1D forced harmonic oscillator. Quick solution of non-homogeneous problem.

Originally appeared at: http://sites.google.com/site/peeterjoot/math2010/1dharmonicOsc.pdf

Peeter Joot — peeter.joot@gmail.com

Feb 19, 2010 1dharmonicOsc.tex

---

1. **Motivation.**

   In [1] equation (25) we have a forced harmonic oscillator equation

$$m\ddot{x} + m\omega^2 x = \gamma(t). \tag{1}$$

   The solution of this equation is provided, but for fun lets derive it.

2. **Guts**

   Writing

$$\omega u = \dot{x}, \tag{2}$$

   we can rewrite the second order equation as a first order linear system

$$\dot{u} + \omega x = \gamma(t)/m\omega \tag{3}$$
$$\dot{x} - \omega u = 0, \tag{4}$$

Or, with $X = (u, x)$, in matrix form

$$\dot{X} + \omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X = \begin{bmatrix} \gamma(t)/m\omega \\ 0 \end{bmatrix}. \tag{5}$$

The two by two matrix has the same properties as the complex imaginary, squaring to the identity matrix, so the equation to solve is now of the form

$$\dot{X} + \omega i X = \Gamma. \tag{6}$$

The homogeneous part of the solution is just the matrix

$$\begin{aligned} X &= e^{-i\omega t} A \\ &= \left(\cos(\omega t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \sin(\omega t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) A, \end{aligned}$$

where A is a two by one column matrix of constants. Assuming for the specific solution $X = e^{-i\omega t} A(t)$, and substuiting we have

$$e^{-i\omega t} \dot{A} = \Gamma(t). \tag{7}$$

This integrates directly, fixing the unknown column vector function $A(t)$

$$A(t) = A(0) + \int_0^t e^{i\omega\tau} \Gamma(\tau). \tag{8}$$

Thus the non-homogeneous solution takes the form

$$X = e^{-i\omega t} A(0) + \int_0^t e^{i\omega(\tau - t)} \Gamma(\tau). \tag{9}$$

Note that $A(0) = (\dot{x}_0/\omega, x_0)$. Multiplying this out, and discarding all but the second row of the matrix product gives $x(t)$, and Feynman's equation (26) follows directly.

References

[1] L.M. Brown, G.D. Carson, L.F. Locke, W.W. Spirduso, S.J. Silverman, D. Holtom, E. Fisher, J.E. Mauch, J.W. Birch, K.L. Turabian, et al. *Feynman's thesis: A New approach to quantum theory*. Houghton Mifflin, 1954. 1