Hoop and spring oscillator problem.

Originally appeared at:
`http://sites.google.com/site/peeterjoot/math2010/hoopSpring.pdf`

Peeter Joot — peeter.joot@gmail.com

June 19, 2010   hoopSpring.tex

---

1. **Motivation.**

Nolan was attempting to setup and solve the equations for the following system (1)

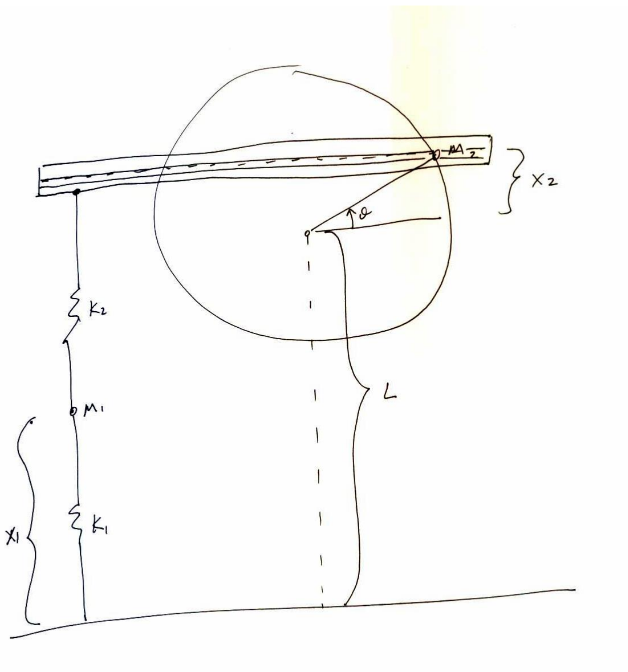

Figure 1: Coupled hoop and spring system.

One mass is connected between two springs to a bar. That bar moves up and down as forced by the motion of the other mass along a immovable hoop. While Nolan didn't include any gravitational force in his potential terms (ie: system lying on a table perhaps) it doesn't take much more to include that, and I'll do so. I also include the distance L to the center of the hoop, which I believe required.

2. **Guts**

The Lagrangian can be written by inspection. Writing $x = x_1$, and $x_2 = R\sin\theta$, we have

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2R^2\dot{\theta}^2 - \frac{1}{2}k_1x^2 - \frac{1}{2}k_2(L + R\sin\theta - x)^2 - m_1gx - m_2g(L + R\sin\theta). \tag{1}$$

Evaluation of the Euler-Lagrange equations gives

$$m_1\ddot{x} = -k_1 x + k_2(L + R\sin\theta - x) - m_1 g \tag{2a}$$
$$m_2 R^2\ddot{\theta} = -k_2(L + R\sin\theta - x)R\cos\theta - m_2 g R\cos\theta, \tag{2b}$$

or

$$\ddot{x} = -x\frac{k_1 + k_2}{m_1} + \frac{k_2 R\sin\theta}{m_1} - g + \frac{k_2 L}{m_1} \tag{3a}$$
$$\ddot{\theta} = -\frac{1}{R}\left(\frac{k_2}{m_2}\left(L + R\sin\theta - x\right) + g\right)\cos\theta. \tag{3b}$$

Just like any other coupled pendulum system, this one is non-linear. There's no obvious way to solve this in closed form, but we could determine a solution in the neighborhood of a point $(x, \theta) = (x_0, \theta_0)$. Let's switch our dynamical variables to ones that express the deviation from the initial point $\delta x = x - x_0$, and $\delta\theta = \theta - \theta_0$, with $u = (\delta x)'$, and $v = (\delta\theta)'$. Our system then takes the form

$$u' = f(x,\theta) = -x\frac{k_1 + k_2}{m_1} + \frac{k_2 R\sin\theta}{m_1} - g + \frac{k_2 L}{m_1} \tag{4a}$$
$$v' = g(x,\theta) = -\frac{1}{R}\left(\frac{k_2}{m_2}\left(L + R\sin\theta - x\right) + g\right)\cos\theta \tag{4b}$$
$$(\delta x)' = u \tag{4c}$$
$$(\delta\theta)' = v. \tag{4d}$$

