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UNIVERSITY OF TORONTO, RELATIVISTIC ELECTRODYNAMICS (PHY450H1S)

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2011 notes and problems

January 2015 – version v.9

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DOCUMENT VERSION

Sources for this notes compilation can be found in the github repository
https://github.com/peeterjoot/physicsplay
The last commit (Jan/19/2015), associated with this pdf was
5096ede4216bff38998eec6e14ce118804452db9

Dedicated to Aurora and Lance, my awesome kids.

PREFACE

These are my personal lecture notes for the Spring 2011, University of Toronto, Relativistic Electrodynamics course (PHY450H1S). This class was taught by Prof. Erich Poppitz, with Simon Freedman handling tutorials (which were excellent lecture style lessons).

Official course description:

Special Relativity, four-vector calculus and relativistic notation, the relativistic Maxwell's Equations, electromagnetic waves in vacuum and conducting and non-conducting materials, electromagnetic radiation from point charges and systems of charges.

The text for the course is [11].

This document contains a few things

- My lecture notes.

  Typos and errors are probably mine (Peeter), and no claim nor attempt of spelling or grammar correctness will be made.

  These notes track along with the Professor's hand written notes very closely, since his lectures follow his notes very closely. While I used the note taking exercise as a way to verify that I understood all the day's lecture materials, the Professor's notes are in many instances a much better study resource, since there are details in his notes that were left for us to read, and not necessarily covered in the lectures. On the other hand, there are details in these notes that I have added when I did not find his approach simplistic enough for me to grasp, or I failed to follow the details in class.

- Some notes from reading of the text. One of the earlier of these was due to unfortunate use of an ancient edition of the text borrowed from the library, since mine had been lost in shipping. That version did not use the upper and lower index quantities that I had expected, so I tried to puzzle out some of what myself from what I knew.

- Some assigned problems, at least the parts of them that I did not hand write. I have corrected some the errors after receiving grading feedback, and where I have not done so I at least recorded some of the grading comments as a reference. Not all the problems were graded, so I make no guarantees of correctness.

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LECTURE NOTES

1

PRINCIPLE OF RELATIVITY

Reading No reading from [11] appears to have been assigned, but relevant stuff can be found in chapter 1. Covering lecture notes RelEM1-11.pdf

1.1 DISTANCE AS A CLOCK

The title of this course is an oxymoron since ELECTRODYNAMICS == RELATIVITY. In classical and quantum physics (non-gravitational) we start by postulating the existence of space and time. These are, in non-gravitational physics, the arena where everything takes place. The space that we work with is the three dimensional Euclidean space $\mathbb{R}^3$. One way of describing it is using three coordinates

$$\mathbb{R}^3 = \{x, y, z; x, y, z \in [-\infty, \infty]\}. \tag{1.1}$$

We define a distance between P and P' as

$$\left|PP'\right| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \tag{1.2}$$

time is a parameter with respect to which positions of free particles particles change at a constant rate.

Mathematically, we describe the motion of free particles by giving $(x(t), y(t), z(t))$: coordinates as functions of t,

$$\frac{d^2 x_i(t)}{dt^2} = 0, \qquad i = 1, 2, 3 \tag{1.3}$$

Here x, y, z are the free particle coordinates in an "internal frame", the frame where $\ddot{\mathbf{r}} = 0$ holds for a free particle ($\ddot{\mathbf{r}} = d^2\mathbf{r}/dt^2$) for a free particle with tragectory such as $x = v_0 t, y = z = 0$.

1.2 THE PRINCIPLE OF RELATIVITY

Definition 1.1: Principle of relativity (Galileo or Einstein)

"Laws of nature are identical in all inertial frames".

Equivalently, "Identical experiments in two inertial frames yield identical results".

What do we mean by laws of nature? Equations that describe dynamics.

Now we need to get more specific. Identical equations means that the equations have the same form in two inertial frames provided, you express them (the equations) via the coordinates $\mathbf{r}, t$ in the given inertial frame.

FIXME: DRAW x,y,z COORDINATE SYSTEM with origin O. And another with origin O' where the origin is moving with velocity v in the y direction.

The Galilean relativity principle states that "equations of motion are invariant under Galilean transformations". What do we mean by transformations? If we have a point $P(t)$ in space with coordinates in both frames that are related. It is pretty clear that the coordinates $x = x'$ and $z = z'$. What about the y' coordinate? For that we have $y' = y - vt$, so that the origins overlap ($O = O'$) at $t = 0$.

In Galilean relativity, time is absolute. i.e. It is the same in all inertial frames. It is now a no-brainer to find the velocities of the particle. Taking derivatives we take time derivatives of

$$x' = x \tag{1.4}$$
$$y' = y - vt \tag{1.5}$$
$$z' = z, \tag{1.6}$$

for

$$v'_x = v_x \tag{1.7}$$
$$v'_y = v_y - v \tag{1.8}$$
$$v'_z = v_z. \tag{1.9}$$

In vector notation we have

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t \tag{1.10}$$
$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 \tag{1.11}$$

The principle of relativity says that the dynamical equations are invariant under such transformations.

Take Newton's law for example applied to two bodies, labeled by their masses M_1 and M_2. These bodies may be interacting. For example, with Newtonian gravitation

$$V(\mathbf{r}_1 - \mathbf{r}_2) = -G_N \frac{M_1 M_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1.12}$$

or the Van Der Waals, interaction

$$V(\mathbf{r}_1 - \mathbf{r}_2) = -\,(\text{const})\,\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|^6}, \tag{1.13}$$

Our interaction is via a gradient $\partial f(\mathbf{r})/\partial \mathbf{r} = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$

$$M_1 \ddot{\mathbf{r}}_1 = -\frac{\partial}{\partial \mathbf{r}_1} V(\mathbf{r}_1 - \mathbf{r}_2) \tag{1.14}$$

$$M_2 \ddot{\mathbf{r}}_2 = -\frac{\partial}{\partial \mathbf{r}_2} V(\mathbf{r}_1 - \mathbf{r}_2) \tag{1.15}$$

In the unprimed frame, these are "the laws of physics". Consider a primed frame $O' : \mathbf{r}'_i = \mathbf{r}_i - \mathbf{v}_0 t$ (for $i = 1, 2$). Taking derivatives we have $\mathbf{v}'_i = \mathbf{v}'_i + \mathbf{v}_0$, and $\dot{\mathbf{v}}'_i = \dot{\mathbf{v}}'_i$.

We note that the distance between the two particles is unchanged in the primed coordinate system

$$\begin{aligned} \mathbf{r}'_1 - \mathbf{r}'_2 &= \mathbf{r}_1 - \mathbf{v}_0 t - (\mathbf{r}_2 - \mathbf{v}_0 t) \\ &= \mathbf{r}_1 - \mathbf{r}_2 \end{aligned} \tag{1.16}$$

Similarly

$$\frac{\partial}{\partial \mathbf{r}_i} = \frac{\partial}{\partial(\mathbf{r}'_i + \mathbf{v}_0 t)} = \frac{\partial}{\partial \mathbf{r}'_i} \tag{1.17}$$

Observe that the interaction eq. (1.14) is unchanged by this change in coordinates.

1.3 ENTER ELECTROMAGNETISM

If the only interactions are $1/r$ gravity and $1/r$ Coulomb, Galilean relativity holds. Electromagnetism came along and Maxwell's prediction that electromagnetic waves exist and propagate with speed

$$c \approx 3 \times 10^8 m/s \tag{1.18}$$

(Note that in SI units $c = 1/\sqrt{\epsilon_0 \mu_0}$).

It was proposed that the speed of light was the speed in a medium (the "aether") through which electrodynamic waves propagate. The idea was that the oscillations of this medium constitute electromagnetic waves. Then "c" would be the speed of light with respect to that medium. This medium would fill all space.

PICTURE: of gradient field, with aether velocity at different points. Superimposed on this is a picture of the Earth's orbit, so that the velocity of the aether could be measured at different points of the earth's orbit by measuring the speed of light at different points in the orbit.

PICTURE: of interferometer.

We can study this effect by rotating this platform to measure at different points of the day and the year.

We note that the speed of the earth is approximately $v_+ = 150 \times 10^6 km/10^7 s \approx 15 km/s$.

Aside: It was not clear to me where these numbers came from. Wolfram alpha says that the Earth's orbital speed is approximately $32 km/s$, although that is still within an order of magnitude of the number used in class.

The shift of fringes would then be $v_+ \approx (v_+/c)^2 \approx 10^{-8}$. What Einstein did was to elevate the principle of relativity to one that applies to electromagnetism, but replacing the transformation relating frames to the Lorentz transformation, a transformation observed by Lorentz and Poincare that leave Maxwell's equations invariant. Einstein did this by postulating that the speed of light is a constant in all frames, and we will see how this is the case.

Question 1.1: Is not this true only outside of matter?

In matter we have electromagnetic wave propagation at speeds less than c.

A: (paraphrasing)

We can consider the in-matter case to be a special case, treating collections of discreet particles as continuous approximations. It is only as a side effect of these approximations that one produces the in-matter Maxwell's equation, and we will consider the "vacuum" Maxwell equation as always true, provided the points of interest do not fall exactly on any specific particle.

Yes we have speed of light different in media. Example, speed of light in water is 3/4 vacuum speed due to high index of refraction. Also note that we can have effects like an electron moving in water can constantly emit light. This is called Cerenkov radiation.

Reading No reading from [11] appears to have been assigned, but relevant stuff can be found in chapter 1.

Covering lecture notes RelEM12-26.pdf.

1.4 EINSTEIN'S RELATIVITY PRINCIPLE

1. Replace Galilean transformations between coordinates in differential inertial frames with Lorentz transforms between $(\mathbf{x}, t)$. Postulate that these constitute the symmetries of physics. Recall that Galilean transformations are symmetries of the laws of non-relativistic physics.

2. Speed of light c is the same in all inertial frames. Phrased in this form, relativity leads to "relativity of simultaneity".

   PICTURE: Three people on a platform, at positions 1, 3, 2, all with equidistant separation. This stationary frame is labeled O. 1 and 2 flash light signals at the same time and in frame O the reception of the light signal by 3 is observed as arriving at 3 simultaneously.

   Now introduce a moving frame with origin O' moving along the positive x axis. To a stationary observer in O' the three guys are seen to be moving in the $-x$ direction. The middle guy (3) is eventually going to be seen to receive the light signal by this O' observer, but less time is required for the light to get from 1 to 3, and more time is required for the light to get from 2 to 1 (3 is moving away from the light according to the O' observer). Because the speed of light is perceived as constant for all observers, the perception is then that the light must arrive at 3 at different times.

   This is very non-intuitive since we are implicitly trained by our surroundings that Galilean transformations govern mechanical behavior.

   In O, 1 and 2 send light signals simultaneously while in O' 1 sends light later than 2. The conclusion, rather surprisingly compared to intuition, is that simultaneity is relative.

Question 1.2: On symmetries.

Q: A comment made that the symmetries impose the dynamics, and the symmetries provided the form of the Lagrangian in classical physics. My question to this comment was

"When we have transformations that leave the Lagrangian unchanged (a symmetry), we have a conserved current. I have done various exercises to compute those currents for various types of transformations (translation, spacetime translation, rotation, boosts, ...), but can not think of a way that the Lagrangian itself is defined these sorts symmetries. Can you elaborate on what you mean by this?"

A: Ah, you see, what I meant by that is the following. For a free particle $\mathcal{L}$ should depend on $\mathbf{x}$, $\dot{\mathbf{x}}$ and t. Homogeneity of space and time do not allow to have x and t dependence and isotropy of space only permits dependence on $|\dot{\mathbf{x}}|$. Finally, Gallilean relativity only allows $\mathcal{L} = \dot{\mathbf{x}}^2$ (times a constant). (See [10] vol 1 or my notes on PHY354 website, p. 23-27).

So what was used is:

- Having only dependence on x and dx/dt.
- Spacetime homogeneity/isotropy.
- Gallilean relativity.

Similar story holds in relativity, as we will see.

2

SPACETIME

We will need to develop some tools to work with these concepts in a concrete fashion. It is convenient to combine space $\mathbb{R}^3$ and time $\mathbb{R}^1$ into a 4d "spacetime". In [11] this is called fictitious spacetime for reasons that are not clear. Points in this space are also called "events", or "spacetime points", or "world point". The "world line" is the trajectory for a particle in spacetime.

PICTURE: $\mathbb{R}^3$ represented as a plane, and t up. For every point we can plot an $\mathbf{x}(t)$ in this combined space.

2.1 INTERVALS FOR LIGHT LIKE BEHAVIOUR

Consider two frames, one moving along the x-axis at a (constant) rate not yet specified.

"events" have coordinates $(t, \mathbf{x})$ in O and $(t', \mathbf{x}')$ in O'. Because we now have to model the mathematics without a notion of simultaneity, we must now also introduce different time coordinates t, and t' in the two frames.

Let us imagine that at at time t_1 light is emitted at $\mathbf{x}_1$, and at time t_2 this light is absorbed. Our space time events are then $(t_1, \mathbf{x}_1)$ and $(t_2, \mathbf{x}_2)$. In the O frame, the light will go a distance $c(t_2 - t_1)$. This same distance can also be expressed as

$$\sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2}. \tag{2.1}$$

These are equal. It is convenient to work without the square roots, so we write

$$(\mathbf{x}_1 - \mathbf{x}_2)^2 = c^2(t_2 - t_1)^2 \tag{2.2}$$

Or

$$\begin{aligned}&c^2(t_2 - t_1)^2 - (\mathbf{x}_1 - \mathbf{x}_2)^2 = \\ &\quad c^2(t_2 - t_1)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 = 0.\end{aligned} \tag{2.3}$$

We can repeat the same argument for the primed frame. In this frame, at time t'_1 light is emitted at $\mathbf{x}'_1$, and at time t'_2 this light is absorbed. Our space time events in this frame are then

$(t'_1, \mathbf{x}'_1)$ and $(t'_2, \mathbf{x}'_2)$. As above, in this O' frame, the light will go a distance $c(t'_2 - t'_1)$, with a similar Euclidean distance involving $\mathbf{x}'_1$ and $\mathbf{x}'_2$. That is

$$
\begin{aligned}
&c^2(t'_2 - t'_1)^2 - (\mathbf{x}'_1 - \mathbf{x}'_2)^2 = \\
&\quad c^2(t'_2 - t'_1)^2 - (x'_1 - x'_2)^2 - (y'_1 - y'_2)^2 - (z'_1 - z'_2)^2 = 0.
\end{aligned}
\tag{2.4}
$$

We get zero for this quantity in any inertial frame 1. This quantity is found to be very important, and want to give this a label. We call this the "interval", or the "spacetime interval", and write this as follows:

$$
s_{12}^2 = c^2(t_2 - t_1)^2 - (\mathbf{r}_2 - \mathbf{r}_1)^2 \tag{2.5}
$$

This is a quantity calculated between any two spacetime points with coordinates $(t_2, \mathbf{r}_2)$ and $(t_1, \mathbf{r}_1)$ in some frame.

So far we have argued that c being the same in any two frames implies that spacetime events "separated by a zero interval" in one frame are "separated by a zero interval" in any other frame.

2.2 INVARIANCE OF INFINITESIMAL INTERVALS

For events that are infinitesimally close to each other. i.e. $t_2 - t_1$ and $\mathbf{r}_2 - \mathbf{r}_1$ are small (infinitesimal), it is convenient to denote $t_2 - t_1$ and $\mathbf{r}_2 - \mathbf{r}_1$ by dt and $d\mathbf{r}$ respectively. We can then define

$$
ds_{12}^2 = c^2 dt^2 - d\mathbf{r}^2, \tag{2.6}
$$

or

$$
ds = \sqrt{c^2 dt^2 - d\mathbf{r}^2}. \tag{2.7}
$$

We will use this a lot.

