Integrating the equation of motion for a one dimensional problem.

Originally appeared at: `http://sites.google.com/site/peeterjoot/math2010/1dpotentialIntegral.pdf`

Peeter Joot — peeter.joot@gmail.com

Jan 1, 2010 1dpotentialIntegral.tex

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1. Motivation.

While linear approximations, such as the small angle approximation for the pendum, are often used to understand the dynamics of non-linear systems, exact solutions may be possible in some cases. Walk through the setup for such an exact solution.

2. Guts

The equation to consider solutions of has the form

$$\frac{d}{dt}\left(m\frac{dx}{dt}\right) = -\frac{\partial U(x)}{\partial x}. \tag{1}$$

We have an unpleasant mix of space and time derivatives, preventing any sort of antidifferentiation. Assuming constant mass m, and employing the chain rule a refactoring in terms of velocities is possible.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{dx}{dt}\right) &= \frac{dx}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right) \\
&= \frac{1}{2}\frac{d}{dx}\left(\frac{dx}{dt}\right)^2
\end{aligned}$$

The one dimensional Newton's law Equation 1 now takes the form

$$\frac{d}{dx}\left(\frac{dx}{dt}\right)^2 = -\frac{2}{m}\frac{\partial U(x)}{\partial x}. \tag{2}$$

This can now be antidifferentiated for

$$\left(\frac{dx}{dt}\right)^2 = \frac{2}{m}(E - U(x)). \tag{3}$$

Taking roots and rearranging produces an implicit differential form x in terms of time

$$dt = \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}}. \tag{4}$$

One can concievably integrate this and invert to solve for position as a function of time, but substitution of a more specific potential is required to go further.

$$t(x) = t(x_0) + \int_{y=x_0}^{x} \frac{dy}{\sqrt{\frac{2}{m}(E - U(y))}}. \tag{5}$$