

Time evolution of some wave functions

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1. Motivation.

In [1] is problem 3.14, Describe the time evolution of the following wavefunctions

$$\psi_1 = A \sin \omega t \cos k(x + ct)$$

$$\psi_2 = A \sin 10^{-5} kx \cos k(x - ct)$$

$$\psi_3 = A \cos k(x - ct) \sin 10^{-5} k(x - ct)$$

This isn't really a QM problem, but seems worthwhile anyways, because it isn't obvious looking at the functions what this is.

2. ψ_3

Let's start in reverse order with ψ_3 , but in a slightly more general form that is less error prone to manipulate. These wavefunctions can be viewed as superpositions, and expanding out as exponentials temporarily gets us to a form that makes this more obvious.

$$\begin{aligned}\psi &= A \sin k_1(x + v_1 t) \cos k_2(x + v_2 t) \\ &= \frac{A}{4i} \left(e^{ik_1(x+v_1 t)} - e^{-ik_1(x+v_1 t)} \right) \left(e^{ik_2(x+v_2 t)} + e^{-ik_2(x+v_2 t)} \right) \\ &= \frac{A}{2} \left(\sin((k_1 + k_2)x + (k_1 v_1 + k_2 v_2)t) + \sin((k_1 - k_2)x + (k_1 v_1 - k_2 v_2)t) \right) \\ &= \frac{A}{2} \left(\sin \left((k_1 + k_2) \left(x + \frac{k_1 v_1 + k_2 v_2}{k_1 + k_2} t \right) \right) + \sin \left((k_1 - k_2) \left(x + \frac{k_1 v_1 - k_2 v_2}{k_1 - k_2} t \right) \right) \right)\end{aligned}$$

Now the problem is simplified to observing how a wave of the form $\phi = \sin \kappa(x + vt)$ propagates, or really the interaction of two such waves moving together or against each other, depending on the signs of the constants. Let's now put in the constants for ψ_3 to get a better feel for it

$$\psi_3 = \frac{A}{2} (\sin(1.00001k(x - ct)) - \sin(0.99999k(x - ct)))$$

It wasn't obvious from the original product of sinusoids form that the question asked about that the resulting wave form stays in phase for its time propagation, but we see that to be the case above. This really just leaves some thought about the standing wave itself to understand what

is happening. For that, at time 0, we have a destructive interference superposition of two almost identical period standing waves. That near perfect cancellation will likely leave an envelope, and a Mathematica plot gives a better feel for this waveform. This in turn will propagate at light speed down the x-axis. Because k_1 is so small we have a nearly linear, and nearly flat, envelope for the $\cos kx$, as can be expected near the origin since we have there

$$\psi_3(x, 0) \approx A10^{-5}kx \cos kx$$

Comparing to a smaller range plot, one sees that it is necessary to increase the plot range significantly before seeing the oscillatory nature of the envelope.

3. ψ_2

For ψ_2 we have almost the same wave function, but our sine term has no time variation. What does this do to the waveform? Let's see if a sum and difference of angles form sheds some light on that. We have

$$\begin{aligned}\psi &= A \sin k_1 x \cos k_2(x + v_2 t) \\ &= \frac{A}{4i} \left(e^{ik_1 x} - e^{-ik_1 x} \right) \left(e^{ik_2(x+v_2 t)} + e^{-ik_2(x+v_2 t)} \right) \\ &= \frac{A}{2} \left(\sin((k_1 + k_2)x + k_2 v_2 t) + \sin((k_1 - k_2)x - k_2 v_2 t) \right) \\ &= \frac{A}{2} \left(\sin \left((k_1 + k_2) \left(x + \frac{k_2 v_2}{k_1 + k_2} t \right) \right) + \sin \left((k_1 - k_2) \left(x - \frac{k_2 v_2}{k_1 - k_2} t \right) \right) \right)\end{aligned}$$

Specifically for $k_1 = 10^{-5}k$, and $k_2 = k$, we have

$$\psi_2 = \frac{A}{2} \left(\sin \left(1.00001k \left(x - \frac{1}{1.00001} ct \right) \right) - \sin \left(0.99999k \left(x + \frac{1}{0.99999} ct \right) \right) \right)$$

Again at $t = 0$ we have a very widely spread envelope with rapid oscillations within it. There is a very small difference in the rate that the two components waveforms will go out of phase, and each of the component waveforms is moving in the opposite directions. What does that phase change do to the evolution? Looking back to the original product of sinusoids form for ψ_2 I believe this will just mean we have the phase shifting with time within the envelope.

4. ψ_1

Finally for the first wave function we have both of the sinusoid factors with time variation. What does that expand out to in terms of superposition of fundamental frequencies?

$$\begin{aligned}\psi_1 &= A \sin \omega t \cos k(x + ct) \\ &= \frac{A}{4i} \left(e^{i\omega t} - e^{-i\omega t} \right) \left(e^{ik(x+ct)} + e^{-ik(x+ct)} \right) \\ &= \frac{A}{2} \left(\sin(k(x + ct) + \omega t) - \sin(k(x + ct) - \omega t) \right),\end{aligned}$$

or

$$\psi_1 = \frac{A}{2} (\sin(kx + (kc + \omega)t) - \sin(kx + (kc - \omega)t))$$

We have the superposition of two $\sin kx$ wave forms, destructively interfering with each other, one with phase changing at the angular rate $kc + \omega$, and the other at the rate $kc - \omega$. What is the overall waveform? It's still not obvious what this is. I actually have the inclination to try to not treat these analytically, but pull out some graphing software. Something like the real mathematica software would be nice since it would allow for the use of sliders to vary parameters and then animate the graphs as time varied.

References

- [1] R. Liboff. *Introductory quantum mechanics*. 2003. [1](#)