\partial_2$$

Our curl is then

$$(\sigma^1\partial_1 + \sigma^2\partial_2) \wedge (\sigma^1 A_1 + \sigma^2 A_2) = (\sigma^1 \wedge \sigma^2)(\partial_1 A_2 - \partial_2 A_1)$$

So we have

$$\partial_1 A_2 = \partial_2 A_1 \tag{2}$$

Now from 1 this means we must have

$$\partial_1\partial_2\psi = \partial_2\partial_1\psi \tag{3}$$

This is just a statement of mixed partial equality, and doesn't look particularily useful for solving for ψ. It really shows that the is redundancy in the problem, and instead of substuiting for both of A_1 and A_2 in 2, we can use one or the other.

Doing so we have two equations, either of which we can solve for

$$\begin{aligned}
\partial_2\partial_1\psi &= \partial_1 A_2 \\
\partial_1\partial_2\psi &= \partial_2 A_1
\end{aligned}$$

Integrating once gives

$$\partial_1 \psi = \int \partial_1 A_2 dy + B(x)$$
$$\partial_2 \psi = \int \partial_2 A_1 dx + C(y)$$

And a second time produces solutions for ψ in terms of the vector coordinates

$$\psi = \iint \frac{\partial A_2}{\partial x} dy dx + \int B(x) dx + D(y) \tag{4}$$
$$\psi = \iint \frac{\partial A_1}{\partial y} dx dy + \int C(y) dy + E(x) \tag{5}$$

Is there a natural way to merge these so that ψ can be expressed more symmetrically in terms of both coordinates? Looking at 4 I am led to guess that its possible to combine these into a single equation expressing ψ in terms of both A_1 and A_2. One way to do so is perhaps just to average the two as in

$$\psi = \alpha \iint \frac{\partial A_2}{\partial x} dy dx + (1 - \alpha) \iint \frac{\partial A_1}{\partial y} dx dy + \int C(y) dy + E(x) + \int B(x) dx + D(y)$$

But that seems pretty arbitrary. Perhaps that's the point?

FIXME: work some examples.

FIXME: look at more than the $\mathbb{R}^2$ case.

2.2 Zero divergence implies curl solution.

Theorem was

$$\begin{array}{lll} \text{If} & \boldsymbol{\nabla} \cdot \mathbf{D} & = 0 \\ \text{there is a} & \mathbf{C} & \\ \text{such that} & \mathbf{D} & = \boldsymbol{\nabla} \times \mathbf{C} \end{array}$$

As above, if $\mathbf{D} = \boldsymbol{\nabla} \times \mathbf{C}$, then we have

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{C})$$

and this has already been shown to be zero. So the problem becomes find $\mathbf{C}$ given $\mathbf{D}$.

Also, as before an equivalent generalized (or de-generalized) problem can be expressed.

That is

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D} &= \langle \boldsymbol{\nabla} \mathbf{D} \rangle \\
&= \langle \boldsymbol{\nabla} (\boldsymbol{\nabla} \times \mathbf{C}) \rangle \\
&= \langle \boldsymbol{\nabla} - i(\boldsymbol{\nabla} \wedge \mathbf{C}) \rangle \\
&= -\langle i\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{C}) \rangle - \langle i\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{C}) \rangle \\
&= -\langle i\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{C}) \rangle
\end{aligned}
$$

So if $\boldsymbol{\nabla} \cdot \mathbf{D}$ it is also true that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{C}) = 0$

Thus the (de)generalized theorem to prove is

$$
\begin{array}{lll}
\text{If} & \boldsymbol{\nabla} \cdot D & = 0 \\
\text{there is a} & C & \\
\text{such that} & D & = \boldsymbol{\nabla} \wedge C
\end{array}
$$

In the $\mathbb{R}^3$ case, to prove the original theorem we want a bivector $D = -i\mathbf{D}$, and seek a vector C such that $D = \boldsymbol{\nabla} \wedge C$ ($\mathbf{D} = -i(\boldsymbol{\nabla} \wedge C)$).

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot D &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge C) \\
&= (\sigma^k \partial_k) \cdot (\sigma^i \wedge \sigma^j \partial_i C_j) \\
&= \sigma^k \cdot (\sigma^i \wedge \sigma^j) \partial_k \partial_i C_j \\
&= (\sigma^j \delta^{ki} - \sigma^i \delta^{kj}) \partial_k \partial_i C_j \\
&= \sigma^j \partial_i \partial_i C_j - \sigma^i \partial_j \partial_i C_j \\
&= \sigma^j \partial_i (\partial_i C_j - \partial_j C_i)
\end{aligned}
$$

If this is to equal zero we must have the following constraint on C

$$
\partial_{ii} C_j = \partial_{ij} C_i \tag{6}
$$

If the following equality was also true

$$
\partial_i C_j = \partial_j C_i
$$

Then this would also work, but would also mean D equals zero so that is not an interesting solution. So, we must go back to 6 and solve for C_k in terms of D.

Suppose we have D explicitly in terms of coordinates

$$
\begin{aligned}
D &= D_{ij} \sigma^i \wedge \sigma^j \\
&= \sum_{i<j} (D_{ij} - D_{ji}) \sigma^i \wedge \sigma^j
\end{aligned}
$$

compare this to $\boldsymbol{\nabla} \wedge C$

$$\begin{aligned} C &= (\partial_i C_j)\sigma^i \wedge \sigma^j \\ &= \sum_{i<j} (\partial_i C_j - \partial_j C_i)\sigma^i \wedge \sigma^j \end{aligned}$$

With the identity

$$\partial_i C_j = D_i j$$

Equation 6 becomes

$$\begin{aligned} &\partial_{ij} C_i = \partial_i D_{ij} \\ &\implies \\ &\partial_j C_i = D_{ij} + \alpha_{ij}(x^{k \neq i}) \end{aligned}$$

Where $\alpha_{ij}(x^{k \neq i})$ is some function of all the $x^k \neq x^i$.
Integrating once more we have

$$C_i = \int \left(D_{ij} + \alpha_{ij}(x^{k \neq i}) \right) dx^j + \beta_{ij}(x^{k \neq j})$$

References

[Feynman et al.(1963)Feynman, Leighton, and Sands] R.P. Feynman, R.B. Leighton, and M.L. Sands. feynman lectures on physics.[Lectures on physics]. 1963.