More problems from Liboff chapter 4

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1. Motivation.

Some more problems from [1].

2. Problem 4.11

Some problems on Hermitian adjoints. The starting point is the definition of the adjoint $A^\dagger$ of A in terms of the inner product

$$\langle \hat{A}^\dagger \phi | \psi \rangle = \langle \phi | \hat{A} \psi \rangle$$

2.1. *4.11 a*

$$\begin{aligned}
\langle \phi | (a\hat{A} + b\hat{B})\psi \rangle &= a\langle \phi | \hat{A}\psi \rangle + b\langle \phi | \hat{B}\psi \rangle \\
&= a\langle \hat{A}^\dagger \phi | \psi \rangle + b\langle \hat{B}^\dagger \phi | \psi \rangle \\
&= \langle a^* \hat{A}^\dagger \phi | \psi \rangle + \langle b^* \hat{B}^\dagger \phi | \psi \rangle \\
&= \langle (a^* \hat{A}^\dagger + b^* \hat{B}^\dagger)\phi | \psi \rangle \\
&\implies \\
(a\hat{A} + b\hat{B})^\dagger = (a^* \hat{A}^\dagger + b^* \hat{B}^\dagger)
\end{aligned}$$

2.2. *4.11 b*

$$\begin{aligned}
\langle \phi | \hat{A}\hat{B}\psi \rangle &= \langle \hat{A}^\dagger \phi | \hat{B}\psi \rangle \\
&= \langle \hat{B}^\dagger \hat{A}^\dagger \phi | \psi \rangle \\
&\implies \\
(\hat{A}\hat{B})^\dagger &= \hat{B}^\dagger \hat{A}^\dagger
\end{aligned}$$

2.3. *4.11 d*

Hermitian adjoint of D^2, where $D = \partial/\partial x$. Here we need the integral form of the inner product

$$\begin{aligned}
\langle \phi | D^2 \psi \rangle &= \int \phi^* \frac{\partial}{\partial x} \frac{\partial \psi}{\partial x} \\
&= -\int \frac{\partial \phi^*}{\partial x} \frac{\partial \psi}{\partial x} \\
&= \int \psi \frac{\partial}{\partial x} \frac{\partial \phi^*}{\partial x} \\
&\implies \\
(D^2)^\dagger &= D^2
\end{aligned}$$

Since the text shows that the square of a Hermitian operator is Hermitian, one perhaps wonders if D is (but we expect not since $\hat{p} = -i\hbar D$ is Hermitian).

Suppose $\hat{A} = aD$, we have

$$\hat{A}^\dagger = -a^* D,$$

so for this to be Hermitian ($\hat{A} = \hat{A}^\dagger$) we must have $-a^* = a$. If $a = re^{i\theta}$, we have

$$-1 = e^{2i\theta}$$

So $\theta = \pi(1/2 + n)$, and $a = \pm ir$. This fixes the scalar multiples of D that are required to form a Hermitian operator

$$\hat{A} = \pm irD$$

where r is any real positive constant.

2.4. *4.11 e*

$$(\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger = -(\hat{A}^\dagger \hat{B}^\dagger - \hat{B}^\dagger \hat{A}^\dagger)$$

2.5. *4.11 f*

$$(\hat{A}\hat{B} + \hat{B}\hat{A})^\dagger = \hat{A}^\dagger \hat{B}^\dagger + \hat{B}^\dagger \hat{A}^\dagger$$

2.6. *4.11 g*

$$i(\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger = i(\hat{A}^\dagger \hat{B}^\dagger - \hat{B}^\dagger \hat{A}^\dagger)$$

2.7. ***4.11 h***

This one was to calculate $(\hat{A}^\dagger)^\dagger$. Intuitively I'd expect that $(\hat{A}^\dagger)^\dagger = \hat{A}$. How could one show this?

Trying to show this with Dirac notation, I got all mixed up initially.

