Notes and problems for Desai chapter III.

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Contents

1. Notes.

Chapter III notes and problems for [1].

FIXME: Some puzzling stuff in the interaction section and superposition of time-dependent states sections. Work through those here.

2. **Problems**

2.1. Problem 1. Virial Theorem.

2.1.1 Statement.

With the assumption that $\langle \mathbf{r}\cdot\mathbf{p} \rangle$ is independent of time, and

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}) = T + V \tag{1}$$

show that

$$2 \langle T \rangle = \langle \mathbf{r}\cdot\boldsymbol{\nabla} V \rangle . \tag{2}$$

2.1.2 Solution.

I floundered with this a bit, but found the required hint in physicsforums. We can start with the Hamiltonian time derivative relation

$$i\hbar \frac{dA_H}{dt} = [A_H, H] \tag{3}$$

So, with the assumption that $\langle \mathbf{r}\cdot\mathbf{p} \rangle$ is independent of time, and the use of a stationary state $|\psi\rangle$ for the expectation calculation we have

$$\begin{aligned}
0 &= \frac{d}{dt} \langle \mathbf{r}\cdot\mathbf{p} \rangle \\
&= \frac{d}{dt} \langle \psi | \mathbf{r}\cdot\mathbf{p} | \psi \rangle \\
&= \langle \psi | \frac{d}{dt} (\mathbf{r}\cdot\mathbf{p}) | \psi \rangle \\
&= \frac{1}{i\hbar} \langle [\mathbf{r}\cdot\mathbf{p}, H] \rangle \\
&= - \left\langle \left[\mathbf{r}\cdot\boldsymbol{\nabla}, \frac{\mathbf{p}^2}{2m} \right] \right\rangle - \langle [\mathbf{r}\cdot\boldsymbol{\nabla}, V(\mathbf{r})] \rangle .
\end{aligned}$$

The exercise now becomes one of evaluating the remaining commutators. For the Laplacian commutator we have

$$\begin{aligned}
\left[\mathbf{r}\cdot\boldsymbol{\nabla}, \boldsymbol{\nabla}^2 \right] \psi &= x_m \partial_m \partial_n \partial_n \psi - \partial_n \partial_n x_m \partial_m \psi \\
&= x_m \partial_m \partial_n \partial_n \psi - \partial_n \partial_n \psi - \partial_n x_m \partial_n \partial_m \psi \\
&= x_m \partial_m \partial_n \partial_n \psi - \partial_n \partial_n \psi - \partial_n \partial_n \psi - x_m \partial_n \partial_n \partial_m \psi \\
&= -2 \boldsymbol{\nabla}^2 \psi
\end{aligned}$$

For the potential commutator we have

$$
\begin{aligned}
[\mathbf{r} \cdot \boldsymbol{\nabla}, V(\mathbf{r})]\, \psi &= x_m \partial_m V \psi - V x_m \partial_m \psi \\
&= x_m (\partial_m V) \psi x_m V \partial_m \psi - V x_m \partial_m \psi \\
&= \Big(\mathbf{r} \cdot (\boldsymbol{\nabla} V) \Big) \psi
\end{aligned}
$$

Putting all the $\hbar$ factors back in, we get

$$
2 \left\langle \frac{\mathbf{p}^2}{2m} \right\rangle = \langle \mathbf{r} \cdot (\boldsymbol{\nabla} V) \rangle \,, \tag{4}
$$

which is the desired result.
Followup: why assume $\langle \mathbf{r} \cdot \mathbf{p} \rangle$ is independent of time?

2.2. ***Problem 2. Application of virial theorem.***

Calculate $\langle T \rangle$ with $V = \lambda \ln(r/a)$.

$$
\begin{aligned}
\mathbf{r} \cdot \boldsymbol{\nabla} V &= r \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \lambda \frac{\partial \ln(r/a)}{\partial r} \\
&= \lambda r \frac{1}{a} \frac{a}{r} \\
&= \lambda \\
\Longrightarrow & \\
\langle T \rangle &= \lambda / 2
\end{aligned}
$$

