PHY356 Problem Set III.

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1. Problem 1.

1.1. Statement

A particle of mass m is free to move along the x-direction such that $V(X) = 0$. The state of the system is represented by the wavefunction Eq. (4.74)

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} e^{-i\omega t} f(k) \tag{1}$$

with $f(k)$ given by Eq. (4.59).

$$f(k) = N e^{-\alpha k^2} \tag{2}$$

Note that I've inserted a $1/\sqrt{2\pi}$ factor above that isn't in the text, because otherwise $\psi(x,t)$ will not be unit normalized (assuming $f(k)$ is normalized in wavenumber space).

- (a) What is the group velocity associated with this state?
- (b) What is the probability for measuring the particle at position $x = x_0 > 0$ at time $t = t_0 > 0$?
- (c) What is the probability per unit length for measuring the particle at position $x = x_0 > 0$ at time $t = t_0 > 0$?
- (d) Explain the physical meaning of the above results.

1.2. Solution

1.2.1 (a). group velocity.

To calculate the group velocity we need to know the dependence of ω on k.

Let's step back and consider the time evolution action on $\psi(x,0)$. For the free particle case we have

$$H = \frac{\mathbf{p}^2}{2m} = -\frac{\hbar^2}{2m} \partial_{xx}. \tag{3}$$

Writing $N' = N/\sqrt{2\pi}$ we have

$$
\begin{aligned}
-\frac{it}{\hbar} H\psi(x,0) &= \frac{it\hbar}{2m} N' \int_{-\infty}^{\infty} dk (ik)^2 e^{ikx-\alpha k^2} \\
&= N' \int_{-\infty}^{\infty} dk \frac{-it\hbar k^2}{2m} e^{ikx-\alpha k^2}
\end{aligned}
$$

Each successive application of $-iHt/\hbar$ will introduce another power of $-it\hbar k^2/2m$, so once we sum all the terms of the exponential series $U(t) = e^{-iHt/\hbar}$ we have

$$
\psi(x,t) = N' \int_{-\infty}^{\infty} dk \exp\left(\frac{-it\hbar k^2}{2m} + ikx - \alpha k^2 \right). \tag{4}
$$

Comparing with 1 we find

$$
\omega(k) = \frac{\hbar k^2}{2m}. \tag{5}
$$

This completes this section of the problem since we are now able to calculate the group velocity

$$
v_g = \frac{\partial \omega(k)}{\partial k} = \frac{\hbar k}{m}. \tag{6}
$$

1.3. ***(b). What is the probability for measuring the particle at position $x = x_0 > 0$ at time $t = t_0 > 0$?***

In order to evaluate the probability, it looks desirable to evaluate the wave function integral 4. Writing $2\beta = i/(\alpha + it\hbar/2m)$, the exponent of that integral is

$$
\begin{aligned}
-k^2 \left(\alpha + \frac{it\hbar}{2m} \right) + ikx &= -\left(\alpha + \frac{it\hbar}{2m} \right) \left(k^2 - \frac{ikx}{\alpha + \frac{it\hbar}{2m}} \right) \\
&= -\frac{i}{2\beta} \left((k - x\beta)^2 - x^2\beta^2 \right)
\end{aligned}
$$

The x^2 portion of the exponential

$$
\frac{ix^2\beta^2}{2\beta} = \frac{ix^2\beta}{2} = -\frac{x^2}{4(\alpha + it\hbar/2m)}
$$

then comes out of the integral. We can also make a change of variables $q = k - x\beta$ to evaluate the remainder of the Gaussian and are left with

$$
\psi(x,t) = N' \sqrt{\frac{\pi}{\alpha + it\hbar/2m}} \exp\left(-\frac{x^2}{4(\alpha + it\hbar/2m)} \right). \tag{7}
$$

Observe that from 2 we can compute $N = (2\alpha/\pi)^{1/4}$, which could be substituted back into 7 if desired.

