Unitary exponential sandwich

Originally appeared at:
http://sites.google.com/site/peeterjoot/math2010/exponentialSandwichCommutator.pdf

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Sept 27, 2010 exponentialSandwichCommutator.tex

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1. Motivation.

One of the chapter II exercises in [1] involves a commutator exponential sandwich of the form

$$e^{iF} B e^{-iF} \tag{1}$$

where F is Hermitian. Asking about commutators on physicsforums I was told that such sandwiches (my term) preserve expectation values, and also have a Taylor series like expansion involving the repeated commutators. Let's derive the commutator relationship.

2. Guts

Let's expand a sandwich of this form in series, and shuffle the summation order so that we sum over all the index plane diagonals $k + m =$ constant. That is

$$\begin{aligned}
e^A B e^{-A} &= \sum_{k,m=0}^{\infty} \frac{1}{k!m!} A^k B (-A)^m \\
&= \sum_{r=0}^{\infty} \sum_{m=0}^{r} \frac{1}{(r-m)!m!} A^{r-m} B (-A)^m \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{m=0}^{r} \frac{r!}{(r-m)!m!} A^{r-m} B (-A)^m \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{m=0}^{r} \binom{r}{m} A^{r-m} B (-A)^m.
\end{aligned}$$

Assuming that these interior sums can be written as commutators, we'll shortly have an induction exercise. Let's write these out for a couple values of r to get a feel for things.

- $r = 1$

$$\binom{1}{0} AB + \binom{1}{1} B(-A) = [A, B]$$

- $r = 2$

$$\binom{2}{0}A^2B+\binom{2}{1}AB(-A)+\binom{2}{2}B(-A)^2 = A^2B-2ABA+BA$$

This compares exactly to the double commutator:

$$\begin{aligned}[A,[A,B]] &= A(AB-BA)-(AB-BA)A \\ &= A^2B-ABA-ABA+BA^2 \\ &= A^2B-2ABA+BA^2\end{aligned}$$

- $r=3$

$$\binom{3}{0}A^3B+\binom{3}{1}A^2B(-A)+\binom{3}{2}AB(-A)^2+\binom{3}{3}B(-A)^3 = A^3B-3A^2BA+3ABA^2-BA^3.$$

And this compares exactly to the triple commutator

$$\begin{aligned}[A,[A,[A,B]]] &= A^3B-2A^2BA+ABA^2-(A^2BA-2ABA^2+BA^3) \\ &= A^3B-3A^2BA+3ABA^2-BA^3\end{aligned}$$

The induction pattern is clear. Let's write the r fold commutator as

$$C_r(A,B) \equiv \underbrace{[A,[A,\cdots,[A,}_{r\text{times}}B]]\cdots] = \sum_{m=0}^{r}\binom{r}{m}A^{r-m}B(-A)^m, \tag{2}$$

and calculate this for the $r+1$ case to verify the induction hypothesis. We have

$$\begin{aligned}C_{r+1}(A,B) &= \sum_{m=0}^{r}\binom{r}{m}\left(A^{r-m+1}B(-A)^m-A^{r-m}B(-A)^mA\right) \\ &= \sum_{m=0}^{r}\binom{r}{m}\left(A^{r-m+1}B(-A)^m+A^{r-m}B(-A)^{m+1}\right) \\ &= A^{r+1}B+\sum_{m=1}^{r}\binom{r}{m}A^{r-m+1}B(-A)^m+\sum_{m=0}^{r-1}\binom{r}{m}A^{r-m}B(-A)^{m+1}+B(-A)^{r+1} \\ &= A^{r+1}B+\sum_{k=0}^{r-1}\binom{r}{k+1}A^{r-k}B(-A)^{k+1}+\sum_{m=0}^{r-1}\binom{r}{m}A^{r-m}B(-A)^{m+1}+B(-A)^{r+1} \\ &= A^{r+1}B+\sum_{k=0}^{r-1}\left(\binom{r}{k+1}+\binom{r}{k}\right)A^{r-k}B(-A)^{k+1}+B(-A)^{r+1}\end{aligned}$$

We now have to sum those binomial coefficients. I like the search and replace technique for this, picking two visibly distinct numbers for r, and k that are easy to manipulate without abstract confusion. How about $r = 7$, and $k = 3$. Using those we have

$$\begin{aligned}
\binom{7}{3+1} + \binom{7}{3} &= \frac{7!}{(3+1)!(7-3-1)!} + \frac{7!}{3!(7-3)!} \\
&= \frac{7!(7-3)}{(3+1)!(7-3)!} + \frac{7!(3+1)}{(3+1)!(7-3)!} \\
&= \frac{7!\,(7-3+3+1)}{(3+1)!(7-3)!} \\
&= \frac{(7+1)!}{(3+1)!((7+1)-(3+1))!}.
\end{aligned}$$

Straight text replacement of 7 and 3 with r and k respectively now gives the harder to follow, but more general identity

$$\begin{aligned}
\binom{r}{k+1} + \binom{r}{k} &= \frac{r!}{(k+1)!(r-k-1)!} + \frac{r!}{k!(r-k)!} \\
&= \frac{r!(r-k)}{(k+1)!(r-k)!} + \frac{r!(k+1)}{(k+1)!(r-k)!} \\
&= \frac{r!\,(r-k+k+1)}{(k+1)!(r-k)!} \\
&= \frac{(r+1)!}{(k+1)!((r+1)-(k+1))!} \\
&= \binom{r+1}{k+1}
\end{aligned}$$

For our commutator we now have

$$\begin{aligned}
C_{r+1}(A,B) &= A^{r+1}B + \sum_{k=0}^{r-1} \binom{r+1}{k+1} A^{r-k}B(-A)^{k+1} + B(-A)^{r+1} \\
&= A^{r+1}B + \sum_{s=1}^{r} \binom{r+1}{s} A^{r+1-s}B(-A)^{s} + B(-A)^{r+1} \\
&= \sum_{s=0}^{r+1} \binom{r+1}{s} A^{r+1-s}B(-A)^{s} \qquad \square
\end{aligned}$$

That completes the inductive proof and allows us to write

$$e^{A}Be^{-A} = \sum_{r=0}^{\infty} \frac{1}{r!} C_r(A,B), \tag{3}$$

Or, in explicit form

$$e^{A}Be^{-A} = B + \frac{1}{1!}\,[A,B] + \frac{1}{2!}\,[A,[A,B]] + \cdots \tag{4}$$

References

[1] BR Desai. *Quantum mechanics with basic field theory*. Cambridge University Press, 2009. 1