2.3. ***Problem 3. Heisenberg Position operator representation.***

2.3.1 Part I.

Express x as an operator x_H for $H = \mathbf{p}^2/2m$.
With

$$
\langle \psi | x | \psi \rangle = \langle \psi_0 | U^\dagger x U | \psi_0 \rangle
$$

We want to expand

$$
\begin{aligned}
x_H &= U^\dagger x U \\
&= e^{iHt/\hbar} x e^{-iHt/\hbar} \\
&= \sum_{k,l=0}^{\infty} \frac{1}{k!} \frac{1}{l!} \left(\frac{iHt}{\hbar} \right)^k x \left(\frac{-iHt}{\hbar} \right)^l .
\end{aligned}
$$

We to evaluate $H^k x H^l$ to proceed. Using $p^n x = -i\hbar n p^{n-1} + x p^n$, we have

$$
\begin{aligned}
H^k x &= \frac{1}{(2m)^k} p^2 k x \\
&= \frac{1}{(2m)^k} \left(-i\hbar (2k) p^{2k-1} + x p^2 k \right) \\
&= x H^k + \frac{1}{2m} (-i\hbar)(2k) p p^{2(k-1)} / (2m)^{k-1} \\
&= x H^k - \frac{i\hbar k}{m} p H^{k-1}.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
x_H &= x - \frac{i\hbar p}{m} \sum_{k,l=0}^{\infty} \frac{k}{k!} \frac{1}{l!} \left(\frac{it}{\hbar} \right)^k H^{k-1+l} \left(\frac{-it}{\hbar} \right)^l \\
&= x - \frac{i\hbar p i t}{m\hbar}
\end{aligned}
$$

Or

$$
x_H = x + \frac{pt}{m} \tag{5}
$$

2.3.2 Part II.

Express x as an operator x_H for $H = \mathbf{p}^2/2m + V$ with $V = \lambda x^m$.

In retrospect, for the first part of this problem, it would have been better to use the series expansion for this exponential sandwich

Or, in explicit form

$$
e^A B e^{-A} = B + \frac{1}{1!} [A, B] + \frac{1}{2!} [A, [A, B]] + \cdots \tag{6}
$$

Doing so, we'd find for the first commutator

$$
\frac{it}{2m\hbar} \left[\mathbf{p}^2, x \right] = \frac{tp}{m}, \tag{7}
$$

so that the series has only the first two terms, and we'd obtain the same result. That seems like a logical approach to try here too. For the first commutator, we get the same tp/m result since $[V, x] = 0$.

Employing

$$
x^n p = i\hbar n x^{n-1} + p x^n, \tag{8}
$$

I find

$$
\begin{aligned}
\left(\frac{it}{\hbar}\right)^2 [H,[H,x]] &= \frac{i\lambda t^2}{\hbar m}\left[x^n, p\right] \\
&= -\frac{n t^2 \lambda}{m} x^{n-1} \\
&= -\frac{n t^2 V}{m x}
\end{aligned}
$$

The triple commutator gets no prettier, and I get

$$
\begin{aligned}
\left(\frac{it}{\hbar}\right)^3 [H,[H,[H,x]]] &= \frac{it}{\hbar}\left[\frac{\mathbf{p}^2}{2m} + \lambda x^n, -\frac{n t^2 V}{m x}\right] \\
&= -\frac{it}{\hbar}\frac{n t^2}{m}\frac{\lambda}{2m}\left[\mathbf{p}^2, x^{n-1}\right] \\
&= \cdots \\
&= \frac{n(n-1)t^3 V}{2m^2 x^3}(i\hbar n + 2px).
\end{aligned}
$$

Putting all the pieces together this gives

$$
x_H = e^{iHt/\hbar} x e^{-iHt/\hbar} = x + \frac{tp}{m} - \frac{n t^2 V}{2mx} + \frac{n(n-1)t^3 V}{12 m^2 x^3}(i\hbar n + 2px) + \cdots \tag{9}
$$

If there is a closed form for this it isn't obvious to me. Would a fixed lower degree potential function shed any more light on this. How about the Harmonic oscillator Hamiltonian

$$
H = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 \tag{10}
$$

... this one works out nicely since there's an even-odd alternation.
Get

$$
x_H = x\cos(\omega^2 t^2/2) + \frac{pt}{m}\frac{\sin(\omega^2 t^2/2)}{\omega^2 t^2/2} \tag{11}
$$

I'd not expect such a tidy result for an arbitrary $V(x) = \lambda x^n$ potential.