We have learned that if $s_{12} = 0$ in one frame, then $s'_{12} = 0$ in any other frame. We generally expect that there is a relation $s'_{12} = F(s_1 2)$ between the intervals in two frames. So far we have learned that $F(0) = 0$.

Let us now consider the case where both of these intervals are infinitesimal. Then we can write

$$
ds'_{12} = F(ds_{12}) = F(0) + F'(0) ds_{12} + \cdots = F'(0) ds_{12} + \cdots. \tag{2.8}
$$

We will neglect terms $O(ds_{12})^2$ and higher. Thus equality of zero intervals between two frames implies that

$$ds'_{12} \sim ds_{12}. \tag{2.9}$$

Now we must invoke an assumption (principle) of homogeneity of time and space and isotropy of space. This interval should not depend on where these events take place, or on the time that the measurements were performed. If this is the case then we conclude that the proportionality constant relating the two intervals is not a function of position or space. We argue that this proportionality can then only be a function of the (absolute) relative speed between the frames.

We write this as

$$ds'_{12} = F(v_{12}) ds_{12} \tag{2.10}$$

This argument can be turned around and we say that $ds_{12} = \tilde{F}(v_{12}) ds'_{12}$. Thus $\tilde{F} = F$, because there is no distinction between O and O'. We want to conclude that

$$ds_{12} = F(v_{12}) ds'_{12} = F(v_{12}) \tilde{F}(v_{12}) ds_{12} \tag{2.11}$$

and then conclude that $F = \tilde{F} = 1$. This argument is to be continued. To complete this conclusion we will need to perform some additional math, once we cover finite intervals.

Reading Still covering chapter 1 material from the text [11], and lecture notes RelEM12-26.pdf.

2.3 GEOMETRY OF SPACETIME: LIGHTLIKE, SPACELIKE, TIMELIKE INTERVALS

Last time we introduced the (squared) interval

$$s_{12}^2 = c^2 dt^2 - d\mathbf{r}^2. \tag{2.12}$$

This spacetime interval is of great importance to relativity, and is as important as the spatial distance $|\mathbf{r}_2 - \mathbf{r}_1|$ in Newtonian physics. This distance determines the Euclidean geometry of space.

Similarly, the interval eq. (2.12) determines the "distance" in space time.

Symmetries are the guiding principles of physics, and this quantity we will see to be related to spacetime symmetries. Last time we argued that the constancy of the speed of light in all frames implies that if $s_{12}^2 = 0$ in one frame, then ${s'_{12}}^2 = 0$.

We were considering infinitesimal $1, 2$ separation with $ds = F(V)ds'$ where V is the relative speed of the two frames. Relating the two incremental intervals we have a function F and its inverse

$$ds' = F(\tilde{V})ds = \tilde{F}(V)F(V)ds, \tag{2.13}$$

But we can also argue that

$$\tilde{F} = F \qquad \text{by } O' \sim O, \tag{2.14}$$

and thus that

$$F^2 = 1, \tag{2.15}$$

or

$$F = \pm 1. \tag{2.16}$$

Since we wish this to hold for $V = 0$, we require the positive root, and can conclude that $F = 1$.

Note that ds (or s_{12}) requires a sign convention, since it is $s_{12}^2 = c^2(t_2 - t_1)^2 - (\mathbf{r}_2 - \mathbf{r}_1)^2$ that is the object which (we will argue) is invariant.

This is similar to the Euclidean case where it is the quantity $(\mathbf{r}_2 - \mathbf{r}_1)^2$ is invariant, and our convention is to always pick the positive sign.

Possible conventions for s_{12} are

$$s_{12} = \sqrt{c^2(t_2 - t_1)^2 - (\mathbf{r}_2 - \mathbf{r}_1)^2}, \tag{2.17}$$

if $s_{12}^2 > 0$, and when $s_{12}^2 < 0$, the alternate convention is

$$s_{12} = i\sqrt{|s_{12}|^2}. \tag{2.18}$$

Later we will argue that $ds = ds'$ implies $s_{12} = s'_{12}$ for any finite interval.

2.4 RELATIVITY PRINCIPLE IN MATHEMATICAL FORMULATION

The Relativity principle (in mathematical formulation): the spacetime interval $s_{12}, \forall 1, 2$ (spacetime points) is the same in all frames.

In other words, the transformations $(t, \mathbf{r}) \to (t', \mathbf{r}')$ have to leave s_{12}^2 invariant for all 1 and 2.

These transformations, that is to say these coordinate transformations

$$\begin{aligned}(t, \mathbf{r}) &\to (t', \mathbf{r}') \\ O &\to O'\end{aligned} \tag{2.19}$$

leave the laws of nature invariant.

We will see later how such invariance, like the spatial invariance in Newtonian physics, defines the dynamics of spacetime. We will also answer the question about what are these transformations that leave the interval invariant. In the Newtonian case those transformations were rotations, and we will be looking for similar transformations. The negative sign in the spacetime interval will complicate things a bit, but not actually too much.

Next week: we will find the "Lorentz transformation".

2.5 GEOMETRY OF SPACETIME

We now want to study a bit of the geometry of spacetime implied by $s_{12}^2 = c^2(t_2 - t_1)^2 - (\mathbf{r}_2 - \mathbf{r}_1)^2$. Consider two spacetime points 1,2, where $(t_1, \mathbf{r}_1), (t_2, \mathbf{r}_2)$ are points in some frame.

PICTURE: two points plotted on the x-axis, with time $t_1 = 0$, and $t = t_2$

The points are

1. $(0, \mathbf{0})$
2. $(t, x, 0, 0)$

The interval is

$$s_{12}^2 = c^2 t^2 - x^2 \tag{2.20}$$

PICTURE: "flat" light cone. 2d cross-section of space time surface $c^2t^2 = x^2 + y^2$.

PICTURE: conic light cone. 3d (2 space + 1 time) cross-section of space time surface $c^2t^2 = x^2 + y^2$. One diagonal for the trajectory $ct = -x$, and another for $ct = x$. The bottom section is the past light cone, since light that is absorbed at the origin must have been emitted at some point in the past. Similarly, light emitted from the origin, takes trajectories on the future light cone.

Observe that on the light cone, $s_{12}^2 = 0$. The intervals $s_{12}^2 = 0$ separates any given set of spacetime points into "lightlike", "spacelike", and "timelike" regions.

For events (or spacetime points) separated by a timelike interval, there always exists a frame such that the occur at same point in space (since $s_{12}^2 = c^2t^2 - \mathbf{r}^2 > 0$ (region II) it is consistent to imagine that there exists a frame where $\mathbf{r}' = 0$ and $s_{12}^2 = c^2t^2 > 0$. This is very much like we can always find a rotation in Euclidean space that orients two points so that they lie along the x (or any other arbitrary) axis.

We have not yet proven this, but will see it shortly. What we will see is that we can never make these two events have the same time ($t' \neq 0$). This is because if we make $t' = 0$ we will get a negative interval in some frame.

For points in spacetime separated by spacelike intervals, one can always choose a frame such that they occur at same t. (i.e. for us $t' = 0$). Since $s_{12}^2 = c^2t^2 - \mathbf{r}^2 < 0$, $s_{12}^2 = -\mathbf{r}^2 < 0$.

Similar to light rays that move along the light cone, particles that move at speeds less than the speed of light propagate along worldlines within region II (in the interior of the light cone). At at arbitrary point in the worldline of a particle draw a 45 degree cone. Tangent to world line should lie inside the figure lightcone of that space time point.

2.6 PROPER TIME

PICTURE4: velocity at $(t, \mathbf{x}) = v$ (say). Consider an inertial frame with speed v, centered at the momentary position of the particle. Call this the primed frame. In this frame $ds^2 = c^2 dt'^2$ (particle is at rest in this frame).

In the original frame $ds^2 = c^2dt^2 - d\mathbf{x}^2$. Since these are equal we have

$$c^2dt^2 - d\mathbf{x}^2 = c^2dt'^2 \tag{2.21}$$

Dividing by c^2 we have

$$dt'^2 = dt^2 - \frac{1}{c^2}d\mathbf{x}^2. \tag{2.22}$$

Here dt^2 is the (squared) time elapsed in the frame where it is moving. The time elapsed in the rest frame of the particle, we call the "proper time", and we have $dt' < dt$ because $1 - \mathbf{v}^2/c^2 < 1$. This is described as

More exactly, we write

$$d\tau^2 = \frac{ds^2}{dt^2} = dt^2\left(1 - \frac{1}{c^2}\left(\frac{d\mathbf{x}}{dt}\right)^2\right) \tag{2.23}$$

In general, for noninfinitesimal dt, to find the proper time one has to integrate

$$\tau_{ab} = \frac{1}{c} \int_a^b ds \tag{2.24}$$

Plan for next class: Talk about causality. Derive the Lorentz transformation.

Reading Still covering chapter 1 material from the text [11], lecture notes RelEM12-26.pdf, and lecture notes RelEM27-44.pdf.

2.7 MORE SPACETIME GEOMETRY

PICTURE: ct,x curvy worldline with tangent vector **v**.

In an inertial frame moving with **v**, whose origin coincides with momentary position of this moving observer $ds^2 = c^2 dt'^2 = c^2 dt^2 - \mathbf{r}^2$

"proper time" is

$$dt' = dt \sqrt{1 - \frac{1}{c^2} \left(\frac{d\mathbf{r}}{dt} \right)^2} = dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \tag{2.25}$$

We see that $dt' < dt$ if $v > 0$, so that $\sqrt{1 - \mathbf{v}^2/c^2} < 1$.

In a manifestly invariant way we define the proper time as

$$d\tau \equiv \frac{ds}{c} \tag{2.26}$$

So that between worldpoints a and b the proper time is a line integral over the worldline

$$d\tau \equiv \frac{1}{c} \int_a^b ds. \tag{2.27}$$

PICTURE: We are splitting up the worldline into many small pieces and summing them up.

2.8 FINITE INTERVAL INVARIANCE

Tomorrow we are going to complete the proof about invariance. We have shown that light like intervals are invariant, and that infinitesimal intervals are invariant. We need to put these pieces together for finite intervals.

2.9 DERIVING THE LORENTZ TRANSFORMATION

Let us find the coordinate transforms that leave s_{12}^2 invariant. This generalizes Galileo's transformations.

We would like to generalize rotations, which leave spatial distance invariant. Such a transformation also leaves the spacetime interval invariant.

In Euclidean space we can generate an arbitrary rotation by composition of rotation around any of the xy, yz, zx axis.

For 4D Euclidean space we would form any rotation by composition of any of the 6 independent rotations for the 6 available planes. For example with x, y, z, w axis we can rotate in any of the xy, xz, xw, yz, yw, zw planes.

For spacetime we can "rotate" in x, t, y, t, z, t "planes". Physically this is motion space (boosting a position).

Consider a x, t transformation The trick (that is in the notes) is to rewrite the time as an analytical continuation of the time coordinate, as follows

$$ds^2 = c^2 dt^2 - dx^2 \tag{2.28}$$

and write

$$t \rightarrow i\tau, \tag{2.29}$$

so that the interval becomes

$$ds^2 = -(c^2 d\tau^2 + dx^2) \tag{2.30}$$

Now we have a structure that is familiar, and we can rotate as we normally do. Prof does not want to go through the details of this "trickery" in class, but says to see the notes. The end result is that we can transform as follows

$$x' = x \cosh\psi + ct \sinh\psi \tag{2.31}$$

$$ct' = x \sinh\psi + ct \cosh\psi \tag{2.32}$$

which is analogous to a spatial rotation

$$x' = x \cos\alpha + y \sin\alpha \tag{2.33}$$

$$y' = -x \sin\alpha + y \cos\alpha \tag{2.34}$$

There are some differences in sign as well, but the important feature to recall is that $\cosh^2 x - \sinh^2 x = (1/4)(e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2) = 1$. We call these hyperbolic rotations, something that is simply a mathematical transformation. Now we want to relate this to something physical.

Q: What is ψ? The origin of O has coordinates $(t, \mathbf{O})$ in the O frame.

PICTURE (pg 32): O' frame translating along x axis with speed v_x. We have

$$\frac{x'}{ct'} = \frac{v_x}{c} \tag{2.35}$$

However, using eq. (2.31) we have for the origin

$$x' = ct \sinh\psi \tag{2.36}$$
$$ct' = ct \cosh\psi \tag{2.37}$$

so that

$$\frac{x'}{ct'} = \tanh\psi = \frac{v_x}{c} \tag{2.38}$$

Using

$$\cosh\psi = \frac{1}{\sqrt{1-\tanh^2\psi}} \tag{2.39}$$
$$\sinh\psi = \frac{\tanh\psi}{\sqrt{1-\tanh^2\psi}} \tag{2.40}$$

Performing all the gory substitutions one gets

$$x' = \frac{1}{\sqrt{1-v_x^2/c^2}}x + \frac{v_x/c}{\sqrt{1-v_x^2/c^2}}ct \tag{2.41}$$
$$y' = y \tag{2.42}$$
$$z' = z \tag{2.43}$$
$$ct' = \frac{v_x/c}{\sqrt{1-v_x^2/c^2}}x + \frac{1}{\sqrt{1-v_x^2/c^2}}ct \tag{2.44}$$

PICTURE: Let us go to the more conventional case, where O is at rest and O' is moving with velocity v_x.

We achieve this by simply changing the sign of v_x in eq. (2.41) above. This gives us

$$x' = \frac{1}{\sqrt{1 - v_x^2/c^2}} x - \frac{v_x/c}{\sqrt{1 - v_x^2/c^2}} ct \tag{2.45}$$

$$y' = y \tag{2.46}$$

$$z' = z \tag{2.47}$$

$$ct' = -\frac{v_x/c}{\sqrt{1 - v_x^2/c^2}} x + \frac{1}{\sqrt{1 - v_x^2/c^2}} ct \tag{2.48}$$

We want some shorthand to make this easier to write and introduce

$$\gamma = \frac{1}{\sqrt{1 - v_x^2/c^2}}, \tag{2.49}$$

so that eq. (2.45) becomes

$$x' = \gamma\left(x - \frac{v_x}{c} ct\right) \tag{2.50}$$

$$ct' = \gamma\left(ct - \frac{v_x}{c} x\right) \tag{2.51}$$

We started the class by saying these would generalize the Galilean transformations. Observe that if we take $c \to \infty$, we have $\gamma \to 1$ and

$$x' = x - v_x t + O((v_x/c)^2) \tag{2.52}$$

$$t' = t + O(v_x/c) \tag{2.53}$$

This is how to remember the signs. We want things to match up with the non-relativistic limit.

Q: How do lines of constant x' and ct' look like on the x, ct spacetime diagram? Our starting point (again) is

$$x' = \gamma\left(x - \frac{v_x}{c} ct\right) \tag{2.54}$$

$$ct' = \gamma\left(ct - \frac{v_x}{c} x\right). \tag{2.55}$$

What are the points with $x' = 0$. Those are the points where $x = (v_x/c)ct$. This is the ct' *axis*. That is the straight worldline

PICTURE: worldline of O' origin.

What are the points with $ct' = 0$. Those are the points where $ct = xv_x/c$. This is the x' *axis*.

Lines that are parallel to the x' axis are lines of constant x', and lines parallel to ct' axis are lines of constant t', but the light cone is the same for both.

What is this good for? We have time to pick from either length contraction or non-causality (how to kill your grandfather). How about length contraction. We can use the diagram to read the x or ct coordinates, or examine causality, but it is hard to read off t' or x' coordinates.

Reading Covering chapter 1 material from the text [11], and lecture notes RelEM27-44.pdf.

2.10 MORE ON PROPER TIME

PICTURE:1: worldline with small interval.

