Notes for Desai Chapter 26.

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Contents

1. Motivation.

Chapter 26 notes for [1].

2. Guts

2.1. Trig relations.

To verify equations 26.3-5 in the text it's worth noting that

$$\begin{aligned}\cos(a+b) &= \Re(e^{ia}e^{ib}) \\ &= \Re((\cos a + i\sin a)(\cos b + i \sin b)) \\ &= \cos a \cos b - \sin a \sin b\end{aligned}$$

and

$$\begin{aligned}\sin(a+b) &= \Im(e^{ia}e^{ib}) \\ &= \Im((\cos a + i\sin a)(\cos b + i \sin b)) \\ &= \cos a \sin b + \sin a \cos b\end{aligned}$$

So, for

$$x = \rho \cos \alpha \tag{1}$$
$$y = \rho \sin \alpha \tag{2}$$

the transformed coordinates are

$$\begin{aligned}
x' &= \rho \cos(\alpha + \phi) \\
&= \rho(\cos\alpha\cos\phi - \sin\alpha\sin\phi) \\
&= x\cos\phi - y\sin\phi
\end{aligned}$$

and

$$\begin{aligned}
y' &= \rho \sin(\alpha + \phi) \\
&= \rho(\cos\alpha\sin\phi + \sin\alpha\cos\phi) \\
&= x\sin\phi + y\cos\phi
\end{aligned}$$

This allows us to read off the rotation matrix. Without all the messy trig, we can also derive this matrix with geometric algebra.

$$\begin{aligned}
\mathbf{v}' &= e^{-\mathbf{e}_1\mathbf{e}_2\phi/2}\mathbf{v}e^{\mathbf{e}_1\mathbf{e}_2\phi/2} \\
&= v_3\mathbf{e}_3 + (v_1\mathbf{e}_1 + v_2\mathbf{e}_2)e^{\mathbf{e}_1\mathbf{e}_2\phi} \\
&= v_3\mathbf{e}_3 + (v_1\mathbf{e}_1 + v_2\mathbf{e}_2)(\cos\phi + \mathbf{e}_1\mathbf{e}_2\sin\phi) \\
&= v_3\mathbf{e}_3 + \mathbf{e}_1(v_1\cos\phi - v_2\sin\phi) + \mathbf{e}_2(v_2\cos\phi + v_1\sin\phi)
\end{aligned}$$

Here we use the Pauli-matrix like identities

$$\mathbf{e}_k^2 = 1 \tag{3}$$

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \quad i \neq j \tag{4}$$

and also note that $\mathbf{e}_3$ commutes with the bivector for the x, y plane $\mathbf{e}_1\mathbf{e}_2$. We can also read off the rotation matrix from this.

2.2. ***Infinitesimal transformations.***

Recall that in the problems of Chapter 5, one representation of spin one matrices were calculated [2]. Since the choice of the basis vectors was arbitrary in that exersize, we ended up with a different representation. For S_x, S_y, S_z as found in (26.20) and (26.23) we can also verify easily that we have eigenvalues $0, \pm\hbar$. We can also show that our spin kets in this non-diagonal representation have the following column matrix representations:

$$|1, \pm 1\rangle_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ \pm i \end{bmatrix} \tag{5}$$

$$|1, 0\rangle_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$|1, \pm 1\rangle_y = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm i \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

$$|1, 0\rangle_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{8}$$

$$|1, \pm 1\rangle_z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix} \tag{9}$$

$$|1, 0\rangle_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

2.3. ***Verifying the commutator relations.***

Given the (summation convention) matrix representation for the spin one operators

$$(S_i)_{jk} = -i\hbar\epsilon_{ijk}, \tag{11}$$

let's demonstrate the commutator relation of (26.25).

$$\begin{aligned}
\left[S_i, S_j\right]_{rs} &= (S_i S_j - S_j S_i)_{rs} \\
&= \sum_t (S_i)_{rt}(S_j)_{ts} - (S_j)_{rt}(S_i)_{ts} \\
&= (-i\hbar)^2 \sum_t \epsilon_{irt}\epsilon_{jts} - \epsilon_{jrt}\epsilon_{its} \\
&= -(-i\hbar)^2 \sum_t \epsilon_{tir}\epsilon_{tjs} - \epsilon_{tjr}\epsilon_{tis}
\end{aligned}$$

Now we can employ the summation rule for sums products of antisymmetic tensors over one free index (4.179)

$$\sum_i \epsilon_{ijk}\epsilon_{iab} = \delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}. \tag{12}$$

Continuing we get

$$\begin{aligned}
\left[S_i, S_j\right]_{rs} &= -(-i\hbar)^2 \left(\delta_{ij}\delta_{rs} - \delta_{is}\delta_{rj} - \delta_{ji}\delta_{rs} + \delta_{js}\delta_{ri}\right) \\
&= (-i\hbar)^2 \left(\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}\right) \\
&= (-i\hbar)^2 \sum_t \epsilon_{tij}\epsilon_{tsr} \\
&= i\hbar \sum_t \epsilon_{tij}(S_t)_{rs} \qquad \square
\end{aligned}$$