2.4. ***Problem 4. Feynman-Hellman relation.***

For continuously parametrized eigenstate, eigenvalue and Hamiltonian $|\psi(\lambda)\rangle$, $E(\lambda)$ and $H(\lambda)$ respectively, we can relate the derivatives

$$\frac{\partial}{\partial\lambda}(H|\psi\rangle) = \frac{\partial}{\partial\lambda}(E|\psi\rangle)$$
$$\Longrightarrow$$
$$\frac{\partial H}{\partial\lambda}|\psi\rangle + H\frac{\partial|\psi\rangle}{\partial\lambda} = \frac{\partial E}{\partial\lambda}|\psi\rangle + E\frac{\partial|\psi\rangle}{\partial\lambda}$$

Left multiplication by $\langle\psi|$ gives

$$\langle\psi|\frac{\partial H}{\partial\lambda}|\psi\rangle + \langle\psi|H\frac{\partial|\psi\rangle}{\partial\lambda} = \langle\psi|\frac{\partial E}{\partial\lambda}|\psi\rangle + E\langle\psi|\frac{\partial|\psi\rangle}{\partial\lambda}$$
$$\Longrightarrow$$
$$\langle\psi|\frac{\partial H}{\partial\lambda}|\psi\rangle + (\langle\psi|E)\frac{\partial|\psi\rangle}{\partial\lambda} = \langle\psi|\frac{\partial E}{\partial\lambda}|\psi\rangle + E\langle\psi|\frac{\partial|\psi\rangle}{\partial\lambda}$$
$$\Longrightarrow$$
$$\langle\psi|\frac{\partial H}{\partial\lambda}|\psi\rangle = \frac{\partial E}{\partial\lambda}\langle\psi|\psi\rangle,$$

which provides the desired identity

$$\frac{\partial E}{\partial\lambda} = \langle\psi(\lambda)|\frac{\partial H}{\partial\lambda}|\psi(\lambda)\rangle \tag{12}$$

2.5. ***Problem 5.***

2.5.1 Description.

With eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$, of H with eigenvalues E_1 and E_2, respectively, and

$$|\chi_1\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle)$$
$$|\chi_2\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle)$$

and $|\psi(0)\rangle = |\chi_1\rangle$, determine $|\psi(t)\rangle$ in terms of $|\phi_1\rangle$ and $|\phi_2\rangle$.

2.5.2 Solution.

$$\begin{aligned}
|\psi(t)\rangle &= e^{-iHt/\hbar}|\psi(0)\rangle \\
&= e^{-iHt/\hbar}|\chi_1\rangle \\
&= \frac{1}{\sqrt{2}}e^{-iHt/\hbar}(|\phi_1\rangle - |\phi_2\rangle) \\
&= \frac{1}{\sqrt{2}}(e^{-iE_1t/\hbar}|\phi_1\rangle - e^{-iE_2t/\hbar}|\phi_2\rangle) \qquad \square
\end{aligned}$$

2.6. ***Problem 6.***

2.6.1 Description.

Consider a Coulomb like potential $-\lambda/r$ with angular momentum $l = 0$. If the eigenfunction is

$$u(r) = u_0 e^{-\beta r} \tag{13}$$

determine u_0, β, and the energy eigenvalue E in terms of λ, and m.

2.6.2 Solution.

We can start with the normalization constant u_0 by integrating

$$\begin{aligned}
1 &= u_0^2 \int_0^\infty dr e^{-\beta r} e^{-\beta r} \\
&= u_0^2 \left.\frac{e^{-2\beta r}}{-2\beta}\right|_{0^\infty} \\
&= u_0^2 \frac{1}{2\beta}
\end{aligned}$$

$$u_0 = \sqrt{2\beta} \tag{14}$$

To go further, we need the Hamiltonian. Note that we can write the Laplacian with the angular momentum operator factored out using

$$\boldsymbol{\nabla}^2 = \frac{1}{\mathbf{x}^2} \left((\mathbf{x} \cdot \boldsymbol{\nabla})^2 + \mathbf{x} \cdot \boldsymbol{\nabla} + (\mathbf{x} \times \boldsymbol{\nabla})^2 \right) \tag{15}$$