Considering a small interval somewhere on the worldline trajectory, we have

$$ds^2 = c^2 dt^2 - dx^2 = c^2 dt'^2, \tag{2.56}$$

where dt' is the proper time elapsed in a frame moving with velocity v, and dt is the time elapsed in a stationary frame.

We have

$$dt' = dt\sqrt{1 - (dx/dt)^2/c^2} = dt\sqrt{1 - v^2/c^2}. \tag{2.57}$$

PICTURE:2: particle at rest.

For the particle at rest

$$c\tau_{21}^{\text{stationary}} = c(t_2 - t_1) = \int_1^2 ds = \int_1^2 cdt \tag{2.58}$$

PICTURE:3: particle with motion.

"length" of 1-2 "curved" worldline

$$\begin{aligned}\int_1^2 ds' &= \int_1^2 cdt' \\ &= \int_1^2 cdt\sqrt{1 - (d\mathbf{v}/dt)^2},\end{aligned} \tag{2.59}$$

where in this case [1, 2] denotes the range of a line integral over the worldline. We see that the multiplier of dt for any point along the curve is smaller than 1, so that the length along a straight line is longest (i.e. for the particle at rest).

We have argued that if 1,2 occur at the same place, the spacetime length of a straight line between them is the longest. This remains the time for all 1,2 timelike separated.

LOTS OF DISCUSSION. See new posted notes for details.

Back to page 18 of the notes.

We have argued that $ds_{12} = ds'_{12} \implies s_{12} = s'_{12}$ for infinitesimal 1,2 even if not infinitesimal.

The idea is to represent the interval between twill not close 1,2 as a sum over small ds's.

P6: $x = x_2 t/t_2$ straight line through origin, with $t \in [0, t_2]$.

P7: zoomed on part of this line.

$$\begin{aligned} ds^2 &= c^2 dt^2 - dx^2 \\ &= c^2 dt^2 - \left(\frac{x_2}{t_2}\right)^2 dt^2 \\ &= c^2 dt^2 \left(1 - \frac{1}{c^2}\left(\frac{x_2}{t_2}\right)^2\right) \end{aligned} \tag{2.60}$$

or

$$\int_0^1 ds = c\int_0^{t_2} dt \sqrt{1 - \frac{1}{c^2}\left(\frac{x_2}{t_2}\right)^2} \tag{2.61}$$

In another frame just replace $t \rightarrow t'$ and $x_2 \rightarrow x'_2$

$$\int_0^1 ds = c\int_0^{t'_2} dt \sqrt{1 - \frac{1}{c^2}\left(\frac{x'_2}{t'_2}\right)^2} \tag{2.62}$$

2.11 LENGTH CONTRACTION

Consider O and O' with O' moving in x with speed $v_x > 0$. Here we have

$$\begin{aligned} x' &= \gamma\left(x - \frac{v_x}{c} ct\right) \\ ct' &= \gamma\left(ct - \frac{v_x}{c} x\right) \end{aligned} \tag{2.63}$$

PICTURE: spacetime diagram with ct' at angle α, where $\tan\alpha = v_x/c$.

Two points $(x_A, 0)$, $(x_B, 0)$, with rest length measured as $L = x_B - x_A$. From the diagram $c(t_B - t_A) = \tan\alpha L$, and from eq. (2.63) we have

$$\begin{aligned} x'_A &= \gamma\left(x_A - \frac{v_x}{c} ct_A\right) \\ x'_B &= \gamma\left(x_B - \frac{v_x}{c} ct_B\right), \end{aligned} \tag{2.64}$$

so that

$$\begin{aligned}
L' &= x'_B - x'_A \\
&= \gamma\left((x_B - x_A) - \frac{v_x}{c}c(t_B - t_A)\right) \\
&= \gamma\left(L - \frac{v_x}{c}\tan\alpha L\right) \\
&= \gamma\left(L - \frac{v_x^2}{c^2}L\right) \\
&= \gamma L\left(1 - \frac{v_x^2}{c^2}\right) \\
&= L\sqrt{1 - \frac{v_x^2}{c^2}}
\end{aligned} \tag{2.65}$$

2.12 SUPERLUMINAL SPEED AND CAUSALITY

If Einstein's relativity holds, superliminal motion is a "no-no". Imagine that some "tachyons" exist that can instantaneously transmit stuff between observers.

PICTURE9: two guys with resting worldlines showing.

Can send info back to A before A sends to B. Superluminal propagation allows sending information not yet available. Can show this for finite superluminal velocities (but hard) as well as infinite velocity superluminal speeds. We see that time ordering can not be changed for events separated by time like separation. Events separated by spacelike separation cannot be ca usually connected.

2.13 PROBLEMS

Exercise 2.1 Transformation of velocities

From the Lorentz transformations of space and time coordinates.

a. Derive the transformation of velocities.
   With a particle moving with $\mathbf{v}$ in the unprimed (stationary) frame, find its velocity $\mathbf{v}'$ in the primed frame. The primed frame is moving with some $\mathbf{V}$ with respect to the unprimed one. Make sure to finally derive the general "addition of velocities" equation in terms of vectors and dot products, as given in [9].
b. Velocities relative to c.
   Then, use the addition of velocities rule to show that:

1. if $v < c$ in one frame, then $v' < c$ in any other frame.
2. If $v = c$ in one frame, then $v' = c$ in any other frame, and
3. if $v > c$ in one frame, than $v' > c$ in any other frame.

Answer for Exercise 2.1

Part a. We need a vector form of the Lorentz transform to start with. Let us write $\boldsymbol{\sigma}$ for a unit vector colinear with the primed frame velocity $\mathbf{V}$, so that $\mathbf{V} = (\mathbf{V} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}$. When our boost was in the x direction, our Lorentz transformation was in terms of $x = \mathbf{x} \cdot \hat{\mathbf{x}}$. The component in the direction of the boost is now $\mathbf{x} \cdot \boldsymbol{\sigma}$, and we have

$$\begin{aligned} ct' &= \gamma\left(ct - (\mathbf{x} \cdot \boldsymbol{\sigma})\frac{\mathbf{V} \cdot \boldsymbol{\sigma}}{c}\right) \\ \mathbf{x}' \cdot \boldsymbol{\sigma} &= \gamma\left(\mathbf{x} \cdot \boldsymbol{\sigma} - \frac{\mathbf{V} \cdot \boldsymbol{\sigma}}{c} ct\right) \\ \mathbf{x}' \wedge \boldsymbol{\sigma} &= \mathbf{x} \wedge \boldsymbol{\sigma}. \end{aligned} \tag{2.66a}$$

We can add the vector components using $\mathbf{x} = (\mathbf{x} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma} + (\mathbf{x} \wedge \boldsymbol{\sigma})\boldsymbol{\sigma}$, leaving

$$\begin{aligned} ct' &= \gamma\left(ct - (\mathbf{x} \cdot \boldsymbol{\sigma})\frac{\mathbf{V} \cdot \boldsymbol{\sigma}}{c}\right) \\ \mathbf{x}' &= (\mathbf{x} \wedge \boldsymbol{\sigma})\boldsymbol{\sigma} + \gamma\left((\mathbf{x} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{\mathbf{V}}{c} ct\right). \end{aligned} \tag{2.67a}$$

Writing $(\mathbf{x} \wedge \boldsymbol{\sigma})\boldsymbol{\sigma} = \mathbf{x} - (\mathbf{x} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}$ we have for the spatial component transformation

$$\mathbf{x}' = \mathbf{x} + (\mathbf{x} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}(\gamma - 1) - \gamma\frac{\mathbf{V}}{c} ct. \tag{2.68}$$

Now we are set to take derivatives to calculate the velocities. This gives us

$$\begin{aligned} \frac{dt'}{dt} &= \gamma\left(1 - \left(\frac{d\mathbf{x}}{dt} \cdot \boldsymbol{\sigma}\right)\frac{\mathbf{V} \cdot \boldsymbol{\sigma}}{c^2}\right) \\ \frac{d\mathbf{x}'}{dt'}\frac{dt'}{dt} &= \frac{d\mathbf{x}}{dt} + \left(\frac{d\mathbf{x}}{dt} \cdot \boldsymbol{\sigma}\right)\boldsymbol{\sigma}(\gamma - 1) - \gamma\frac{\mathbf{V}}{c} c. \end{aligned} \tag{2.69a}$$

Dividing this pair of equations, and using $\mathbf{v} = d\mathbf{x}/dt$, and $\mathbf{v}' = d\mathbf{x}'/dt'$, this is

$$\mathbf{v}' = \frac{\gamma^{-1}\mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}(1 - \gamma^{-1}) - \mathbf{V}}{1 - (\mathbf{v} \cdot \boldsymbol{\sigma})(\mathbf{V} \cdot \boldsymbol{\sigma})/c^2}. \tag{2.70}$$

Since $\mathbf{V}$ and our direction vector $\boldsymbol{\sigma}$ are colinear, we have $(\mathbf{v} \cdot \boldsymbol{\sigma})(\mathbf{V} \cdot \boldsymbol{\sigma}) = \mathbf{v} \cdot \boldsymbol{\sigma}$, and can simplify this last expression slightly

$$\boxed{\mathbf{v}' = \frac{\gamma^{-1}\mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}(1 - \gamma^{-1}) - \mathbf{V}}{1 - \mathbf{v} \cdot \mathbf{V}/c^2}.} \tag{2.71}$$

Finally, if we are to compare to the text, we note that the inverse expression requires replacement of $\mathbf{V}$ with $-\mathbf{V}$ and switching $\mathbf{v}$ with $\mathbf{v}'$. That gives us

$$\mathbf{v} = \frac{\gamma^{-1}\mathbf{v}' + (\mathbf{v}' \cdot \boldsymbol{\sigma})\boldsymbol{\sigma}(1 - \gamma^{-1}) + \mathbf{V}}{1 + \mathbf{v}' \cdot \mathbf{V}/c^2}. \tag{2.72}$$

The expression in the text is also a small velocity approximation. For $|\mathbf{V}| \ll c$, we have $\gamma^{-1} \approx 1$, and $(1 + \mathbf{v}' \cdot \mathbf{V}/c^2)^{-1} \approx 1 - \mathbf{v}' \cdot \mathbf{V}/c^2$. This gives us

$$\mathbf{v} \approx (\mathbf{v}' + \mathbf{V})(1 - \mathbf{v}' \cdot \mathbf{V}/c^2) \approx \mathbf{V} + \mathbf{v}' - \mathbf{v}'(\mathbf{v}' \cdot \mathbf{V})/c^2 \tag{2.73}$$

One additional approximation was made dropping the $\mathbf{V}(\mathbf{v}' \cdot \mathbf{V})/c^2$ term which is quadratic in $\mathbf{V}/c$, which leave us with equation 5.3 in the text as desired.

Part b. In eq. (2.72), let us write $\mathbf{v}' = u\mathbf{u}$, where $\mathbf{u}$ is a unit vector, $V = \mathbf{V} \cdot \boldsymbol{\sigma}$, and $\alpha = \mathbf{u} \cdot \boldsymbol{\sigma}$ for the direction cosine between the primed frame's direction of motion and the particle's velocity direction (also in the unprimed frame). The stationary frame's particle velocity is then

$$\mathbf{v} = \frac{\gamma^{-1}u\mathbf{u} + u\alpha\boldsymbol{\sigma}(1 - \gamma^{-1}) + V\boldsymbol{\sigma}}{1 + \alpha u V/c^2}. \tag{2.74}$$

As a check, note that for $1 = \alpha = \mathbf{u} \cdot \boldsymbol{\sigma} = \cos(0)$, we recover the familiar addition of velocities formula

$$\mathbf{v} = \mathbf{u}\frac{u + V}{1 + uV/c^2}. \tag{2.75}$$

We want to put eq. (2.74) into a form that renders it more tractable for general angles too. Factoring out the γ^{-1} term appears to do the job, yielding

$$\mathbf{v} = \frac{u\gamma^{-1}(\mathbf{u} - \alpha\boldsymbol{\sigma}) + (u\alpha + V)\boldsymbol{\sigma}}{1 + \alpha u V/c^2}. \tag{2.76}$$

After a bit of reduction and rearranging we can dot this with itself to calculate

$$\mathbf{v}^2 = \frac{V^2(1-\alpha^2)(1-u^2/c^2) + (u+\alpha V)^2}{(1+\alpha u V/c^2)^2} \tag{2.77}$$

Note that for $u = c$, we have $\mathbf{v}^2 = c^2$, regardless of the direction of $\mathbf{V}$ with respect to the motion of the particle in the unprimed frame. This should not be surprising since this light like invariance is exactly what the Lorentz transformation is designed to maintain. It is however slightly comforting to know that the algebra appears to be still be kosher after all this. This also answers part (b) of this question, since we have tackled the $v = c$ case in the primed frame, and seen that the speed remains $v = c$ in the unprimed frame (and thus any frame moving at constant speed relative to another).

Observe that since $1 - \alpha^2 = \sin^2\theta$, and $u \le c$, this is positive definite as expected. If one allowed $u > c$ in some frame, our speed could go imaginary!

For the $u < c$ and $u > c$ cases, let $x = u/c$ and $y = V/c$. This allows eq. (2.77) to be casted in a simpler form

$$\mathbf{v}^2 = c^2 \frac{y^2(1-\alpha^2)(1-x^2) + (x+\alpha y)^2}{(1+\alpha x y)^2} \tag{2.78}$$

We wish to verify that (a) given any $x \in (-1, 1)$, we have $\mathbf{v}^2 < c^2$ for all $y \in (-1, 1)$, $\alpha \in (-1, 1)$, and (c) given any $|x| > 1$, we have $\mathbf{v}^2 > c^2$ for all $y \in (-1, 1)$, $\alpha \in (-1, 1)$.

Considering (a) first, this requires a demonstration that

$$y^2(1-\alpha^2)(1-x^2) + (x+\alpha y)^2 < (1+\alpha x y)^2. \tag{2.79}$$

Expanding out the products and canceling terms, we want to show that for (a) that if $|x|, |y| < 1$ we have

$$x^2(1-y^2) + y^2 < 1, \tag{2.80}$$

and for (c) that if $|x| > 1$, we have for any $|y| < 1$

$$x^2(1-y^2) + y^2 > 1. \tag{2.81}$$

Observe that the frame velocity orientation direction cosines have completely dropped out, leaving just the (relative to c) velocity terms.

To get an initial feel for this function $f(x,y) = x^2(1-y^2)+y^2$, notice that when graphed we have a bowl with a minimum (zero) at the origin, and what appears to be a uniform value of one on the boundary (case (b)). Then provided $|y| < 1$ it appears that the function f increases monotonically to a value greater than one (case (c)). While looking at a plot is not any sort of rigorous proof, let us move on to some of the other problems for now, and return to this last loose thread later if time permits.

Exercise 2.2 Toy GPS model

A toy model of a GPS system has satellites moving in a straight line with constant velocity V_x and at a constant height h (measured, e.g., along the y-axis) above "ground" (the x-axis). The satellites broadcast the time in their rest frame as well as their location at a time of broadcast. Imagine a person on the ground receives simultaneously broadcasts from two satellites, A and B, reporting their locations x'_A and x'_B as well as times of broadcast (which happen to be equal), $t'_A = t'_B$.

a. Find a condition determining your position in x. Evaluate it to find your deviation from the midpoint between the satellites to first order in V_x/c.
b. For some real numbers, note that in reality there are 24 satellites, moving with V 4km$/s$, a distance $R \approx 2.7\times10^4$km. Use these numbers and the result from the previous problem (assuming a flat Earth, to be sure...) to get an idea whether (special) relativistic effects are important for the typical modern GPS accuracy of order 10 m (or less)?