We can use a first order Taylor approximation of the form $f(x,\theta) = f(x_0,\theta_0) + f_x(x_0,\theta_0)(\delta x) + f_\theta(x_0,\theta_0)(\delta\theta)$. So, to first order, our system has the approximation

$$u' = -x_0\frac{k_1 + k_2}{m_1} + \frac{k_2 R\sin\theta_0}{m_1} - g + \frac{k_2 L}{m_1} - (\delta x)\frac{k_1 + k_2}{m_1} + \frac{k_2 R\cos\theta_0}{m_1}(\delta\theta) \tag{5a}$$
$$v' = -\frac{1}{R}\left(\frac{k_2}{m_2}\left(L + R\sin\theta_0 - x_0\right) + g\right)\cos\theta_0 + \frac{k_2\cos\theta_0}{m_2 R}(\delta x) - \frac{1}{R}\left(\frac{k_2}{m_2}\left(\left(L - x_0\right)\sin\theta_0 + R\right) + g\sin\theta_0\right)(\delta\theta) \tag{5b}$$
$$(\delta x)' = u \tag{5c}$$
$$(\delta\theta)' = v. \tag{5d}$$

This would be tidier in matrix form with $\mathbf{x} = (u, v, \delta x, \delta\theta)$

$$\mathbf{x}' = \begin{bmatrix} -x_0\frac{k_1+k_2}{m_1} + \frac{k_2 R\sin\theta_0}{m_1} - g + \frac{k_2 L}{m_1} \\ -\frac{1}{R}\left(\frac{k_2}{m_2}\left(L + R\sin\theta_0 - x_0\right) + g\right)\cos\theta_0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\frac{k_1+k_2}{m_1} & \frac{k_2 R\cos\theta_0}{m_1} \\ 0 & 0 & \frac{k_2\cos\theta_0}{m_2 R} & -\frac{1}{R}\left(\frac{k_2}{m_2}\left(\left(L - x_0\right)\sin\theta_0 + R\right) + g\sin\theta_0\right) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\mathbf{x}. \tag{6}$$

This reduces the problem to the solutions of first order equations of the form

$$\mathbf{x}' = \mathbf{a} + \begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} \mathbf{x} = \mathbf{a} + \mathbf{B}\mathbf{x} \tag{7}$$

where $\mathbf{a}$, and A are constant matrices. Such a matrix equation has the solution

$$\mathbf{x} = e^{Bt}\mathbf{x}_0 + (e^{Bt} - I)B^{-1}\mathbf{a}, \tag{8}$$

but the zeros in B should allow the exponential and inverse to be calculated with less work. That inverse is readily verified to be

$$B^{-1} = \begin{bmatrix} 0 & I \\ A^{-1} & 0 \end{bmatrix}. \tag{9}$$

It is also not hard to show that

$$B^{2n} = \begin{bmatrix} A^n & 0 \\ 0 & A^n \end{bmatrix} \tag{10a}$$

$$B^{2n+1} = \begin{bmatrix} 0 & A^{n+1} \\ A^n & 0 \end{bmatrix}. \tag{10b}$$

Together this allows for the power series expansion

$$e^{Bt} = \begin{bmatrix} \cosh(t\sqrt{A}) & \sinh(t\sqrt{A}) \\ \sinh(t\sqrt{A})\frac{1}{\sqrt{A}} & \cosh(t\sqrt{A}) \end{bmatrix}. \tag{11}$$

All of the remaining sub matrix expansions should be straightforward to calculate provided the eigenvalues and vectors of A are calculated. Specifically, suppose that we have

$$A = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U^{-1}. \tag{12}$$

Then all the perhaps non-obvious functions of matrixes expand to just

$$A^{-1} = U \begin{bmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{bmatrix} U^{-1} \tag{13a}$$

$$\sqrt{A} = U \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} U^{-1} \tag{13b}$$

$$\cosh(t\sqrt{A}) = U \begin{bmatrix} \cosh(t\sqrt{\lambda_1}) & 0 \\ 0 & \cosh(t\sqrt{\lambda_2}) \end{bmatrix} U^{-1} \tag{13c}$$

$$\sinh(t\sqrt{A}) = U \begin{bmatrix} \sinh(t\sqrt{\lambda_1}) & 0 \\ 0 & \sinh(t\sqrt{\lambda_2}) \end{bmatrix} U^{-1} \tag{13d}$$

$$\sinh(t\sqrt{A})\frac{1}{\sqrt{A}} = U \begin{bmatrix} \sinh(t\sqrt{\lambda_1})/\sqrt{\lambda_1} & 0 \\ 0 & \sinh(t\sqrt{\lambda_2})/\sqrt{\lambda_2} \end{bmatrix} U^{-1}. \tag{13e}$$

An interesting question would be how are the eigenvalues and eigenvectors changed with each small change to the initial position $\mathbf{x}_0$ in phase space. Can these be related to each other?