With zero for the angular momentum operator $\mathbf{x} \times \boldsymbol{\nabla}$, and switching to spherical coordinates, we have

$$\begin{aligned}
\boldsymbol{\nabla}^2 &= \frac{1}{r}\partial_r + \frac{1}{r}\partial_r r \partial_r \\
&= \frac{1}{r}\partial_r + \frac{1}{r}\partial_r + \frac{1}{r} r \partial_{rr} \\
&= \frac{2}{r}\partial_r + \partial_{rr}
\end{aligned}$$

We can now write the Hamiltonian for the zero angular momentum case

$$H = -\frac{\hbar^2}{2m} \left(\frac{2}{r}\partial_r + \partial_{rr} \right) - \frac{\lambda}{r} \tag{16}$$

With application of this Hamiltonian to the eigenfunction we have

$$\begin{aligned}
E u_0 e^{-\beta r} &= \left(-\frac{\hbar^2}{2m} \left(\frac{2}{r}\partial_r + \partial_{rr} \right) - \frac{\lambda}{r} \right) u_0 e^{-\beta r} \\
&= \left(-\frac{\hbar^2}{2m} \left(\frac{2}{r}(-\beta) + \beta^2 \right) - \frac{\lambda}{r} \right) u_0 e^{-\beta r}.
\end{aligned}$$

In particular for $r = \infty$ we have

$$-\frac{\hbar^2 \beta^2}{2m} = E \tag{17}$$

$$\begin{aligned}
-\frac{\hbar^2 \beta^2}{2m} &= \left(-\frac{\hbar^2}{2m} \left(\frac{2}{r}(-\beta) + \beta^2 \right) - \frac{\lambda}{r} \right) \\
&\Longrightarrow \\
\frac{\hbar^2}{2m}\frac{2}{r}\beta &= \frac{\lambda}{r}
\end{aligned}$$

Collecting all the results we have

$$\beta = \frac{\lambda m}{\hbar^2} \tag{18}$$

$$E = -\frac{\lambda^2 m}{2\hbar^2} \tag{19}$$

$$u_0 = \frac{\sqrt{2\lambda m}}{\hbar} \tag{20}$$

2.7. ***Problem 7.***

2.7.1 Description.

A particle in a uniform field $\mathbf{E}_0$. Show that the expectation value of the position operator $\langle \mathbf{r} \rangle$ satisfies

$$m \frac{d^2 \langle \mathbf{r} \rangle}{dt^2} = e\mathbf{E}_0. \tag{21}$$

2.7.2 Solution.

This follows from Ehrehfest's theorem once we formulate the force $e\mathbf{E}_0 = -\boldsymbol{\nabla}\phi$, in terms of a potential ϕ. That potential is

$$\phi = -e\mathbf{E}_0 \cdot (x, y, z) \tag{22}$$

The Hamiltonian is therefore

$$H = \frac{\mathbf{p}^2}{2m} - e\mathbf{E}_0 \cdot (x, y, z). \tag{23}$$

Ehrehfest's theorem gives us

$$\begin{aligned}
\frac{d}{dt} \langle x_k \rangle &= \frac{1}{m} \langle p_k \rangle \\
\frac{d}{dt} \langle p_k \rangle &= - \left\langle \frac{\partial V}{\partial x_k} \right\rangle,
\end{aligned}$$

or

$$\frac{d^2}{dt^2} \langle x_k \rangle = -\frac{1}{m} \left\langle \frac{\partial V}{\partial x_k} \right\rangle. \tag{24}$$

$$\frac{\partial V}{\partial x_k} = -e(\mathbf{E}_0)_k$$

Putting all the last bits together, and summing over the directions $\mathbf{e}_k$ we have

$$m\frac{d^2}{dt^2}\mathbf{e}_k \langle x_k \rangle = \mathbf{e}_k \langle e(\mathbf{E}_0)_k \rangle = e\mathbf{E}_0 \qquad \square$$

2.8. ***Problem 8.***

2.8.1 Description.

For Hamiltonian eigenstates $|E_n\rangle$, $C = AB$, $A = [B, H]$, obtain the matrix element $\langle E_m|C|E_n\rangle$ in terms of the matrix element of A.