Answer for Exercise 2.2

Part a. We are looking for a worldpoints (ct', x', y') in satellite frame on the light cone emanating from the satellite worldpoints (ct'_A, x'_A, y'_A), and (ct'_B, x'_B, y'_B). These are

$$\begin{aligned} c^2(t'_A - t')^2 &= (x'_A - x')^2 + (y'_A - y')^2 \\ c^2(t'_B - t')^2 &= (x'_B - x')^2 + (y'_B - y')^2, \end{aligned} \tag{2.82}$$

where the worldpoints (ct', x', y') are related to the stationary frame by

$$\begin{bmatrix} ct' \\ x' \\ y' \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \end{bmatrix}. \tag{2.83}$$

The problem has been artificially simplified by stating that $t'_A = t'_B$, and we can eliminate the y' terms since we want $y'_A - y' = h = y'_B - y'$ at the point where the signal is received.

Suppose that in the observer frame the light signals are received with event coordinates $(ct_0, x_0, 0)$. In the satellites rest frame these are

$$\begin{aligned} ct' &= \gamma(ct_0 - \beta x_0) \\ x' &= \gamma(x_0 - \beta ct_0) \end{aligned} \tag{2.84}$$

We can make these substitutions above, yielding

$$\begin{aligned} (ct'_A - \gamma ct_0 + \gamma\beta x_0)^2 &= (x'_A - \gamma x_0 + \gamma\beta ct_0)^2 + h^2 \\ (ct'_A - \gamma ct_0 + \gamma\beta x_0)^2 &= (x'_B - \gamma x_0 + \gamma\beta ct_0)^2 + h^2 \end{aligned} \tag{2.85}$$

Observe that the $t'_A = t'_B$ condition allows us to equate the pair of RHS terms and thus have

$$x'_A - \gamma x_0 + \gamma\beta ct_0 = \pm(x'_B - \gamma x_0 + \gamma\beta ct_0) \tag{2.86}$$

If we pick the positive root, then we have $x'_A = x'_B$, a perfectly valid mathematical solution, but not one that can be used for triangularization. Taking the negative root instead and rearranging we have

$$\gamma\beta ct_0 = \gamma x_0 - \frac{1}{2}(x'_A + x'_B) \tag{2.87}$$

As a sanity check observe that if $\beta = 0$ we have $x_0 = \frac{1}{2}(x'_A + x'_B) = x'_m$, the midpoint in the satellite (also the observer frame for $\beta = 0$). This is what we would expect if a simultaneous signal is received that emanated at the same time when both sources are at rest at the same height.

When $\beta \neq 0$ we have

$$\gamma ct_0 = \frac{1}{\beta}(\gamma x_0 - x'_m), \tag{2.88}$$

allowing us to eliminate γct_0 terms from the equations we wish to solve

$$\begin{aligned} \left(ct'_A - \frac{1}{\beta}(\gamma x_0 - x'_m) + \gamma\beta x_0\right)^2 &= (x'_A - x'_m)^2 + h^2 \\ \left(ct'_A - \frac{1}{\beta}(\gamma x_0 - x'_m) + \gamma\beta x_0\right)^2 &= (x'_B - x'_m)^2 + h^2. \end{aligned} \tag{2.89}$$

We can group the γx_0 terms on the LHS nicely

$$\begin{aligned} -\frac{1}{\beta}\gamma x_0 \gamma\beta x_0 &= \gamma x_0(-\frac{1}{\beta} + \beta) \\ &= \frac{1}{\beta}\gamma x_0(-1 + \beta^2) \\ &= -\frac{1}{\beta} x_0, \end{aligned} \tag{2.90}$$

leaving

$$\left(ct'_A - \frac{1}{\beta}x_0 + \frac{1}{\beta}x'_m\right)^2 = (x'_A - x'_m)^2 + h^2 = (x'_B - x'_m)^2 + h^2. \tag{2.91}$$

The value $\left|x'_A - x'_m\right| = \left|x'_B - x'_m\right| = \left|x'_A - x'_B\right|/2$ is half the separation L' of the satellites in their rest frame, so we have

$$ct'_A - \frac{1}{\beta}x_0 + \frac{1}{\beta}x'_m = \pm\sqrt{L'^2/4 + h^2}, \tag{2.92}$$

or

$$x_0 = x'_m + \beta c t'_A \mp \beta\sqrt{L'^2/4 + h^2} \tag{2.93}$$

Utilizing the inverse transformation we have for a x-axis spatial coordinate in the observer frame

$$x = \gamma(x' + \beta c t'), \tag{2.94}$$

allowing the t'_A term to be eliminated in favour of the position that the midpoint between the satellites would have been observed at time t'_A. This gives us

$$x_0 = \frac{1}{\gamma}x_m \mp \beta\sqrt{L'^2/4 + h^2} \tag{2.95}$$

FIXME: Which sign is correct for this problem? I had guess the negative sign. Fixing that is probably the toughest part of this problem!

Part b. FIXME: Had hand written notes for this part of the problem, with how I'd attempted it first (considering the actual geometric problem in 3D.)

3

FOUR VECTORS AND TENSORS

3.1 INTRODUCING FOUR VECTORS

(From tutorial 1)

A 3-vector:

$$\begin{aligned} \mathbf{a} &= (a_x, a_y, a_z) = (a^1, a^2, a^3) \\ \mathbf{b} &= (b_x, b_y, b_z) = (b^1, b^2, b^3) \end{aligned} \tag{3.1}$$

For this we have the dot product

$$\mathbf{a} \cdot \mathbf{b} = \sum_{\alpha=1}^{3} a^\alpha b^\alpha \tag{3.2}$$

Greek letters in this course (opposite to everybody else in the world, because of Landau and Lifshitz) run from 1 to 3, whereas roman letters run through the set $\{0, 1, 2, 3\}$.

We want to put space and time on an equal footing and form the composite quantity (four vector)

$$x^i = (ct, \mathbf{r}) = (x^0, x^1, x^2, x^3), \tag{3.3}$$

where

$$\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z. \end{aligned} \tag{3.4}$$

It will also be convenient to drop indices when referring to all the components of a four vector and we will use lower or upper case non-bold letters to represent such four vectors. For example

$$X = (ct, \mathbf{r}), \tag{3.5}$$

or

$$u = \gamma \left(1, \mathbf{v}/c\right). \tag{3.6}$$

Three vectors will be represented as letters with over arrows $\vec{a}$ or (in text) bold face **a**. Recall that the squared spacetime interval between two events X_1 and X_2 is defined as

$$S_{X_1,X_2}{}^2 = (ct_1 - ct_2)^2 - (\mathbf{x}_1 - \mathbf{x}_2)^2. \tag{3.7}$$

In particular, with one of these zero, we have an operator which takes a single four vector and spits out a scalar, measuring a “distance” from the origin

$$s^2 = (ct)^2 - \mathbf{r}^2. \tag{3.8}$$

This motivates the introduction of a dot product for our four vector space.

$$X \cdot X = (ct)^2 - \mathbf{r}^2 = (x^0)^2 - \sum_{\alpha=1}^{3} (x^\alpha)^2 \tag{3.9}$$

Utilizing the spacetime dot product of eq. (3.9) we have for the dot product of the difference between two events

$$\begin{aligned}(X - Y) \cdot (X - Y) &= (x^0 - y^0)^2 - \sum_{\alpha=1}^{3} (x^\alpha - y^\alpha)^2 \\ &= X \cdot X + Y \cdot Y - 2x^0 y^0 + 2 \sum_{\alpha=1}^{3} x^\alpha y^\alpha.\end{aligned} \tag{3.10}$$

From this, assuming our dot product eq. (3.9) is both linear and symmetric, we have for any pair of spacetime events

$$X \cdot Y = x^0 y^0 - \sum_{\alpha=1}^{3} x^\alpha y^\alpha. \tag{3.11}$$

How do our four vectors transform? This is really just a notational issue, since this has already been discussed. In this new notation we have

$$\begin{aligned}x^{0\prime} &= ct' = \gamma(ct - \beta x) = \gamma(x^0 - \beta x^1) \\ x^{1\prime} &= x' = \gamma(x - \beta ct) = \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3\end{aligned} \tag{3.12}$$

where $\beta = V/c$, and $\gamma^{-2} = 1 - \beta^2$.

In order to put some structure to this, it can be helpful to express this dot product as a quadratic form. We write

$$A \cdot B = \begin{bmatrix} a^0 & \mathbf{a}^\mathrm{T} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b^0 \\ \mathbf{b} \end{bmatrix} = A^\mathrm{T} G B. \tag{3.13}$$

We can write our Lorentz boost as a matrix

$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

so that the dot product between two transformed four vectors takes the form

$$A' \cdot B' = A^\mathrm{T} O^\mathrm{T} G O B \tag{3.15}$$

Reading Covering chapter 1 material from the text [11], and lecture notes RelEM27-44.pdf.

3.2 THE SPECIAL ORTHOGONAL GROUP (FOR EUCLIDEAN SPACE)

Lorentz transformations are like "rotations" for (t, x, y, z) that preserve $(ct)^2 - x^2 - y^2 - z^2$. There are 6 continuous parameters:

- 3 rotations in x, y, z space
- 3 "boosts" in x or y or z.

For rotations of space we talk about a group of transformations of 3D Euclidean space, and call this the $S0(3)$ group. Here S is for Special, O for Orthogonal, and 3 for the dimensions.

For a transformed vector in 3D space we write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}' = O \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{3.16}$$

Here O is an orthogonal 3×3 matrix, and has the property

$$O^T O = \mathbf{1}. \tag{3.17}$$

Taking determinants, we have

$$\det O^T \det O = 1, \tag{3.18}$$

and since $\det O^{\mathrm{T}} = \det O$, we have

$$(\det O)^2 = 1, \tag{3.19}$$

so our determinant must be

$$\det O = \pm 1. \tag{3.20}$$

We work with the positive case only, avoiding the transformations that include reflections.

The Unitary condition $O^{\mathrm{T}} O = 1$ is an indication that the inner product is preserved. Observe that in matrix form we can write the inner product

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \\ x_3 \end{bmatrix}. \tag{3.21}$$

For a transformed vector $X' = OX$, we have $X'^{\mathrm{T}} = X^{\mathrm{T}} O^{\mathrm{T}}$, and

$$X' \cdot X' = (X^{\mathrm{T}} O^{\mathrm{T}})(OX) = X^{\mathrm{T}}(O^{\mathrm{T}} O)X = X^T X = X \cdot X \tag{3.22}$$

3.3 THE SPECIAL ORTHOGONAL GROUP (FOR SPACETIME)

This generalizes to Lorentz boosts! There are two differences

1. Lorentz transforms should be 4×4 not 3×3 and act in (ct, x, y, z), and NOT (x, y, z).
2. They should leave invariant NOT $\mathbf{r}_1 \cdot \mathbf{r}_2$, but $c2t_2t_1 - \mathbf{r}_2 \cdot \mathbf{r}_1$.

Do not get confused that I demanded $c^2 t_2 t_1 - \mathbf{r}_2 \cdot \mathbf{r}_1 =$ invariant rather than $c^2(t_2 - t_1)^2 - (\mathbf{r}_2 - \mathbf{r}_1)^2 =$ invariant. Expansion of this (squared) interval, provides just this four vector dot product and its invariance condition

$$\begin{aligned} \text{invariant} &= c^2(t_2 - t_1)^2 - (\mathbf{r}_2 - \mathbf{r}_1)^2 \\ &= (c^2 t_2^2 - \mathbf{r}_2^2) + (c^2 t_2^2 - \mathbf{r}_2^2) - 2c^2 t_2 t_1 + 2\mathbf{r}_1 \cdot \mathbf{r}_2. \end{aligned} \tag{3.23}$$

Observe that we have the sum of two invariants plus our new cross term, so this cross term, (-2 times our dot product to be defined), must also be an invariant.

Introduce the four vector

$$\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned} \tag{3.24}$$

Or $(x^0, x^1, x^2, x^3) = \{x^i, i = 0, 1, 2, 3\}$.
We will also write

$$\begin{aligned} x^i &= (ct, \mathbf{r}) \\ \tilde{x}^i &= (c\tilde{t}, \tilde{\mathbf{r}}) \end{aligned} \tag{3.25}$$

Our inner product is

$$c^2 t\tilde{t} - \mathbf{r} \cdot \tilde{\mathbf{r}} \tag{3.26}$$

Introduce the 4×4 matrix

$$\left\| g_{ij} \right\| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.27}$$

This is called the Minkowski spacetime metric.

Then

$$\begin{aligned} c^2 t\tilde{t} - \mathbf{r}\cdot\tilde{\mathbf{r}} &\equiv \sum_{i,j=0}^{3} \tilde{x}^i g_{ij} x^j \\ &= \sum_{i,j=0}^{3} \tilde{x}^i g_{ij} x^j \\ &\tilde{x}^0 x^0 - \tilde{x}^1 x^1 - \tilde{x}^2 x^2 - \tilde{x}^3 x^3 \end{aligned} \tag{3.28}$$

Einstein summation convention . Whenever indices are repeated that are assumed to be summed over.

We also write

$$X = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{3.29}$$

$$\tilde{X} = \begin{bmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{bmatrix} \tag{3.30}$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.31}$$

Our inner product

$$\begin{aligned} c^2 t\tilde{t} - \tilde{\mathbf{r}}\cdot\mathbf{r} &= \tilde{X}^{\mathrm{T}} G X \\ &= \begin{bmatrix} \tilde{x}^0 & \tilde{x}^1 & \tilde{x}^2 & \tilde{x}^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{bmatrix} \end{aligned} \tag{3.32}$$

Under Lorentz boosts, we have

$$X = \hat{O}X', \tag{3.33}$$

where

$$\hat{O} = \begin{bmatrix} \gamma & -\gamma v_x/c & 0 & 0 \\ -\gamma v_x/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.34}$$

(for x-direction boosts)

$$\begin{aligned} \tilde{X} &= \hat{O}\tilde{X}' \\ \tilde{X}^{\mathrm{T}} &= \tilde{X}'^{\mathrm{T}}\hat{O}^{\mathrm{T}} \end{aligned} \tag{3.35}$$

But $\hat{O}$ must be such that $\tilde{X}^{\mathrm{T}}GX$ is invariant. i.e.

$$\tilde{X}^{\mathrm{T}}GX = \tilde{X}'^{\mathrm{T}}(\hat{O}^{\mathrm{T}}G\hat{O})X' = X'^{\mathrm{T}}(G)X' \qquad \forall X' \text{ and } \tilde{X}' \tag{3.36}$$

This implies

$$\boxed{\hat{O}^{\mathrm{T}}G\hat{O} = G} \tag{3.37}$$

Such $\hat{O}$'s are called “pseudo-orthogonal”.
Lorentz transformations are represented by the set of all 4×4 pseudo-orthogonal matrices.
In symbols

$$\hat{O}^{T}G\hat{O} = G \tag{3.38}$$

Just as before we can take the determinant of both sides. Doing so we have

$$\det(\hat{O}^{T}G\hat{O}) = \det(\hat{O}^{T})\det(G)\det(\hat{O}) = \det(G) \tag{3.39}$$

The det(G) terms cancel, and since $\det(\hat{O}^T) = \det(\hat{O})$, this leaves us with $(\det(\hat{O}))^2 = 1$, or

$$\det(\hat{O}) = \pm 1 \tag{3.40}$$

We take the $\det 0 = +1$ case only, so that the transformations do not change orientation (no reflection in space or time). This set of transformation forms the group

$$SO(1,3)$$

Special orthogonal, one time, 3 space dimensions. Note that when the -1 determinant is also allowed the group is called the $O(1,3)$ set of transformations.

Einstein relativity can be defined as the "laws of physics that leave four vectors invariant in the

$$SO(1,3) \times T^4$$

symmetry group.

Here T^4 is the group of translations in spacetime with 4 continuous parameters. The complete group of transformations that form the group of relativistic physics has $10 = 3 + 3 + 4$ continuous parameters.

This group is called the Poincare group of symmetry transforms.