2.4. *General infinitesimal rotation.*

Equation (26.26) has for an infinitesimal rotation counterclockwise around the unit axis of rotation vector $\mathbf{n}$

$$\mathbf{V}' = \mathbf{V} + \epsilon\mathbf{n} \times \mathbf{V}. \tag{13}$$

Let's derive this using the geometric algebra rotation expression for the same

$$\begin{aligned}
\mathbf{V}' &= e^{-I\mathbf{n}\alpha/2}\mathbf{V}e^{I\mathbf{n}\alpha/2} \\
&= e^{-I\mathbf{n}\alpha/2}\left((\mathbf{V}\cdot\mathbf{n})\mathbf{n} + (\mathbf{V}\wedge\mathbf{n})\mathbf{n}\right)e^{I\mathbf{n}\alpha/2} \\
&= (\mathbf{V}\cdot\mathbf{n})\mathbf{n} + (\mathbf{V}\wedge\mathbf{n})\mathbf{n}e^{I\mathbf{n}\alpha}
\end{aligned}$$

We note that $I\mathbf{n}$ and thus the exponential commutes with $\mathbf{n}$, and the projection component in the normal direction. Similarily $I\mathbf{n}$ anticommutes with $(\mathbf{V}\wedge\mathbf{n})\mathbf{n}$. This leaves us with

$$\mathbf{V}' = (\mathbf{V}\cdot\mathbf{n})\mathbf{n}\left(+(\mathbf{V}\wedge\mathbf{n})\mathbf{n}\right)(\cos\alpha + I\mathbf{n}\sin\alpha)$$

For $\alpha = \epsilon \to 0$, this is

$$\begin{aligned}
\mathbf{V}' &= (\mathbf{V}\cdot\mathbf{n})\mathbf{n} + (\mathbf{V}\wedge\mathbf{n})\mathbf{n}(1 + I\mathbf{n}\epsilon) \\
&= (\mathbf{V}\cdot\mathbf{n})\mathbf{n} + (\mathbf{V}\wedge\mathbf{n})\mathbf{n} + \epsilon I^2(\mathbf{V}\times\mathbf{n})\mathbf{n}^2 \\
&= \mathbf{V} + \epsilon(\mathbf{n}\times\mathbf{V}) \qquad \square
\end{aligned}$$

2.5. *Position and angular momentum commutator.*

Equation (26.71) is

$$\left[x_i, L_j\right] = i\hbar\epsilon_{ijk}x_k. \tag{14}$$

Let's derive this. Recall that we have for the position-momentum commutator

$$\left[x_i, p_j\right] = i\hbar\delta_{ij}, \tag{15}$$

and for each of the angular momentum operator components we have

$$L_m = \epsilon_{mab}x_a p_b. \tag{16}$$

The commutator of interest is thus

$$\begin{aligned}
\left[x_i, L_j\right] &= x_i \epsilon_{jab} x_a p_b - \epsilon_{jab} x_a p_b x_i \\
&= \epsilon_{jab} x_a \left(x_i p_b - p_b x_i\right) \\
&= \epsilon_{jab} x_a i\hbar \delta_{ib} \\
&= i\hbar \epsilon_{jai} x_a \\
&= i\hbar \epsilon_{ija} x_a \qquad \square
\end{aligned}$$

2.6. *A note on the angular momentum operator exponential sandwiches.*

In (26.73-74) we have

$$e^{i\epsilon L_z/\hbar} x e^{-i\epsilon L_z/\hbar} = x + \frac{i\epsilon}{\hbar} \left[L_z, x\right] \tag{17}$$

Observe that

$$\left[x, \left[L_z, x\right]\right] = 0 \tag{18}$$

so from the first two terms of (10.99)

$$e^A B e^{-A} = B + \left[A, B\right] + \frac{1}{2} \left[A, \left[A, B\right]\right] \cdots \tag{19}$$

we get the desired result.

2.7. *Trace relation to the determinant.*

Going from (26.90) to (26.91) we appear to have a mystery identity

$$\det \left(\mathbf{1} + \mu \mathbf{A}\right) = 1 + \mu \operatorname{Tr} \mathbf{A} \tag{20}$$

According to wikipedia, under derivative of a determinant, [3], this is good for small μ, and related to something called the Jacobi identity. Someday I should really get around to studying determinants in depth, and will take this one for granted for now.

References

[1] BR Desai. *Quantum mechanics with basic field theory*. Cambridge University Press, 2009. 1

[2] Peeter Joot. Notes and problems for Desai Chapter V. [online]. Available from: http://sites.google.com/site/peeterjoot/math2010/desaiCh5.pdf. 2.2

[3] Wikipedia. Determinant — wikipedia, the free encyclopedia [online]. 2010. [Online; accessed 10-December-2010]. Available from: http://en.wikipedia.org/w/index.php?title=Determinant&oldid=400983667. 2.7