2.8.2 Solution.

I was able to get most of what was asked for here, with a small exception. I started with the matrix element for A, which is

$$\langle E_m|A|E_n\rangle = \langle E_m|BH - HB|E_n\rangle = (E_n - E_m)\langle E_m|B|E_n\rangle \tag{25}$$

Next, computing the matrix element for C we have

$$
\begin{aligned}
\langle E_m|C|E_n\rangle &= \langle E_m|BHB - HB^2|E_n\rangle \\
&= \sum_a \langle E_m|BH|E_a\rangle\langle E_a|B|E_n\rangle - E_m\langle E_m|B|E_a\rangle\langle E_a|B|E_n\rangle \\
&= \sum_a E_a\langle E_m|B|E_a\rangle\langle E_a|B|E_n\rangle - E_m\langle E_m|B|E_a\rangle\langle E_a|B|E_n\rangle \\
&= \sum_a (E_a - E_m)\langle E_m|B|E_a\rangle\langle E_a|B|E_n\rangle \\
&= \sum_a \langle E_m|A|E_a\rangle\langle E_a|B|E_n\rangle \\
&= \langle E_m|A|E_n\rangle\langle E_n|B|E_n\rangle + \sum_{a\neq n} \langle E_m|A|E_a\rangle\langle E_a|B|E_n\rangle \\
&= \langle E_m|A|E_n\rangle\langle E_n|B|E_n\rangle + \sum_{a\neq n} \langle E_m|A|E_a\rangle \frac{\langle E_a|A|E_n\rangle}{E_n - E_a}
\end{aligned}
$$

Except for the $\langle E_n|B|E_n\rangle$ part of this expression, the problem as stated is complete. The relationship 25 is no help for with $n = m$, so I see no choice but to leave that small part of the expansion in terms of B.

*2.9. **Problem 9.***

2.9.1 Description.

Operator A has eigenstates $|a_i\rangle$, with a unitary change of basis operation $U|a_i\rangle = |b_i\rangle$. Determine in terms of U, and A the operator B and its eigenvalues for which $|b_i\rangle$ are eigenstates.

2.9.2 Solution.

Consider for motivation the matrix element of A in terms of $|b_i\rangle$. We will also let $A|a_i\rangle = \alpha_i|a_i\rangle$. We then have

$$
\langle a_i|A|a_j\rangle = \langle b_i|UAU^\dagger|b_j\rangle
$$

We also have

$$
\begin{aligned}
\langle a_i|A|a_j\rangle &= a_j\langle a_i||a_j\rangle \\
&= a_j\delta_{ij}
\end{aligned}
$$

So it appears that the operator $UAU^\dagger$ has the orthonormality relation required. In terms of action on the basis $\{|b_i\rangle\}$, let's see how it behaves. We have

$$
\begin{aligned}
UAU^\dagger|b_i\rangle &= UA|a_i\rangle \\
&= U\alpha_i|a_i\rangle \\
&= \alpha_i|b_i\rangle
\end{aligned}
$$

So we see that the operators A and $B = UAU^\dagger$ have common eigenvalues.

*2.10. **Problem 10.***

2.10.1 Description.

With $H|n\rangle = E_n|n\rangle$, $A = [H, F]$ and $\langle 0|F|0\rangle = 0$, show that

$$\sum_{n\neq 0} \frac{\langle 0|A|n\rangle\langle n|A|0\rangle}{E_n - E_0} = \langle 0|AF|0\rangle \tag{26}$$

2.10.2 Solution.

$$\begin{aligned}
\langle 0|AF|0\rangle &= \langle 0|HFF - FHF|0\rangle \\
&= \sum_n E_0\langle 0|F|n\rangle\langle n|F|0\rangle - E_n\langle 0|F|n\rangle\langle n|F|0\rangle \\
&= \sum_n (E_0 - E_n)\langle 0|F|n\rangle\langle n|F|0\rangle \\
&= \sum_{n\neq 0} (E_0 - E_n)\langle 0|F|n\rangle\langle n|F|0\rangle
\end{aligned}$$