3.4 LOWER INDEX NOTATION

Our inner product is written

$$\tilde{x}^i g_{ij} x^j \tag{3.41}$$

but this is very cumbersome. The convenient way to write this is instead

$$\tilde{x}^i g_{ij} x^j = \tilde{x}_j x^j = \tilde{x}^i x_i \tag{3.42}$$

where

$$x_i = g_{ij} x^j = g_{ji} x^j \tag{3.43}$$

Note: A check that we should always be able to make. Indexes that are not summed over should be conserved. So in the above we have a free i on the LHS, and should have a non-summed i index on the RHS too (also lower matching lower, or upper matching upper).

Non-matched indices are bad in the same sort of sense that an expression like

$$\mathbf{r} = 1 \tag{3.44}$$

is not well defined (assuming a vector space $\mathbf{r}$ and not a multivector Clifford algebra that is;)

Expanded out explicitly (noting that all off diagonal terms of the metric tensor are zero):

$$\begin{aligned}
x_0 &= g_{00} x^0 = ct \\
x_1 &= g_{1j} x^j = g_{11} x^1 = -x^1 \\
x_2 &= g_{2j} x^j = g_{22} x^2 = -x^2 \\
x_3 &= g_{3j} x^j = g_{33} x^3 = -x^3
\end{aligned} \tag{3.45}$$

We would not have objects of the form

$$x^i x^i = (ct)^2 + \mathbf{r}^2 \tag{3.46}$$

for example. This is not a Lorentz invariant quantity.

Lorentz scalar example: $\tilde{x}^i x_i$

Lorentz vector example: x^i

This last is also called a rank-1 tensor.

Lorentz rank-2 tensors: ex: g_{ij}

or other 2-index objects.

Why in the world would we ever want to consider two index objects. We are not just trying to be hard on ourselves. Recall from classical mechanics that we have a two index object, the inertial tensor.

In mechanics, for a rigid body we had the energy

$$T = \sum_{ij=1}^{3} \Omega_i I_{ij} \Omega_j \tag{3.47}$$

The inertial tensor was this object

$$I_{ij} = \sum_{a=1}^{N} m_a \left(\delta_{ij} \mathbf{r}_a^2 - r_{a_i} r_{a_j} \right) \tag{3.48}$$

or for a continuous body

$$I_{ij} = \int \rho(\mathbf{r}) \left(\delta_{ij} \mathbf{r}^2 - r_i r_j \right) \tag{3.49}$$

In electrostatics we have the quadrupole tensor, ... and we have other such objects all over physics.

Note that the energy T of the body above cannot depend on the coordinate system in use. This is a general property of tensors. These are object that transform as products of vectors, as I_{ij} does.

We call I_{ij} a rank-2 3-tensor. rank-2 because there are two indices, and 3 because the indices range from 1 to 3.

The point is that tensors have the property that the transformed tensors transform as

$$I'_{ij} = \sum_{l,m=1,2,3} O_{il} O_{jm} I_{lm} \tag{3.50}$$

Another example: the completely antisymmetric rank 3, 3-tensor

$$\epsilon_{ijk} \tag{3.51}$$

3.5 PROBLEMS

Exercise 3.1 Photon Energy flux in other frames

In a source's rest frame S emits radiation isotropically with a frequency ω with number flux $f(\text{photons}/\text{cm}^2 s)$. Moves along x'-axis with speed V in an observer frame (O). What does the energy flux in O look like?

Answer for Exercise 3.1

Simon (our TA) blasted through a problem from Hartle [5], section 5.17 (all the while apologizing for going so slow). It took me a while to work through my notes to come up with something that was comprehensible to me.

At one point he asked if anybody was completely lost. Nobody said yes, but given the class title, I had the urge to say "No, just relatively lost".

We will work in momentum space, where we have

$$\begin{aligned}
p^i &= (p^0, \mathbf{p}) = \left(\frac{E}{c}, \mathbf{p}\right) \\
p^2 &= \frac{E^2}{c^2} - \mathbf{p}^2 \\
\mathbf{p} &= \hbar\mathbf{k} \\
E &= \hbar\omega \\
p^i &= \hbar k^i \\
k^i &= \left(\frac{\omega}{c}, \mathbf{k}\right)
\end{aligned} \tag{3.52}$$

We set up the x'-axis to be the direction of motion, and we call α the angle from it, or the azimuthal angle. The wavevector, **k**, is the direction the wave travels. Therefore, if we want to find the angle the radiation makes to the direction of motion, you need the projection of the wavevector onto the x-axis, or $k^1/|\mathbf{k}|$. In other words, imagine a piece of radiation emitted in a certain direction, the angle it makes with the x'-axis is the cosine of the projection on the x'-axis over the magnitude.

This azimuthal angle in the unprimed frame is

$$\cos\alpha = \frac{k^1}{|\mathbf{k}|} = \frac{k^1}{\omega/c}, \tag{3.53}$$

In the observer's reference frame (the primed coordinates), the source is moving in the $+x$ direction, and therefore, we must boost in the $-x$ from the source's frame, or $-\beta$. Transforming out wave four vector in the same fashion as regular mechanical position and momentum four vectors, we have for the observer

$$\cos\alpha' = \frac{k^{1\prime}}{\omega'/c} = \frac{\gamma(k^1 + \beta\omega/c)}{\gamma(\omega/c + \beta k^1)} \tag{3.54}$$

check 1 as $\beta \to 1$ (ie: our primed frame velocity approaches the speed of light relative to the rest frame), $\cos\alpha' \to 1$, $\alpha' = 0$. The surface gets more and more compressed.

In the original reference frame the radiation was isotropic. In the new frame how does it change with respect to the angle? This is really a question to find this number flux rate

$$f'(\alpha') =? \tag{3.55}$$

In our rest frame the total number of photons traveling through the surface in a given interval of time is

$$N = \int d\Omega dt f(\alpha) = \int d\phi \sin\alpha d\alpha = -2\pi \int d(\cos\alpha) dt f(\alpha) \tag{3.56}$$

Here we utilize the spherical solid angle $d\Omega = \sin\alpha d\alpha d\phi = -d(\cos\alpha)d\phi$, and integrate ϕ over the $[0, 2\pi]$ interval. We also have to assume that our number flux density is not a function of horizontal angle ϕ in the rest frame.

In the moving frame we similarly have

$$N' = -2\pi \int d(\cos\alpha')dt' f'(\alpha'), \tag{3.57}$$

and we again have had to assume that our transformed number flux density is not a function of the horizontal angle ϕ. This seems like a reasonable move since $k^{2\prime} = k^2$ and $k^{3\prime} = k^3$ as they are perpendicular to the boost direction.

$$f'(\alpha') = \frac{d(\cos\alpha)}{d(\cos\alpha')}\left(\frac{dt}{dt'}\right) f(\alpha) \tag{3.58}$$

Now, utilizing a conservation of mass argument, we can argue that $N = N'$. Regardless of the motion of the frame, the same number of particles move through the surface. Taking ratios, and examining an infinitesimal time interval, and the associated flux through a small patch, we have

$$\left(\frac{d(\cos\alpha)}{d(\cos\alpha')}\right) = \left(\frac{d(\cos\alpha')}{d(\cos\alpha)}\right)^{-1} = \gamma^2(1 + \beta\cos\alpha)^2 \tag{3.59}$$

Part of the statement above was a do-it-yourself. First recall that $ct' = \gamma(ct + \beta x)$, so dt/dt' evaluated at $x = 0$ is $1/\gamma$.

The rest is messier. We can calculate the $d(\cos)$ values in the ratio above using eq. (3.53). For example, for $d(\cos(\alpha))$ we have

$$\begin{aligned} d(\cos\alpha) &= d\left(\frac{k^1}{\omega/c}\right) \\ &= dk^1 \frac{1}{\omega/c} - c\frac{1}{\omega^2} d\omega. \end{aligned} \tag{3.60}$$

If one does the same thing for $d(\cos\alpha')$, after a whole whack of messy algebra one finds that the differential terms and a whole lot more mystically cancels, leaving just

$$\frac{d\cos\alpha'}{d\cos\alpha} = \frac{\omega^2/c^2}{(\omega/c + \beta k^1)^2}(1 - \beta^2) \tag{3.61}$$

A bit more reduction with reference back to eq. (3.54) verifies eq. (3.59).

Also note that again from eq. (3.54) we have

$$\cos\alpha' = \frac{\cos\alpha + \beta}{1 + \beta\cos\alpha} \tag{3.62}$$

and rearranging this for $\cos\alpha'$ gives us

$$\cos\alpha = \frac{\cos\alpha' - \beta}{1 - \beta\cos\alpha'}, \tag{3.63}$$

which we can sum to find that

$$1 + \beta\cos\alpha = \frac{1}{\gamma^2(1 - \beta\cos\alpha')}, \tag{3.64}$$

so putting all the pieces together we have

$$f'(\alpha') = \frac{1}{\gamma^3}\frac{f(\alpha)}{(1 - \beta\cos\alpha')^2} \tag{3.65}$$

The question asks for the energy flux density. We get this by multiplying the number density by the frequency of the light in question. This is, as a function of the polar angle, in each of the frames.

$$\begin{aligned} L(\alpha) &= \hbar\omega(\alpha) f(\alpha) = \hbar\omega f \\ L'(\alpha') &= \hbar\omega'(\alpha') f'(\alpha') = \hbar\omega' f' \end{aligned} \tag{3.66}$$

But we have

$$\omega'(\alpha')/c = \gamma(\omega/c + \beta k^1) = \gamma\omega/c(1 + \beta\cos\alpha) \tag{3.67}$$

Aside, $\beta << 1$,

$$\omega' = \omega(1 + \beta\cos\alpha) + O(\beta^2) = \omega + \delta\omega \tag{3.68}$$

$$\begin{aligned} &\delta\omega = \beta, \alpha = 0 \qquad \text{blue shift} \\ &\delta\omega = -\beta, \alpha = \pi \qquad \text{red shift} \end{aligned} \tag{3.69}$$

The energy flux density in the unprimed observer frame is now found to be

$$L'(\alpha') = \frac{L/\gamma}{(\gamma(1-\beta\cos\alpha'))^3} \tag{3.70}$$

And the forward backward ratio is

$$L'(0)/L'(\pi) = \left(\frac{1+\beta}{1-\beta}\right)^3, \tag{3.71}$$

allowing us to conclude that the forward radiation is bigger than the backwards radiation (and much bigger when the motion approaches the speed of light).

Exercise 3.2 Trajectory of particle with hyperbolic worldline

A particle moves on the x-axis along a world line described by

$$\begin{aligned} ct(\sigma) &= \frac{1}{a}\sinh(\sigma) \\ x(\sigma) &= \frac{1}{a}\cosh(\sigma) \end{aligned} \tag{3.72}$$

where the dimension of the constant $[a] = \frac{1}{L}$, is inverse length, and our parameter takes any values $-\infty < \sigma < \infty$.

Find the

a. trajectory
   $x^i(\tau)$,
b. proper velocity
   $u^i(\tau)$, and
c. proper acceleration
   $a^i(\tau)$.

Answer for Exercise 3.2

Parametrize by time First note that we can re-parametrize $x = x^1$ in terms of t. That is

$$\begin{aligned} \cosh(\sigma) &= \sqrt{1+\sinh^2(\sigma)} \\ &= \sqrt{1+(act)^2} \\ &= a\sqrt{a^{-2}+(ct)^2} \end{aligned} \tag{3.73}$$

So

$$x(t) = \sqrt{a^{-2} + (ct)^2} \tag{3.74}$$

Asymptotes Squaring and rearranging, shows that our particle is moving through half of a hyperbolic arc in spacetime (two such paths are possible, one for strictly positive x and one for strictly negative).

$$x^2 - (ct)^2 = a^{-2} \tag{3.75}$$

Observe that the asymptotes of this curve are the lightcone boundaries. Taking derivatives we have

$$2x\frac{dx}{d(ct)} - 2(ct) = 0, \tag{3.76}$$

and rearranging we have

$$\begin{aligned}\frac{dx}{d(ct)} &= \frac{ct}{x} \\ &= \frac{ct}{\sqrt{(ct)^2 + a^{-2}}} \\ &\rightarrow \pm 1\end{aligned} \tag{3.77}$$

Is this timelike? Let us compute the interval between two worldpoints. That is

$$\begin{aligned}s_{12}^2 &= (ct(\sigma_1) - ct(\sigma_2))^2 - (x(\sigma_1) - x(\sigma_2))^2 \\ &= a^{-2}(\sinh\sigma_1 - \sinh\sigma_2)^2 - a^{-2}(\cosh\sigma_1 - \cosh\sigma_2)^2 \\ &= 2a^{-2}\left(-1 - \sinh\sigma_1\sinh\sigma_2 + \cosh\sigma_1\cosh\sigma_2\right) \\ &= 2a^{-2}\left(\cosh(\sigma_2 - \sigma_1) - 1\right) \geq 0\end{aligned} \tag{3.78}$$

Yes, this is timelike. That is what we want for a particle that is realistic moving along a worldline in spacetime. If the spacetime interval between any two points were to be negative, we would be talking about something of tachyon like hypothetical nature.

Part a. Reparametrize by proper time. Our first task is to compute $x^i(\tau)$. We have $x^i(\sigma)$ so we need the relation between our proper length τ and the worldline parameter σ. Such a relation is implicitly provided by the differential spacetime interval

$$\begin{aligned}\left(\frac{d\tau}{d\sigma}\right)^2 &= \frac{1}{c^2}\left(\frac{ds}{d\sigma}\right)^2 \\ &= \frac{1}{c^2}\left(\left(\frac{d(x^0)}{d\sigma}\right)^2 - \left(\frac{d(x^1)}{d\sigma}\right)^2\right) \\ &= \frac{1}{c^2}\left(a^{-2}\cosh^2\sigma - a^{-2}\sinh^2\sigma\right) \\ &= \frac{1}{a^2c^2}.\end{aligned} \tag{3.79}$$

Taking roots we have

$$\frac{d\tau}{d\sigma} = \pm\frac{1}{ac}, \tag{3.80}$$

We take the positive root, so that the worldline is traversed in a strictly increasing fashion, and then integrate once

$$\tau = \frac{1}{ac}\sigma + \tau_s. \tag{3.81}$$

We are free to let $\tau_s = 0$, effectively starting our proper time at $t = 0$.

$$x^i(\tau) = (a^{-1}\sinh(ac\tau), a^{-1}\cosh(ac\tau), 0, 0) \tag{3.82}$$

As noted already this is a hyperbola (or degenerate hyperboloid) in spacetime, with asymptote 1 (ie: approaching the speed of light).

Part b. Proper velocity The next computational task is now simple.

$$u^i = \frac{dx^i}{ds} = \frac{1}{c}\frac{dx^i}{d\tau} = (\cosh(ac\tau), \sinh(ac\tau), 0, 0) \tag{3.83}$$

Is this light like or time like? We can answer this by considering the four vector square

$$u \cdot u \tag{3.84}$$

Time like vectors What is a light like or a time like vector?

Recall that we have defined lightlike, spacelike, and timelike intervals. A lightlike interval between two world points had $(ct - c\tilde{t})^2 - (\mathbf{x} - \tilde{\mathbf{x}})^2 = 0$, whereas a timelike interval had $(ct - c\tilde{t})^2 - (\mathbf{x} - \tilde{\mathbf{x}})^2 > 0$. Taking the vector $(c\tilde{t}, \tilde{\mathbf{x}})$ as the origin, the distance to any single four vector from the origin is then just that vector's square, so it logically makes sense to call a vector light like if it has a zero square, and time like if it has a positive square.