Using the more straightforward and old fashioned integral notation (as in [2]), this is more straightforward. We have the Hermitian conjugate defined by

$$\int \psi_2^* (\hat{A}\psi_1) = \int (\hat{A}^\dagger \psi_2^*) \psi_1,$$

Or, more symmetrically, using braces to indicate operator direction

$$\int \psi_2^* (\hat{A}\psi_1) = \int (\psi_2^* \hat{A}^\dagger) \psi_1.$$

Introduce a couple of variable substuitions for clarity

$$\begin{aligned}
\phi_1 &= \psi_1^* \\
\phi_2 &= \psi_2^* \\
\hat{B} &= \hat{A}^\dagger.
\end{aligned}$$

We then have

$$\begin{aligned}
\int \psi_2^* (\hat{A}\psi_1) &= \int (\psi_2^* \hat{A}^\dagger) \psi_1 \\
&= \int (\phi_2 \hat{B}) \phi_1^* \\
&= \int \phi_1^* (\hat{B}\phi_2) \\
&= \int (\phi_1^* \hat{B}^\dagger) \phi_2 \\
&= \int \phi_2 (\hat{B}^\dagger \phi_1^*) \\
&= \int \psi_2^* (\hat{A}^{\dagger\dagger} \psi_1)
\end{aligned}$$

Since this is true for all ψ_k, we have $\hat{A} = \hat{A}^{\dagger\dagger}$ as expected.

Having figured out the problem in the simpleton way, it's now simple to go back and translate this into the Dirac inner product notation without getting muddled. We have

$$
\begin{aligned}
\langle \psi_2 | \hat{A} \psi_1 \rangle &= \langle \hat{A}^\dagger \psi_2 | \psi_1 \rangle \\
&= \langle \hat{B} \phi_2^* | \phi_1^* \rangle \\
&= \langle \phi_1 | \hat{B}^* \phi_2 \rangle^* \\
&= \langle (\hat{B}^*)^\dagger \phi_1 | \phi_2 \rangle^* \\
&= \langle \phi_2^* | \hat{B}^\dagger \phi_1^* \rangle \\
&= \langle \psi_2 | \hat{A}^{\dagger\dagger} \psi_1 \rangle
\end{aligned}
$$

2.8. ***4.11 i***

$$
(\hat{A}\hat{A}^\dagger)^\dagger = (\hat{A}^\dagger)^\dagger \hat{A}^\dagger
$$

since $(\hat{A}^\dagger)^\dagger = \hat{A}$

$$
(\hat{A}\hat{A}^\dagger)^\dagger = \hat{A}\hat{A}^\dagger.
$$

3. Problem 4.12 d

If $\hat{A}$ is not Hermitian, is the product $\hat{A}^\dagger \hat{A}$ Hermitian? To start we need to verify that $\langle \psi | \hat{A}^\dagger \phi \rangle = \langle \hat{A} \psi | \phi \rangle$.

$$
\begin{aligned}
\langle \psi | \hat{A}^\dagger \phi \rangle &= \langle (\hat{A}^\dagger)^* \phi^* | \psi^* \rangle^* \\
&= \langle \phi^* | \hat{A}^* \psi^* \rangle^* \\
&= \langle \psi | \hat{A} \psi \rangle.
\end{aligned}
$$

With that verified we have

$$
\begin{aligned}
\langle \psi | \hat{A}^\dagger \hat{A} \phi \rangle &= \langle \hat{A} \psi | \hat{A} \phi \rangle \\
&= \langle \hat{A}^\dagger \hat{A} \psi | \phi \rangle,
\end{aligned}
$$

so, the answer is yes. Provided the adjoint exists, that product will be Hermitian.

4. Problem 4.14

Show that $\langle \hat{A} \rangle = \langle \hat{A} \rangle^*$ (that it is real), if $\hat{A}$ is Hermitian. This follows by expansion of that conjuagate

$$
\begin{aligned}
\langle \hat{A} \rangle^* &= \left(\int \psi^* \hat{A} \psi \right)^* \\
&= \int \psi \hat{A}^* \psi^* \\
&= \int (\hat{A}\psi)^* \psi \\
&= \langle \hat{A}\psi | \psi \rangle \\
&= \langle \psi | \hat{A}^\dagger \psi \rangle \\
&= \langle \psi | \hat{A}\psi \rangle \\
&= \langle \hat{A} \rangle
\end{aligned}
$$

References

[1] R. Liboff. *Introductory quantum mechanics*. 2003. 1

[2] D. Bohm. *Quantum Theory*. Courier Dover Publications, 1989. 2.7