Our probability density is

$$
\begin{aligned}
|\psi(x,t)|^2 &= \frac{1}{2\pi} N^2 \left| \frac{\pi}{\alpha + it\hbar/2m} \right| \exp\left(-\frac{x^2}{4} \left(\frac{1}{(\alpha + it\hbar/2m)} + \frac{1}{(\alpha - it\hbar/2m)} \right) \right) \\
&= \frac{1}{2\pi} \sqrt{\frac{2\alpha}{\pi}} \frac{\pi}{\sqrt{\alpha^2 + (t\hbar/2m)^2}} \exp\left(-\frac{x^2}{4} \frac{1}{\alpha^2 + (t\hbar/2m)^2} \left(\alpha - it\hbar/2m + \alpha + it\hbar/2m \right) \right) \\
&=
\end{aligned}
$$

With a final regrouping of terms, this is

$$
|\psi(x,t)|^2 = \sqrt{\frac{\alpha}{2\pi(\alpha^2 + (t\hbar/2m)^2})} \exp\left(-\frac{x^2}{2} \frac{\alpha}{\alpha^2 + (t\hbar/2m)^2} \right). \tag{8}
$$

As a sanity check we observe that this integrates to unity for all t as desired. The probability that we find the particle at position $x > x_0$ is then

$$
P_{x>x_0}(t) = \sqrt{\frac{\alpha}{2\pi(\alpha^2 + (t\hbar/2m)^2})} \int_{x=x_0}^{\infty} dx \exp\left(-\frac{x^2}{2} \frac{\alpha}{\alpha^2 + (t\hbar/2m)^2} \right) \tag{9}
$$

The only simplification we can make is to rewrite this in terms of the complementary error function

$$
\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \tag{10}
$$

Writing

$$
\beta(t) = \frac{\alpha}{\alpha^2 + (t\hbar/2m)^2}, \tag{11}
$$

we have

$$
P_{x>x_0}(t_0) = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\beta(t_0)/2} x_0 \right) \tag{12}
$$

Sanity checking this result, we note that since $\operatorname{erfc}(0) = 1$ the probability for finding the particle in the $x > 0$ range is $1/2$ as expected.

1.4. ***(c). What is the probability per unit length for measuring the particle at position*** $x = x_0 > 0$ ***at time*** $t = t_0 > 0$***?***

This unit length probability is thus

$$
P_{x>x_0+1/2}(t_0) - P_{x>x_0-1/2}(t_0) = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{\beta(t_0)}{2}} \left(x_0 + \frac{1}{2} \right) \right) - \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{\beta(t_0)}{2}} \left(x_0 - \frac{1}{2} \right) \right) \tag{13}
$$

1.5. *(d). Explain the physical meaning of the above results.*

To get an idea what the group velocity means, observe that we can write our wavefunction 1 as

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ik(x-v_g t)} f(k) \tag{14}$$

We see that the phase coefficient of the Gaussian $f(k)$ "moves" at the rate of the group velocity v_g. Also recall that in the text it is noted that the time dependent term 11 can be expressed in terms of position and momentum uncertainties $(\Delta x)^2$, and $(\Delta p)^2 = \hbar^2 (\Delta k)^2$. That is

$$\frac{1}{\beta(t)} = (\Delta x)^2 + \frac{(\Delta p)^2}{m^2} t^2 \equiv (\Delta x(t))^2 \tag{15}$$

This makes it evident that the probability density flattens and spreads over time with the rate equal to the uncertainty of the group velocity $\Delta p/m = \Delta v_g$ (since $v_g = \hbar k/m$). It is interesting that something as simple as this phase change results in a physically measurable phenomena. We see that a direct result of this linear with time phase change, we are less able to find the particle localized around it's original time $x = 0$ position as more time elapses.

2. Problem 2.

2.1. *Statement*

A particle with intrinsic angular momentum or spin $s = 1/2$ is prepared in the spin-up with respect to the z-direction state $|f\rangle = |z+\rangle$. Determine

$$\left(\langle f| \left(S_z - \langle f|S_z|f\rangle \mathbf{1}\right)^2 |f\rangle \right)^{1/2} \tag{16}$$

and

$$\left(\langle f| \left(S_x - \langle f|S_x|f\rangle \mathbf{1}\right)^2 |f\rangle \right)^{1/2} \tag{17}$$

and explain what these relations say about the system.

2.2. *Solution: Uncertainty of S_z with respect to $|z+\rangle$*

Noting that $S_z|f\rangle = S_z|z+\rangle = \hbar/2|z+\rangle$ we have

$$\langle f|S_z|f\rangle = \frac{\hbar}{2} \tag{18}$$

The average outcome for many measurements of the physical quantity associated with the operator S_z when the system has been prepared in the state $|f\rangle = |z+\rangle$ is $\hbar/2$.

$$\Big(S_z - \langle f|S_z|f\rangle\mathbf{1}\Big)|f\rangle = \frac{\hbar}{2}|f\rangle - \frac{\hbar}{2}|f\rangle = 0 \tag{19}$$

We could also compute this from the matrix representations, but it is slightly more work.