Consider the very simplest example of a time like trajectory, that of a particle at rest at a fixed position $\mathbf{x}_0$. Such a particle's worldline is

$$X = (ct, \mathbf{x}_0) \tag{3.85}$$

While we interpret t here as time, it functions as a parametrization of the curve, just as σ does in this question. If we want to compute the proper time interval between two points on this worldline we have

$$\begin{aligned}
\tau_b - \tau_0 &= \frac{1}{c}\int_{\lambda=t_a}^{t_b} \sqrt{\frac{dX(\lambda)}{d\lambda}\cdot\frac{dX(\lambda)}{d\lambda}}d\lambda \\
&= \frac{1}{c}\int_{\lambda=t_a}^{t_b} \sqrt{(c,0)^2}d\lambda \\
&= \frac{1}{c}\int_{\lambda=t_a}^{t_b} c d\lambda \\
&= t_b - t_a
\end{aligned} \tag{3.86}$$

The conclusion (arrived at the hard way, but methodologically) is that proper time on this worldline is just the parameter t itself.

Now examine the proper velocity for this trajectory. This is

$$u = \frac{dX}{ds} = (1, 0, 0, 0) \tag{3.87}$$

We can compute the dot product $u \cdot u = 1 > 0$ easily enough, and in this case for the particle at rest (but moving in time) we see that this four-vector velocity does have a time like separation from the origin, and it therefore makes sense to label the four-velocity vector itself as time like.

Now, let us return to our non-inertial system. Is our four velocity vector time like? Let us compute its square to check

$$u \cdot u = \cosh^2 - \sinh^2 = 1 > 0 \tag{3.88}$$

Yes, it is timelike.

Spatial velocity Now, let us calculate our spatial velocity

$$v^\alpha = \frac{dx^\alpha}{dt} = \frac{dx^\alpha}{ds} c \frac{d\tau}{dt} \tag{3.89}$$

Since $ct = \sinh(ac\tau)/a$ we have

$$c = \frac{1}{a} ac \cosh(ac\tau) \frac{d\tau}{dt}, \tag{3.90}$$

or

$$\frac{d\tau}{dt} = \frac{1}{\cosh(ac\tau)} \tag{3.91}$$

Similarly from eq. (3.82), we have

$$\frac{dx^1}{ds} = \frac{1}{c} \frac{dx^1}{d\tau} = \sinh(ac\tau) \tag{3.92}$$

So our spatial velocity is sinh/cosh = tanh, and we have

$$v^\alpha = (c \tanh(ac\tau), 0, 0) \tag{3.93}$$

Note how tricky this index notation is. For our four vector velocity we use $u^i = dx^i/ds$, whereas our spatial velocity is distinguished by a change of letter as well as the indices, so when we write v^α we are taking our derivatives with respect to time and not proper time (i.e. $v^\alpha = dx^\alpha/dt$).

Part c. Four-acceleration From eq. (3.83), we have

$$w^i = \frac{du^i}{ds} = ax^i \tag{3.94}$$

Observe that our four-velocity square is

$$w \cdot w = a^2 a^{-1}(-1) \tag{3.95}$$

What does this really signify? Think on this. A check to verify that things are okay is to see if this four-acceleration is orthogonal to our four-velocity as expected

$$\begin{aligned} w \cdot u &= a(a^{-1} \sinh(ac\tau), a^{-1} \cosh(ac\tau), 0, 0) \cdot (\cosh(ac\tau), \sinh(ac\tau), 0, 0) \\ &= (\sinh(ac\tau) \cosh(ac\tau) - \cosh(ac\tau) \sinh(ac\tau)) \\ &= 0 \end{aligned} \tag{3.96}$$

Spatial acceleration A last beastie that we can compute is the spatial acceleration.

$$
\begin{aligned}
a^\alpha &= \frac{d^2 x^\alpha}{dt^2} \\
&= \frac{d}{dt}\frac{dx^\alpha}{dt} \\
&= \frac{d}{dt}\left(\frac{dx^\alpha}{ds} c \frac{d\tau}{dt}\right) \\
&= \frac{d}{dt}\left(c u^\alpha \frac{d\tau}{dt}\right) \\
&= \frac{d}{dt}\left(c \frac{\sinh(ac\tau)}{\cosh(ac\tau)}\right) \\
&= \frac{d}{d\tau}\left(c \frac{\sinh(ac\tau)}{\cosh(ac\tau)}\right)\frac{d\tau}{dt} \\
&= \frac{ac^2}{\cosh^2(ac\tau)} \frac{1}{\cosh(ac\tau)} \\
&= \frac{ac^2}{\cosh^3(ac\tau)}
\end{aligned}
\tag{3.97}
$$

Summary Collecting all results we have

$$
\begin{aligned}
x^i(\tau) &= \left(a^{-1}\sinh(ac\tau), a^{-1}\cosh(ac\tau), 0, 0\right) \\
u^i(\tau) &= \left(\cosh(ac\tau), \sinh(ac\tau), 0, 0\right) \\
v^\alpha(\tau) &= \left(c\tanh(ac\tau), 0, 0\right) \\
w^i(\tau) &= a x^i(\tau) \\
a^\alpha(\tau) &= \left(\frac{ac^2}{\cosh^3(ac\tau)}, 0, 0\right).
\end{aligned}
\tag{3.98}
$$

XX

Exercise 3.3 Motion in an constant uniform Electric field

Given

$$\mathbf{E} = E\hat{\mathbf{x}}, \tag{3.99}$$

We want to solve the problem

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = e\mathbf{E}. \tag{3.100}$$

Unlike second year classical physics, we will use relativistic momentum, so for only a constant electric field, our Lorentz force equation to solve becomes

$$\frac{d\mathbf{p}}{dt} = \frac{d(m\gamma\mathbf{v})}{dt} = e\mathbf{E}. \tag{3.101}$$

Answer for Exercise 3.3

In components this is

$$\begin{aligned}\dot{p}_x &= eE \\ \dot{p}_y &= \text{constant}\end{aligned} \tag{3.102}$$

Integrating the x component we have

$$eEt + p_x(0) = \frac{m\dot{x}}{\sqrt{1 - (\dot{x}^2 + \dot{y}^2)/c^2}} \tag{3.103}$$

If we let $p_x(0) = 0$, square and rearrange a bit we have

$$\frac{m^2}{(eEt)^2}\dot{x}^2 = 1 - \frac{\dot{x}^2 + \dot{y}^2}{c^2} \tag{3.104}$$

For

$$\dot{x}^2 = \frac{c^2 - \dot{y}^2}{1 + (\frac{mc}{eEt})^2}. \tag{3.105}$$

Now for the y components, with $p_y(0) = p_0$, our equation to solve is

$$\frac{m\dot{y}}{\sqrt{1 - (\dot{x}^2 + \dot{y}^2)/c^2}} = p_0. \tag{3.106}$$

Squaring this one we have

$$\frac{c^2m^2}{p_0^2}\dot{y}^2 = c^2 - \dot{x}^2 - \dot{y}^2, \tag{3.107}$$

and

$$\dot{y}^2 = \frac{c^2 - \dot{x}^2}{1 + \frac{m^2c^2}{p_0^2}} \tag{3.108}$$

Observe that our energy is

$$\mathcal{E}^2 = p^2c^2 + m^2c^4, \tag{3.109}$$

and for $t = 0$

$$\mathcal{E}_0^2 = p_0^2c^2 + m^2c^4. \tag{3.110}$$

We can then write

$$\dot{y}^2 = \frac{c^2 p_0^2 (c^2 - \dot{x}^2)}{\mathcal{E}_0^2}. \tag{3.111}$$

Some messy substitution, using eq. (3.105), yields

$$\boxed{\begin{aligned} \dot{x} &= \frac{c^2 eEt}{\sqrt{\mathcal{E}_0^2 + (ecEt)^2}} \\ \dot{y} &= \frac{c^2 p_0}{\sqrt{\mathcal{E}_0^2 + (ecEt)^2}} \end{aligned}} \tag{3.112}$$

Solving for x we have

$$x(t) = c^2 eE \int \frac{dt' t'}{\sqrt{\mathcal{E}_0^2 + (ecEt')^2}} \tag{3.113}$$

Can solve with hyperbolic substitution or

$$x(t) = c^2 eE \int \frac{dt' t'}{\sqrt{\mathcal{E}_0^2 + (ecEt')^2}} \tag{3.114}$$

$$d(u^2) = 2udu \implies udu = \frac{1}{2} d(u^2) \tag{3.115}$$

$$x(t) = \frac{c^2 eE}{2\mathcal{E}_0} \int \frac{d(u^2)}{\sqrt{1 + \left(\frac{ecE}{\mathcal{E}_0}\right)^2 u^2}} \tag{3.116}$$

Now we have something of the form

$$\int \frac{dv}{\sqrt{1+av}} = \frac{2}{a}\sqrt{1+av}, \tag{3.117}$$

so our final solution for $x(t)$ is

$$x(t) = \frac{1}{eE}\sqrt{\mathcal{E}_0^2 + (ecEt)^2} \tag{3.118}$$

or

$$x^2 - c^2t^2 = \frac{\mathcal{E}_0^2}{e^2E^2} = a^{-2}. \tag{3.119}$$

Now for $y(t)$ we have

$$y(t) = c^2 p_0 \int \frac{dt}{\sqrt{\mathcal{E}_0^2 + (ecEt)^2}} \tag{3.120}$$

$$t = \frac{\mathcal{E}_0}{ecE}\sinh(u) \tag{3.121}$$

$$dt = \frac{\mathcal{E}_0}{ecE}\cosh(u)du \tag{3.122}$$

$$\begin{aligned} y(t) &= \frac{c^2 p_0}{\mathcal{E}_0}\int \frac{dt}{\sqrt{1+(\frac{ecE}{\mathcal{E}_0})^2 t^2}} \\ &= \frac{c^2 p_0}{\mathcal{E}_0}\frac{\mathcal{E}_0}{ecE}\int \frac{du\cosh u}{\sqrt{1+\sinh^2 u}} \\ &= \frac{cp_0}{eE}u \end{aligned} \tag{3.123}$$

A final bit of substitution, including a sort of odd seeming parametrization of x in terms of y in terms of t, we have

$$\boxed{\begin{aligned} y(t) &= \frac{cp_0}{eE}\sinh^{-1}\left(\frac{ecEt}{\mathcal{E}_0}\right) \\ x(y) &= \frac{\mathcal{E}_0}{cE\cosh\left(\frac{yeE}{cp_0}\right)} \end{aligned}} \tag{3.124}$$

Checks FIXME: check the checks.

$$v \rightarrow c, t \rightarrow \infty \tag{3.125}$$

$$v << c, t \rightarrow 0 \tag{3.126}$$

$$\begin{aligned} mv_x &= eEt + ... \\ x &\sim t^2 \end{aligned} \tag{3.127}$$

$$mv_y = p_0 \rightarrow y \sim t \tag{3.128}$$

$$x(y) \sim y^2 \tag{3.129}$$

(a parabola)

An alternate way There is also a tricky way (as in the text), with

$$\begin{aligned} \mathbf{p} &= m\gamma\mathbf{v} \\ \mathcal{E} &= \gamma m c^2 \end{aligned} \tag{3.130}$$

We can solve this for **p**

$$\begin{aligned} m\gamma &= \frac{\mathbf{p}\cdot\mathbf{v}}{\mathbf{v}^2} = \frac{\mathcal{E}}{c^2} \\ \mathbf{p}\times\mathbf{v} &= 0 \end{aligned} \tag{3.131}$$

With the cross product zero, **p** has only a component in the direction of **v**, and we can invert to yield

$$\mathbf{p} = \frac{\mathcal{E}\mathbf{v}}{c^2}. \tag{3.132}$$

This implies

$$\dot{x} = \frac{c^2 p_x}{\mathcal{E}}, \tag{3.133}$$

and one can work from there as well.

Exercise 3.4 Motion in an constant uniform Magnetic field

Calculate a particle motion in a uniform magnetic field.

Answer for Exercise 3.4

Work by the magnetic field Note that the magnetic field does no work

$$\mathbf{F} = \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{3.134}$$

$$\begin{aligned} dW &= \mathbf{F} \cdot d\mathbf{l} \\ &= \frac{e}{c}(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \\ &= \frac{e}{c}(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt \\ &= 0 \end{aligned} \tag{3.135}$$

Because $\mathbf{v}$ and $\mathbf{v} \times \mathbf{B}$ are necessarily perpendicular we are reminded that the magnetic field does no work (even in this relativistic sense).

Initial energy of the particle Because no work is done, the particle's energy is only the initial time value

$$\mathcal{E} = ... + eA^0 \tag{3.136}$$

Simon asked if we would calculated this (i.e. the Hamiltonian in class). We would calculated the conservation for time invariance, the Hamiltonian (and called it E). We would also calculated the Hamiltonian for the free particle

$$\mathcal{E}^2 = \mathbf{p}^2 c^2 + (mc^2)^2. \tag{3.137}$$

We had not done this calculation for the Lorentz force Lagrangian, so lets do it now. Recall that this Lagrangian was

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - e\phi + \frac{e}{c}\mathbf{v} \cdot \mathbf{A}, \tag{3.138}$$

with generalized momentum of

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} + \frac{e}{c}\mathbf{A}. \tag{3.139}$$

Our Hamiltonian is thus

$$\mathcal{E} = \frac{m\mathbf{v}^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} + \frac{e}{c}\mathbf{A} \cdot \mathbf{v} + mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} + e\phi - \frac{e}{c}\mathbf{v} \cdot \mathbf{A}, \tag{3.140}$$

which gives us

$$\mathcal{E} = e\phi + \frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \tag{3.141}$$

So we see that our “energy”, defined as a quantity that is conserved, as a result of the symmetry of time translation invariance, has a component due to the electric field (but not the vector potential field $\mathbf{A}$), plus the free particle “energy”.

Is this right? With $\mathbf{A}$ and ϕ being functions of space and time, perhaps we need to be more careful with this argument. Perhaps this actually only applies to a statics case where $\mathbf{A}$ and ϕ are constant.

Since it was hinted to us that the energy component of the Lorentz force equation was proportional to $F^{0j}u_j$, and we can peek ahead to find that $F^{ij} = \partial^i A^j - \partial^j A^i$, let us compare to that

$$\begin{aligned} eF^{0j}u_j &= e(\partial^0 A^j - \partial^j A^0)u_j \\ &= e(\partial^0 A^\alpha - \partial^\alpha A^0)u_\alpha \\ &= e\left(\frac{1}{c}\frac{\partial A^\alpha}{\partial t} + \partial_\alpha A^0\right)\frac{1}{c}\frac{dx_\alpha}{d\tau} \\ &= -e\left(\frac{1}{c}\frac{\partial A^\alpha}{\partial t} + \frac{\partial \phi}{\partial x^\alpha}\right)\frac{1}{c}\frac{dx^\alpha}{dt}\gamma, \end{aligned} \tag{3.142}$$

which is

$$eF^{0j}u_j = e\left(\mathbf{E} \cdot \frac{\mathbf{v}}{c}\right)\gamma. \tag{3.143}$$

So if we have

$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{3.144}$$

I had guess that we have

$$\frac{d(\mathcal{E}/c)}{d\tau} \sim eF^{0j}u_j, \tag{3.145}$$

which is, using eq. (3.143)

$$\frac{d(\mathcal{E}/c)}{dt} \sim e\left(\mathbf{E} \cdot \frac{\mathbf{v}}{c}\right) \tag{3.146}$$

Can the left hand side be integrated to yield $e\phi$? Yes, but only in the statics case when $\partial\mathbf{A}/\partial t = 0$, and $\phi(\mathbf{x}, t) = \phi(\mathbf{x})$ for which we have

$$\begin{aligned}
\mathcal{E} &\sim e\int_a^b \mathbf{E} \cdot \mathbf{v} dt \\
&= -e\int_a^b (\boldsymbol{\nabla}\phi) \cdot \frac{d\mathbf{x}}{dt} dt \\
&= -e\int_a^b (\boldsymbol{\nabla}\phi) \cdot d\mathbf{x} \\
&= -e\int_a^b \frac{\partial\phi}{\partial x^\alpha} dx^\alpha \\
&= -e(\phi_b - \phi_a)
\end{aligned} \tag{3.147}$$

FIXME: My suspicion is that the result eq. (3.146), is generally true, but that we have dropped terms from the Hamiltonian calculation that need to be retained when ϕ and $\mathbf{A}$ are functions of time.

