Effect of sinusoid operators

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1. **Problem 3.19.**

[1] problem 3.19 is
What is the effect of operating on an arbitrary function $f(x)$ with the following two operators

$$\hat{O}_1 \equiv \partial^2/\partial x^2 - 1 + sin^2(\partial^3/\partial x^3) + cos^2(\partial^3/\partial x^3) \tag{1a}$$

$$\hat{O}_2 \equiv +cos(2\partial/\partial x) + sin^2(\partial/\partial x) + \int_a^b dx \tag{1b}$$

On the surface with $\sin^2 y + \cos^2 y = 1$ and $\cos 2y + 2\sin^2 y = 1$ it appears that we have just

$$\hat{O}_1 \equiv \partial^2/\partial x^2 \tag{2a}$$

$$\hat{O}_2 \equiv 1 + \int_a^b dx \tag{2b}$$

but it this justified when the sinusoids are functions of operators? Let's look at the first case. For some operator $\hat{f}$ we have

$$\begin{aligned}
\sin^2\hat{f} + \cos^2\hat{f} &= -\frac{1}{4}\left(e^{i\hat{f}} - e^{-i\hat{f}}\right)\left(e^{i\hat{f}} - e^{-i\hat{f}}\right) + \frac{1}{4}\left(e^{i\hat{f}} + e^{-i\hat{f}}\right)\left(e^{i\hat{f}} + e^{-i\hat{f}}\right) \\
&= \frac{1}{2}\left(e^{i\hat{f}}e^{-i\hat{f}} + e^{-i\hat{f}}e^{i\hat{f}}\right)
\end{aligned}$$

Can we assume that these cancel for general operators? How about for our specific differential operator $\hat{f} = \partial^3/\partial x^3$? For that one we have

$$e^{i\partial^3/\partial x^3}e^{-i\partial^3/\partial x^3}g(x) = \sum_{k=0}^\infty \frac{1}{k!}\left(\frac{\partial^3}{\partial x^3}\right)^k \sum_{m=0}^\infty \frac{1}{m!}\left(\frac{\partial^3}{\partial x^3}\right)^m g(x)$$

Since the differentials commute, so do the exponentials and we can write the slightly simpler

$$\sin^2\hat{f} + \cos^2\hat{f} = e^{i\hat{f}}e^{-i\hat{f}}$$

I'm pretty sure the commutative property of this differential operator would also allow us to say (in this case at least)

$$\sin^2\hat{f} + \cos^2\hat{f} = 1$$

Will have to look up the combinatoric argument that allows one to write, for numbers,

$$e^x e^y = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \sum_{m=0}^{\infty} \frac{1}{m!} y^m = \sum_{j=0}^{\infty} \frac{1}{j!} (x + y)^j = e^{x+y}$$

If this only assumes that x and y commute, and not any other numeric properties then we have the supposed result 2. We also know of algebraic objects where this does not hold. One example is exponentials of non-commuting square matrices, and other is non-commuting bivector exponentials.

References

[1] R. Liboff. *Introductory quantum mechanics*. 2003. 1