Operating once more with $S_z - \langle f|S_z|f\rangle\mathbf{1}$ on the zero ket vector still gives us zero, so we have zero in the root for 16

$$\Big(\langle f|\left(S_z - \langle f|S_z|f\rangle\mathbf{1}\right)^2|f\rangle\Big)^{1/2} = 0 \tag{20}$$

What does 20 say about the state of the system? Given many measurements of the physical quantity associated with the operator $V = (S_z - \langle f|S_z|f\rangle\mathbf{1})^2$, where the initial state of the system is always $|f\rangle = |z+\rangle$, then the average of the measurements of the physical quantity associated with V is zero. We can think of the operator $V^{1/2} = S_z - \langle f|S_z|f\rangle\mathbf{1}$ as a representation of the observable, "how different is the measured result from the average $\langle f|S_z|f\rangle$".

So, given a system prepared in state $|f\rangle = |z+\rangle$, and performance of repeated measurements capable of only examining spin-up, we find that the system is never any different than its initial spin-up state. We have no uncertainty that we will measure any difference from spin-up on average, when the system is prepared in the spin-up state.

2.3. *Solution: Uncertainty of S_x with respect to $|z+\rangle$*

For this second part of the problem, we note that we can write

$$|f\rangle = |z+\rangle = \frac{1}{\sqrt{2}}(|x+\rangle + |x-\rangle). \tag{21}$$

So the expectation value of S_x with respect to this state is

$$\begin{aligned}
\langle f|S_x|f\rangle &= \frac{1}{2}(|x+\rangle + |x-\rangle)S_x(|x+\rangle + |x-\rangle)\\
&= \hbar(|x+\rangle + |x-\rangle)(|x+\rangle - |x-\rangle)\\
&= \hbar(1 + 0 + 0 - 1)\\
&= 0
\end{aligned}$$

After repeated preparation of the system in state $|f\rangle$, the average measurement of the physical quantity associated with operator S_x is zero. In terms of the eigenstates for that operator $|x+\rangle$ and $|x-\rangle$ we have equal probability of measuring either given this particular initial system state.

For the variance calculation, this reduces our problem to the calculation of $\langle f|S_x^2|f\rangle$, which is

$$\begin{aligned}
\langle f|S_x^2|f\rangle &= \frac{1}{2}\left(\frac{\hbar}{2}\right)^2(|x+\rangle + |x-\rangle)((+1)^2|x+\rangle + (-1)^2|x-\rangle)\\
&= \left(\frac{\hbar}{2}\right)^2,
\end{aligned}$$

so for 22 we have

$$\left(\langle f | \left(S_x - \langle f | S_x | f \rangle \mathbf{1} \right)^2 | f \rangle \right)^{1/2} = \frac{\hbar}{2} \tag{22}$$

The average of the absolute magnitude of the physical quantity associated with operator S_x is found to be $\hbar/2$ when repeated measurements are performed given a system initially prepared in state $|f\rangle = |z+\rangle$. We saw that the average value for the measurement of that physical quantity itself was zero, showing that we have equal probabilities of measuring either $\pm\hbar/2$ for this experiment. A measurement that would show the system was in the x-direction spin-up or spin-down states would find that these states are equi-probable.

3. **Grading comments.**

I lost one mark on the group velocity response. Instead of 23 he wanted

$$v_g = \left. \frac{\partial \omega(k)}{\partial k} \right|_{k=k_0} = \frac{\hbar k_0}{m} = 0 \tag{23}$$

since $f(k)$ peaks at $k = 0$.

I'll have to go back and think about that a bit, because I'm unsure of the last bits of the reasoning there.

I also lost 0.5 and 0.25 (twice) because I didn't explicitly state that the probability that the particle is at x_0, a specific single point, is zero. I thought that was obvious and didn't have to be stated, but it appears expressing this explicitly is what he was looking for.

Curiously, one thing that I didn't loose marks on was, the wrong answer for the probability per unit length. What he was actually asking for was the following

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{x_0 - \epsilon/2}^{x_0 + \epsilon/2} |\Psi(x_0, t_0)|^2 dx = |\Psi(x_0, t_0)|^2 \tag{24}$$

That's a whole lot more sensible seeming quantity to calculate than what I did, but I don't think that I can be faulted too much since the phrase was never used in the text nor in the lectures.