Expressing the field and the force equation We will align our field with the z axis, and write

$$\mathbf{B} = H\hat{\mathbf{z}}, \tag{3.148}$$

or, in components

$$\delta_{\alpha 3} H = H_\alpha. \tag{3.149}$$

Because the energy is only due to the initial value, we write

$$\mathcal{E}(t) = \mathcal{E}_0 \tag{3.150}$$

$$\mathbf{p} = \mathcal{E}\frac{\mathbf{v}}{c^2} = \mathcal{E}_0\frac{\mathbf{v}}{c^2} \tag{3.151}$$

implies

$$\mathbf{v} = \mathbf{p}\frac{c^2}{\mathcal{E}_0} \tag{3.152}$$

$$\dot{\mathbf{v}} = \dot{\mathbf{p}}\frac{c^2}{\mathcal{E}_0} \tag{3.153}$$

$$\dot{v}_\alpha = \frac{ec}{\mathcal{E}_0}\epsilon_{\alpha\beta\gamma}v_\beta H_\gamma \tag{3.154}$$

write

$$\omega = \frac{ecH}{\mathcal{E}_0} \tag{3.155}$$

Evaluating the delta

$$\dot{v}_\alpha = \omega\epsilon_{\alpha\beta 3}v_\beta \tag{3.156}$$

$$\begin{aligned}
\dot{v}_1 &= \omega\epsilon_{1\beta 3}v_\beta = \omega v_2 \\
\dot{v}_2 &= \omega\epsilon_{2\beta 3}v_\beta = -\omega v_1 \\
\dot{v}_3 &= \omega\epsilon_{3\beta 3}v_\beta = 0
\end{aligned} \tag{3.157}$$

Looks like circular motion, so it is natural to use complex variables. With

$$z = v_1 + i v_2 \tag{3.158}$$

Using this we have

$$\begin{aligned}\frac{d}{dt}(v_1 + i v_2) &= \omega v_2 - i\omega v_1 \\ &= -i\omega(v_1 + i v_2).\end{aligned} \tag{3.159}$$

which comes out nicely

$$\frac{dz}{dt} = -i\omega z \tag{3.160}$$

for

$$z = V_0 e^{-i\omega z t + i\alpha} \tag{3.161}$$

Real and imaginary parts

$$\begin{aligned}v_1(t) &= V_0 \cos(\omega z t + \alpha) \\ v_2(t) &= -V_0 \sin(\omega z t + \alpha)\end{aligned} \tag{3.162}$$

Integrating

$$\begin{aligned}x_1(t) &= x_1(0) + V_0 \sin(\omega z t + \alpha) \\ x_2(t) &= x_2(0) + V_0 \cos(\omega z t + \alpha)\end{aligned} \tag{3.163}$$

Which is a helix. PICTURE: ...

Exercise 3.5 Particle collision

A particle of rest mass m whose energy is three times its rest energy collides with an identical particle at rest. Suppose they stick together. Use conservation laws to find the mass of the resulting particle and its velocity. Is its mass greater or smaller than $2m$? Comment.

Answer for Exercise 3.5

The energy of the initially moving particle before collision is

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = 3mc^2. \tag{3.164}$$

Solving for the velocity we have

$$\left|\frac{\mathbf{v}}{c}\right| = \frac{2\sqrt{2}}{3}. \tag{3.165}$$

Our four velocity is

$$u^i = \gamma\left(1, \frac{\mathbf{v}}{c}\right) = (3, 2\sqrt{2}). \tag{3.166}$$

Designate the four momentum for this particle as

$$p^i_{(1)} = mc(3, 2\sqrt{2}). \tag{3.167}$$

For the second particle we have

$$p^i_{(2)} = mc(1, 0). \tag{3.168}$$

Our initial and final four momentum will be equal, and our resulting velocity can only be in the direction of the initial particle. This leaves us with

$$\begin{aligned} p^i_{(f)} &= Mc\frac{1}{\sqrt{1 - \frac{\mathbf{v}_f^2}{c^2}}}\left(1, \frac{\mathbf{v}_f}{c}\right) \\ &= mc(1, 0) + mc(3, 2\sqrt{2}) \\ &= mc(4, 2\sqrt{2}) \\ &= 4mc\left(1, \frac{1}{\sqrt{2}}\right) \end{aligned} \tag{3.169}$$

Our final velocity is $v_f = c/\sqrt{2}$.
We have $M\gamma = 4$ for the final particle, but we have

$$\gamma = \frac{1}{\sqrt{1 - 1/2}} = \sqrt{2}, \tag{3.170}$$

so our final mass is

$$M = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2. \tag{3.171}$$

Relativistically, we have conservation of four-momentum, not conservation of mass, so a composite body will not necessarily have a mass measurement that is the sum of the parts. One possible way to reconcile this statement with intuition is to define mass in terms of the four momentum

$$m^2 = \frac{p^i p_i}{c^2}, \tag{3.172}$$

and think of it as a derived quantity, not fundamental.

Exercise 3.6 Particle in an electromagnetic field

This problem has three parts

a. Express the "normal" (i.e. not 4-, but 3-) acceleration, equal to $\dot{\mathbf{v}}$, or a particle in terms of its velocity, **E**, and **B**, using the equation of motion of a relativistic particle in an external electromagnetic field.

b. Consider now a beam of electrons, moving along the x direction with a known energy $\mathcal{E}$, entering a region with constant homogeneous **E** and **B** fields. The fields are perpendicular, **E** is along the y direction while **B** is along the z direction.

   1. Show that by tuning the values of **E** and **B** it is possible to balance electric and magnetic forces so that the beam does not deviate from its original direction (and, say, hits a screen directly ahead).
   2. Find a relation determining the mass of the electron using $\mathcal{E}$ and the measured values of the fields for which no deviation occurs. Do not assume a non-relativistic limit and elucidate which part of this problem (a way to measure the mass of the electron) is affected by relativity.

c. Solve for the motion (i.e. find the trajectories) of a relativistic charged particle in perpendicular constant and homogeneous electric and magnetic fields; do not assume $\mathbf{E} = \mathbf{B}$.

Answer for Exercise 3.6

Part a. Finding $\dot{\mathbf{v}}$. With the particle's energy given by

$$\mathcal{E} = \gamma m c^2, \tag{3.173}$$

we note that

$$\mathcal{E}\mathbf{v} = (\gamma m \mathbf{v})c^2 = \mathbf{p}c^2. \tag{3.174}$$

Taking derivatives we have

$$\begin{aligned} c^2 \frac{d\mathbf{p}}{dt} &= \mathbf{v} \frac{d\mathcal{E}}{dt} + \frac{d\mathbf{v}}{dt}\mathcal{E} \\ &= \mathbf{v}(e\mathbf{E} \cdot \mathbf{v}) + \frac{d\mathbf{v}}{dt}\mathcal{E} \end{aligned} \tag{3.175}$$

Rearranging we have

$$\frac{d\mathbf{v}}{dt} = \frac{c^2 e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) - \mathbf{v}(e\mathbf{E} \cdot \mathbf{v})}{\mathcal{E}} \tag{3.176}$$

which leaves us with the desired result

$$\boxed{\dot{\mathbf{v}} = \frac{e}{m}\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} - \frac{\mathbf{v}}{c}\left(\mathbf{E} \cdot \frac{\mathbf{v}}{c}\right)\right)} \tag{3.177}$$

Part b. On the energy change rate. Note that when the problem set was assigned, the relation

$$\frac{d\mathcal{E}}{dt} = e\mathbf{E} \cdot \mathbf{v} \tag{3.178}$$

had not been demonstrated. To show this observe that we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E} &= mc^2 \frac{d\gamma}{dt} \\ &= mc^2 \frac{d}{dt} \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \\ &= mc^2 \frac{\frac{\mathbf{v}}{c^2} \cdot \frac{d\mathbf{v}}{dt}}{\left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{3/2}} \\ &= \frac{m\gamma\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}}{1 - \frac{\mathbf{v}^2}{c^2}} \end{aligned} \tag{3.179}$$

We also have

$$\begin{aligned}
\mathbf{v} \cdot \frac{d\mathbf{p}}{dt} &= \mathbf{v} \cdot \frac{d}{dt} \frac{m\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \\
&= m\mathbf{v}^2 \frac{d\gamma}{dt} + m\gamma \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \\
&= m\mathbf{v}^2 \frac{d\gamma}{dt} + mc^2 \frac{d\gamma}{dt} \left(1 - \frac{\mathbf{v}^2}{c^2}\right) \\
&= mc^2 \frac{d\gamma}{dt}.
\end{aligned} \tag{3.180}$$

Utilizing the Lorentz force equation, we have

$$\mathbf{v} \cdot \frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \mathbf{v} = e\mathbf{E} \cdot \mathbf{v} \tag{3.181}$$

and are able to assemble the above, and find that we have

$$\frac{d(mc^2\gamma)}{dt} = e\mathbf{E} \cdot \mathbf{v} \tag{3.182}$$

2. (a). Tuning $\mathbf{E}$ *and* $\mathbf{B}$ Using our previous result with $\mathbf{E} = E\hat{\mathbf{y}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$, our system of equations takes the form

$$\dot{\mathbf{v}} = \frac{e}{m} \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \left(E\hat{\mathbf{y}} + \hat{\mathbf{x}}\frac{v_y}{c}B - \hat{\mathbf{y}}\frac{y_x}{B} - \frac{\mathbf{v}}{c}E\frac{v_y}{c}\right) \tag{3.183}$$

This is really three equations, but they are coupled with the nasty $\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}$ term. However, since it is specified that the particles have a known energy $\mathcal{E}$, and that energy is

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}, \tag{3.184}$$

we can write

$$\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} = \frac{mc^2}{\mathcal{E}} \tag{3.185}$$

This eliminates the worst of the coupling, leaving three less hairy equations to solve

$$\begin{aligned}
\dot{v}_x &= \frac{ec^2}{\mathcal{E}}\left(\frac{v_y}{c}B - \frac{v_x v_y}{c^2}E\right) \\
\dot{v}_y &= \frac{ec^2}{\mathcal{E}}\left(E - \frac{v_x}{c}B - \frac{v_y^2}{c^2}E\right) \\
\dot{v}_z &= \frac{ec^2}{\mathcal{E}}\left(-\frac{v_y v_z}{c^2}E\right)
\end{aligned} \tag{3.186}$$

We do not actually want to compute general solutions for these equations. Instead we just wish to examine the constraints on E and B that will keep $v_y = v_z = 0$.

First off we see from the $\dot{v}_z$ equation above that if $v_y = 0$ or $v_z = 0$ initially, then $\dot{v}_z = 0$, and $v_z(t) = \text{constant} = v_z(0) = 0$. So, if the beam is initially aligned with the x direction, it will not deviate towards the z axis (in the direction of the magnetic field) at all.

Next, if we initially have $v_y = 0$, then at that point of time, our equation for $\dot{v}_x$ and $\dot{v}_y$ are respectively

$$\begin{aligned}
\dot{v}_x &= 0 \\
\dot{v}_y &= \frac{ec^2}{\mathcal{E}}\left(E - \frac{v_x}{c}B\right)
\end{aligned} \tag{3.187}$$

We are able to solve for the time evolution of the velocities directly

$$\begin{aligned}
v_x(t) &= \text{constant} = v_x(0) \\
v_y(t) &= \frac{ec^2}{\mathcal{E}}\left(E - \frac{v_x(0)}{c}B\right)t
\end{aligned} \tag{3.188}$$

We can maintain zero deviation in the y direction ($v_y(t) = 0$) provided we pick

$$E = \frac{v_x(0)}{c}B \tag{3.189}$$

3.5.0.1 *2. (b). Finding the mass of the electron*

After measuring the fields that once adjusted produce no deviation in the y and z directions, our particles velocity must then be

$$\frac{v_x}{c} = \frac{E}{B} \tag{3.190}$$

If the energy has also been measured, we have a relation between the mass from

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - v_x^2/c^2}} = \frac{mc^2}{\sqrt{1 - E^2/B^2}} \tag{3.191}$$

With a slight rearrangement, our mass can then be calculated from the energy $\mathcal{E}$, and field measurements

$$m = \frac{\mathcal{E}}{c^2}\sqrt{1 - E^2/B^2}. \tag{3.192}$$

Part c. Solve for the relativistic trajectory of a particle in perpendicular fields. Our equation to solve is

$$\frac{du^i}{ds} = \frac{e}{mc^2} F^{ij} g_{jk} u^k, \tag{3.193}$$

where

$$\left\| F^{ij} g_{jk} \right\| = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}. \tag{3.194}$$

However, with the fields being perpendicular, we are free to align them with our choice of axis. As above, let us use $\mathbf{E} = E\hat{\mathbf{y}}$, and $\mathbf{B} = B\hat{\mathbf{z}}$. Writing u for the column vector with components u^i we have a matrix equation to solve

$$\frac{du}{ds} = \frac{e}{mc^2} \begin{bmatrix} 0 & 0 & E & 0 \\ 0 & 0 & B & 0 \\ E & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} u = Fu. \tag{3.195}$$

It is simple to verify that our characteristic equation is

$$
\begin{aligned}
0 &= |F - \lambda I| \\
&= \begin{vmatrix} -\lambda & 0 & E & 0 \\ 0 & -\lambda & B & 0 \\ E & -B & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} \\
&= -\lambda^2(-\lambda^2 - B^2 + E^2)
\end{aligned}
\tag{3.196}
$$

so that our eigenvalues are

$$
\lambda = 0, 0, \pm\sqrt{E^2 - B^2}. \tag{3.197}
$$

Since the fields are constant, we can diagonalize this, and solve by exponentiation. Let

$$
D = \sqrt{E^2 - B^2}. \tag{3.198}
$$

To solve for the eigenvector e_D for $\lambda = D$ we need solutions to

$$
\begin{bmatrix} -D & 0 & E & 0 \\ 0 & -D & B & 0 \\ E & -B & -D & 0 \\ 0 & 0 & 0 & -D \end{bmatrix}
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0, \tag{3.199}
$$

and it is straightforward to compute

$$
e_D = \frac{1}{\sqrt{2}E} \begin{bmatrix} E \\ B \\ D \\ 0 \end{bmatrix}. \tag{3.200}
$$

Similarly for the $\lambda = -D$ eigenvector e_{-D} we wish to solve

$$\begin{bmatrix} D & 0 & E & 0 \\ 0 & D & B & 0 \\ E & -B & D & 0 \\ 0 & 0 & 0 & D \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0, \tag{3.201}$$

and find that

$$e_{-D} = \frac{1}{\sqrt{2}E} \begin{bmatrix} E \\ B \\ -D \\ 0 \end{bmatrix}. \tag{3.202}$$

We can also pick orthonormal eigenvectors for the degenerate zero eigenvalues from the null space of the matrix

$$\begin{bmatrix} 0 & 0 & E & 0 \\ 0 & 0 & B & 0 \\ E & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.203}$$

By inspection, two such eigenvectors are

$$\frac{1}{\sqrt{E^2 + B^2}} \begin{bmatrix} B \\ E \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.204}$$

Unfortunately, the first is not generally orthonormal to either of $e_{\pm D}$, so our similarity transformation matrix is not invertible by Hermitian transposition. Regardless, we are now well on track to putting the matrix equation we wish to solve into a much simpler form. With

