Classical Electrodynamic gauge interaction.

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Contents

1. Motivation.

In [1] chapter 6, we have a statement that in classical mechanics the electromagnetic interaction is due to a transformation of the following form

$$\mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c}\mathbf{A} \tag{1}$$

$$E \rightarrow E - e\phi \tag{2}$$

Let's verify that this does produce the classical interaction law. Putting a more familiar label on this, we should see that we obtain the Lorentz force law from a transformation of the Hamiltonian.

2. Hamiltonian equations.

Recall that the Hamiltonian was defined in terms of conjugate momentum components p_k as

$$H(x_k, p_k) = \dot{x}_k p_k - \mathcal{L}(x_k, \dot{x}_k), \tag{3}$$

we can take x_k partials to obtain the first of the Hamiltonian system of equations for the motion

$$\begin{aligned}\frac{\partial H}{\partial x_k} &= -\frac{\partial \mathcal{L}}{\partial x_k} \\ &= -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_k}\end{aligned}$$

With $p_k \equiv \partial \mathcal{L}/\partial \dot{x}_k$, and taking p_k partials too, we have the system of equations

$$\frac{\partial H}{\partial x_k} = -\frac{dp_k}{dt} \tag{4a}$$

$$\frac{\partial H}{\partial p_k} = \dot{x}_k \tag{4b}$$

3. **Classical interaction**

Starting with the free particle Hamiltonian

$$H = \frac{\mathbf{p}}{2m}, \tag{5}$$

we make the transformation required to both the energy and momentum terms

$$H - e\phi = \frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}{2m} = \frac{1}{2m}\mathbf{p}^2 - \frac{e}{mc}\mathbf{p}\cdot\mathbf{A} + \frac{1}{2m}\left(\frac{e}{c}\right)^2\mathbf{A}^2 \tag{6}$$

From 4b we find

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial p_k} = \frac{1}{m}\left(p_k - \frac{e}{c}A_k\right), \tag{7}$$

or

$$p_k = m\frac{dx_k}{dt} + \frac{e}{c}A_k. \tag{8}$$

Taking derivatives and employing 4a we have

$$\begin{aligned}
\frac{dp_k}{dt} &= m\frac{d^2x_k}{dt^2} + \frac{e}{c}\frac{dA_k}{dt} \\
&= -\frac{\partial H}{\partial x_k} \\
&= \frac{1}{m}\frac{e}{c}p_n\frac{\partial A_n}{\partial x_k} - e\frac{\partial \phi}{\partial x_k} - \frac{1}{m}\left(\frac{e}{c}\right)^2 A_k\frac{\partial A_k}{\partial x_k} \\
&= \frac{1}{m}\frac{e}{c}\left(m\frac{dx_n}{dt} + \frac{e}{c}A_n\right)\frac{\partial A_n}{\partial x_k} - e\frac{\partial \phi}{\partial x_k} - \frac{1}{m}\left(\frac{e}{c}\right)^2 A_k\frac{\partial A_k}{\partial x_k} \\
&= \frac{e}{c}\frac{dx_n}{dt}\frac{\partial A_n}{\partial x_k} - e\frac{\partial \phi}{\partial x_k}
\end{aligned}$$

Rearranging and utilizing the convective derivative expansion $d/dt = (dx_a/dt)\partial/\partial x_a$ (ie: chain rule), we have

$$m\frac{d^2x_k}{dt^2} = \frac{e}{c}\frac{dx_n}{dt}\left(\frac{\partial A_n}{\partial x_k} - \frac{\partial A_k}{\partial x_n}\right) - e\frac{\partial \phi}{\partial x_k} \tag{9}$$

We guess and expect that the first term of 9 is $e(\mathbf{v}/c \times \mathbf{B})_k$. Let's verify this

$$\begin{aligned}
(\mathbf{v}\times\mathbf{B})_k &= \dot{x}_m B_d \epsilon_{kmd} \\
&= \dot{x}_m(\epsilon_{dab}\partial_a A_b)\epsilon_{kmd} \\
&= \dot{x}_m \partial_a A_b \epsilon_{dab}\epsilon_{dkm}
\end{aligned}$$

Since $\epsilon_{dab}\epsilon_{dkm} = \delta_{ak}\delta_{bm} - \delta_{am}\delta_{bk}$ we have

$$
\begin{aligned}
(\mathbf{v} \times \mathbf{B})_k &= \dot{x}_m \partial_a A_b \epsilon_{dab} \epsilon_{dkm} \\
&= \dot{x}_m \partial_a A_b \delta_{ak}\delta_{bm} - \dot{x}_m \partial_a A_b \delta_{am}\delta_{bk} \\
&= \dot{x}_m (\partial_k A_m - \partial_m A_k)
\end{aligned}
$$

This matches what we had in 9, and we are able to put that into the traditional Lorentz force vector form

$$
m \frac{d^2 \mathbf{x}}{dt^2} = e \frac{\mathbf{v}}{c} \times \mathbf{B} + e\mathbf{E}. \tag{10}
$$

It's good to see that we get the classical interaction from this transformation before moving on to the trickier seeming QM interaction.

References

[1] BR Desai. *Quantum mechanics with basic field theory*. Cambridge University Press, 2009. 1