We also have

$$\begin{aligned}
\langle 0|A|n\rangle\langle n|A|0\rangle &= \langle 0|HF - FH|n\rangle\langle n|A|0\rangle \\
&= (E_0 - E_n)\langle 0|F|n\rangle\langle n|HF - FH|0\rangle \\
&= -(E_0 - E_n)^2\langle 0|F|n\rangle\langle n|F|0\rangle
\end{aligned}$$

Or, for $n \neq 0$,

$$\langle 0|F|n\rangle\langle n|F|0\rangle = -\frac{\langle 0|A|n\rangle\langle n|A|0\rangle}{(E_0 - E_n)^2}.$$

This gives

$$\begin{aligned}
\langle 0|AF|0\rangle &= -\sum_{n\neq 0} (E_0 - E_n)\frac{\langle 0|A|n\rangle\langle n|A|0\rangle}{(E_0 - E_n)^2} \\
&= \sum_{n\neq 0} \frac{\langle 0|A|n\rangle\langle n|A|0\rangle}{E_n - E_0} \qquad \square
\end{aligned}$$

*2.11. **Problem 11. commutator of angular momentum with Hamiltonian.***

Show that $[\mathbf{L}, H] = 0$, where $H = \mathbf{p}^2/2m + V(r)$.
This follows by considering $\left[\mathbf{L}, \mathbf{p}^2\right]$, and $[\mathbf{L}, V(r)]$. Let

$$L_{jk} = x_j p_k - x_k p_j, \tag{27}$$

so that

$$\mathbf{L} = \mathbf{e}_i \epsilon_{ijk} L_{jk}. \tag{28}$$

We now need to consider the commutators of the operators L_{jk} with $\mathbf{p}^2$ and $V(r)$. Let's start with p^2. In particular

$$\begin{aligned}
\mathbf{p}^2 x_m p_n &= p_k p_k x_m p_n \\
&= p_k (p_k x_m) p_n \\
&= p_k (-i\hbar \delta_{km} + x_m p_k) p_n \\
&= -i\hbar p_m p_n + (p_k x_m) p_k p_n \\
&= -i\hbar p_m p_n + (-i\hbar \delta_{km} + x_m p_k) p_k p_n \\
&= -2i\hbar p_m p_n + x_m p_n \mathbf{p}^2.
\end{aligned}$$

So our commutator with $\mathbf{p}^2$ is

$$\left[L_{jk}, \mathbf{p}^2\right] = (x_j p_k - x_j p_k)\mathbf{p}^2 - (-2i\hbar p_j p_k + x_j p_k \mathbf{p}^2 + 2i\hbar p_k p_j - x_k p_j \mathbf{p}^2).$$

Since $p_j p_k = p_k p_j$, all terms cancel out, and the problem is reduced to showing that

$$[\mathbf{L}, H] = [\mathbf{L}, V(r)] = 0.$$

Now assume that $V(r)$ has a series representation

$$V(r) = \sum_j a_j r^j = \sum_j a_j (x_k x_k)^{j/2}$$

We'd like to consider the action of $x_m p_n$ on this function

$$\begin{aligned}
x_m p_n V(r)\Psi &= -i\hbar x_m \sum_j a_j \partial_n (x_k x_k)^{j/2} \Psi \\
&= -i\hbar x_m \sum_j a_j (j x_n (x_k x_k)^{j/2-1} + r^j \partial_n \Psi) \\
&= -\frac{i\hbar x_m x_n}{r^2} \sum_j a_j j r^j + x_m V(r) p_n \Psi
\end{aligned}$$

$$\begin{aligned}
L_{mn} V(r) &= (x_m p_n - x_n p_m) V(r) \\
&= -\frac{i\hbar x_m x_n}{r^2} \sum_j a_j j r^j + \frac{i\hbar x_n x_m}{r^2} \sum_j a_j j r^j + V(r)(x_m p_n - x_n p_m) \\
&= V(r) L_{mn}
\end{aligned}$$

Thus $[L_{mn}, V(r)] = 0$ as expected, implying $[\mathbf{L}, H] = 0$.

References

[1] BR Desai. *Quantum mechanics with basic field theory*. Cambridge University Press, 2009. 1