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}E} \begin{bmatrix} E \\ B \\ D \\ 0 \end{bmatrix} & \frac{1}{\sqrt{2}E} \begin{bmatrix} E \\ B \\ -D \\ 0 \end{bmatrix} & \frac{1}{\sqrt{E^2 + B^2}} \begin{bmatrix} B \\ E \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}, \tag{3.205}$$

and

$$\Sigma = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & -D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.206}$$

observe that our Lorentz force equation can now be written

$$\frac{du}{ds} = \frac{e}{mc^2} S \Sigma S^{-1} u. \tag{3.207}$$

This we can rearrange, leaving us with a diagonal system that has a trivial solution

$$\frac{d}{ds}(S^{-1}u) = \frac{e}{mc^2} \Sigma (S^{-1}u). \tag{3.208}$$

Let us write

$$v = S^{-1}u, \tag{3.209}$$

and introduce a sort of proper distance wave number

$$k = \frac{e\sqrt{E^2 - B^2}}{mc^2}. \tag{3.210}$$

With this the Lorentz force equation is left in the form

$$\frac{dv}{ds} = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} v. \tag{3.211}$$

Integrating once, our solution is

$$v(s) = \begin{bmatrix} e^{ks} & 0 & 0 & 0 \\ 0 & e^{-ks} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} v(s = 0) \tag{3.212}$$

Our proper velocity is thus given by

$$u = \frac{dX}{ds} = S \begin{bmatrix} e^{ks} & 0 & 0 & 0 \\ 0 & e^{-ks} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} S^{-1} u(s = 0). \tag{3.213}$$

We can integrate once more for our trajectory, parametrized by proper distance on the worldline of the particle. That is

$$X(s) - X(0) = S \left(\int_{s'=0}^{s} ds' \begin{bmatrix} e^{ks'} & 0 & 0 & 0 \\ 0 & e^{-ks'} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) S^{-1} u(s = 0). \tag{3.214}$$

With $u(0) = \gamma_0(1, \mathbf{v}_0/c)$, and $X = (ct_0, \mathbf{x}_0)$, plus the defining relations eq. (3.205), and eq. (3.210) our parametric equation for the trajectory is fully specified

$$\begin{aligned} &\begin{bmatrix} ct(s) \\ \mathbf{x}^{\mathrm{T}}(s) \end{bmatrix} - \begin{bmatrix} ct_0 \\ \mathbf{x}_0^{\mathrm{T}} \end{bmatrix} \\ &= S \begin{bmatrix} \frac{1}{k}(e^{ks} - 1) & 0 & 0 & 0 \\ 0 & -\frac{1}{k}(e^{-ks} - 1) & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix} S^{-1} \frac{1}{\sqrt{1 - (\mathbf{v}_0)^2/c^2}} \begin{bmatrix} 1 \\ \mathbf{v}_0^{\mathrm{T}}/c \end{bmatrix}. \end{aligned} \tag{3.215}$$

Observe that for the case $E^2 > B^2$, our value k is real, so the solution is entirely composed of linear combinations of the hyperbolic functions $\cosh(ks)$ and $\sinh(ks)$. However, for the $E^2 < B^2$ case where our eigenvalues are purely imaginary, the constant k is also purely imaginary (and our eigenvectors $e_{\pm D}$ are complex). In that case, we can take the real part of this equation, and will be left with a solution that is formed of linear combinations of $\sin(ks)$ and $\cos(ks)$ terms. The $E = B$ case would have to be handled separately, and this is done in depth in the text, so there is little value repeating it here.

Exercise 3.7 Transformation of fields.

In class, we introduced the 4-vector potential A^i and its transformation law under Lorentz transformations. While we have not yet discussed how $\mathbf{E}$ and $\mathbf{B}$ transform, knowing how A^i

transforms is enough to solve some concrete problems. Suppose in one (unprimed) frame there is a charge at rest, which creates an electrostatic field: $A^0 = \phi = \frac{q}{r}, \mathbf{A} = 0$.

a. Find the values of **E** and **B** in this frame.
b. Consider now the same field in a (primed) frame moving in the x-direction with velocity v. Using the transformation law of the vector potential, find $A^{i'}$ in the primed frame.
c. Use the relations between electric and magnetic field strengths and vector potential (valid in every frame) to find the electric and magnetic fields in the primed frame (i.e. find the electromagnetic field of a moving charge). Sketch the lines of constant electric and magnetic field and comment on the result.

Answer for Exercise 3.7

Part a. In the unprimed frame we have

$$\begin{aligned}\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ &= -\boldsymbol{\nabla}\phi \\ &= -\hat{\mathbf{r}} q \partial_r (1/r) \\ &= \hat{\mathbf{r}}\frac{q}{r^2},\end{aligned} \tag{3.216}$$

and

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = 0 \tag{3.217}$$

Part b. The coordinates in the moving frame, assuming the frames are overlapping at $t = 0$, are related to the unprimed coordinates by

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{3.218}$$

Our four vector potential also transforms in the same fashion, and we have

$$\begin{bmatrix} \phi' \\ A'_x \\ A'_y \\ A'_z \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi \\ 0 \\ 0 \\ 0 \end{bmatrix} = \gamma\phi(1, -\beta, 0, 0) \tag{3.219}$$

So in the primed frame we have

$$\begin{aligned}
\phi' &= \gamma \frac{q}{r} \\
A'_x &= -\gamma\beta \frac{q}{r} \\
A'_y &= 0 \\
A'_z &= 0
\end{aligned} \tag{3.220}$$

Part c. In the primed frame our electric and magnetic fields are

$$\begin{aligned}
\mathbf{E}' &= -\boldsymbol{\nabla}'\phi' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} \\
\mathbf{B}' &= \boldsymbol{\nabla}' \times \mathbf{A}'
\end{aligned} \tag{3.221}$$

We have ϕ' and $\mathbf{A}'$ expressed in terms of the unprimed coordinates, so need to calculate the transformation of the gradient and time partial too. These partials transform as

$$\begin{aligned}
\frac{\partial}{\partial ct'} &= \frac{\partial ct}{\partial ct'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial ct'}\frac{\partial}{\partial x} \\
\frac{\partial}{\partial x'} &= \frac{\partial ct}{\partial x'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z'} &= \frac{\partial}{\partial z}
\end{aligned} \tag{3.222}$$

Utilizing the inverse transformation

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} \tag{3.223}$$

we have

$$
\begin{aligned}
\frac{\partial}{\partial ct'} &= \gamma \frac{\partial}{\partial ct} + \gamma\beta \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x'} &= \gamma\beta \frac{\partial}{\partial ct} + \gamma \frac{\partial}{\partial x} \\
\frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z'} &= \frac{\partial}{\partial z}
\end{aligned}
\tag{3.224}
$$

Since neither ϕ' nor $\mathbf{A}'$ have time dependence, we have for electric field in the primed frame

$$
\begin{aligned}
\mathbf{E}' &= -\boldsymbol{\nabla}'\phi' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} \\
&= -\left(\gamma \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi' - \gamma\beta \frac{\partial \mathbf{A}'}{\partial x} \\
&= -\left(\gamma \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\gamma \frac{q}{r} - \gamma\beta \frac{\partial}{\partial x}\left(-\gamma\beta \frac{q}{r}, 0, 0\right) \\
&= -q\left(\gamma^2(1-\beta^2)\frac{\partial}{\partial x}, \gamma \frac{\partial}{\partial y}, \gamma \frac{\partial}{\partial z}\right)\frac{1}{r} \\
&= -q\left(\frac{\partial}{\partial x}, \gamma \frac{\partial}{\partial y}, \gamma \frac{\partial}{\partial z}\right)\frac{1}{r}
\end{aligned}
\tag{3.225}
$$

Our electric field in the primed frame is thus

$$
\mathbf{E}' = \frac{q}{r^3}\left(x, \gamma y, \gamma z\right)
\tag{3.226}
$$

Now for the magnetic field. We want

$$
\begin{aligned}
\mathbf{B}' &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_{x'} & \partial_{y'} & \partial_{z'} \\ -\gamma\beta q/r & 0 & 0 \end{vmatrix} \\
&= \left(0, \partial_{z'}, -\partial_{y'}\right)\frac{-\gamma\beta q}{r}
\end{aligned}
\tag{3.227}
$$

$$
\mathbf{B}' = \frac{q\gamma\beta}{r^3}\left(0, -z, y\right)
\tag{3.228}
$$

FIXME: sketch and comment.

Notes on grading of my solution I lost two marks for not reducing my solution for the trajectory in eq. (3.215) to $x(t), y(t)$ or $x(y)$ form. That is difficult in the form that I solved this for arbitrary initial conditions (this is easy for $u^i = (1, 0, 0, 0)$ when $\mathbf{B} = 0$). I will be curious to see the Professor's approach later.

FIXME: I had expanded out the trajectory in the way that appears to have been desired on paper for the special case above. Re-do this and include it here (at least as a check of my final result since I switched the orientation of the fields when I typed it up). Also include a similar special case expansion for the case where the invariant $E^2 - B^2$ is negative.

Exercise 3.8 Fun with $\epsilon_{\alpha\beta\gamma}$, ϵ^{ijkl}, F_{ij}, and the duality of Maxwell's equations in vacuum

a. rank 3 spatial antisymmetric tensor identities
Prove that

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\gamma} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} \tag{3.229}$$

and use it to find the familiar relation for

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) \tag{3.230}$$

Also show that

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\beta\gamma} = 2\delta_{\alpha\mu}. \tag{3.231}$$

(Einstein summation implied all throughout this problem).

b. Determinant of three by three matrix
Prove that for any 3×3 matrix $\left\|A_{\alpha\beta}\right\|$: $\epsilon_{\mu\nu\lambda}A_{\alpha\mu}A_{\beta\nu}A_{\gamma\lambda} = \epsilon_{\alpha\beta\gamma} \det A$ and that $\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\lambda}A_{\alpha\mu}A_{\beta\nu}A_{\gamma\lambda} = 6 \det A$.

c. Rotational invariance of 3D antisymmetric tensor
Use the previous results to show that $\epsilon_{\mu\nu\lambda}$ is invariant under rotations.

d. Rotational invariance of 4D antisymmetric tensor
Use the previous results to show that ϵ_{ijkl} is invariant under Lorentz transformations.

e. Sum of contracting symmetric and antisymmetric rank 2 tensors
Show that $A^{ij}B_{ij} = 0$ if A is symmetric and B is antisymmetric.

f. Characteristic equation for the electromagnetic strength tensor
Show that $P(\lambda) = \det \left\|F_{ij} - \lambda g_{ij}\right\|$ is invariant under Lorentz transformations. Consider the polynomial of $P(\lambda)$, also called the characteristic polynomial of the matrix $\left\|F_{ij}\right\|$. Find the coefficients of the expansion of $P(\lambda)$ in powers of λ in terms of the components of $\left\|F_{ij}\right\|$. Use the result to argue that $\mathbf{E} \cdot \mathbf{B}$ and $\mathbf{E}^2 - \mathbf{B}^2$ are Lorentz invariant.

g. Show that the pseudoscalar invariant has only boundary effects
Use integration by parts to show that $\int d^4 x \epsilon^{ijkl} F_{ij} F_{kl}$ only depends on the values of $A^i(x)$ at the "boundary" of spacetime (e.g. the "surface" depicted on page 105 of the notes) and hence does not affect the equations of motion for the electromagnetic field.

h. Electromagnetic duality transformations
Show that the Maxwell equations in vacuum are invariant under the transformation: $F_{ij} \rightarrow \tilde{F}_{ij}$, where $\tilde{F}_{ij} = \frac{1}{2}\epsilon_{ijkl} F^{kl}$ is the dual electromagnetic stress tensor. Replacing F with $\tilde{F}$ is known as "electric-magnetic duality". Explain this name by considering the transformation in terms of **E** and **B**. Are the Maxwell equations with sources invariant under electric-magnetic duality transformations?

Answer for Exercise 3.8

Part a. We can explicitly expand the (implied) sum over indices γ. This is

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\gamma} = \epsilon_{\alpha\beta 1}\epsilon_{\mu\nu 1} + \epsilon_{\alpha\beta 2}\epsilon_{\mu\nu 2} + \epsilon_{\alpha\beta 3}\epsilon_{\mu\nu 3} \tag{3.232}$$

For any $\alpha \neq \beta$ only one term is non-zero. For example with $\alpha, \beta = 2, 3$, we have just a contribution from the $\gamma = 1$ part of the sum

$$\epsilon_{231}\epsilon_{\mu\nu 1}. \tag{3.233}$$

The value of this for $(\mu, \nu) = (\alpha, \beta)$ is

$$(\epsilon_{231})^2 \tag{3.234}$$

whereas for $(\mu, \nu) = (\beta, \alpha)$ we have

$$-(\epsilon_{231})^2 \tag{3.235}$$

Our sum has value one when (α, β) matches (μ, ν), and value minus one for when (μ, ν) are permuted. We can summarize this, by saying that when $\alpha \neq \beta$ we have

$$\boxed{\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\gamma} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}.} \tag{3.236}$$

However, observe that when $\alpha = \beta$ the RHS is

$$\delta_{\alpha\mu}\delta_{\alpha\nu} - \delta_{\alpha\nu}\delta_{\alpha\mu} = 0, \tag{3.237}$$

as desired, so this form works in general without any $\alpha \neq \beta$ qualifier, completing this part of the problem.

$$
\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\epsilon_{\alpha\beta\gamma}\mathbf{e}^\alpha A^\beta B^\gamma) \cdot (\epsilon_{\mu\nu\sigma}\mathbf{e}^\mu C^\nu D^\sigma) \\
&= \epsilon_{\alpha\beta\gamma} A^\beta B^\gamma \epsilon_{\alpha\nu\sigma} C^\nu D^\sigma \\
&= (\delta_{\beta\nu}\delta_{\gamma\sigma} - \delta_{\beta\sigma}\delta_{\gamma\nu}) A^\beta B^\gamma C^\nu D^\sigma \\
&= A^\nu B^\sigma C^\nu D^\sigma - A^\sigma B^\nu C^\nu D^\sigma.
\end{aligned}
\tag{3.238}
$$

This gives us

$$
\boxed{(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).}
\tag{3.239}
$$

We have one more identity to deal with.

$$
\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\beta\gamma}
\tag{3.240}
$$

We can expand out this (implied) sum slow and dumb as well

$$
\begin{aligned}
\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\beta\gamma} &= \epsilon_{\alpha 12}\epsilon_{\mu 12} + \epsilon_{\alpha 21}\epsilon_{\mu 21} \\
&+ \epsilon_{\alpha 13}\epsilon_{\mu 13} + \epsilon_{\alpha 31}\epsilon_{\mu 31} \\
&+ \epsilon_{\alpha 23}\epsilon_{\mu 23} + \epsilon_{\alpha 32}\epsilon_{\mu 32} \\
&= 2\epsilon_{\alpha 12}\epsilon_{\mu 12} + 2\epsilon_{\alpha 13}\epsilon_{\mu 13} + 2\epsilon_{\alpha 23}\epsilon_{\mu 23}
\end{aligned}
\tag{3.241}
$$

Now, observe that for any $\alpha \in (1, 2, 3)$ only one term of this sum is picked up. For example, with no loss of generality, pick $\alpha = 1$. We are left with only

$$
2\epsilon_{123}\epsilon_{\mu 23}
\tag{3.242}
$$

This has the value

$$
2(\epsilon_{123})^2 = 2
\tag{3.243}
$$

when $\mu = \alpha$ and is zero otherwise. We can therefore summarize the evaluation of this sum as

$$
\boxed{\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\beta\gamma} = 2\delta_{\alpha\mu},}
\tag{3.244}
$$

completing this problem.

Part b. In class Simon showed us how the first identity can be arrived at using the triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det(\mathbf{abc})$. It occurred to me later that I had seen the identity to be proven in the context of Geometric Algebra, but hhad not recognized it in this tensor form. Basically, a wedge product can be expanded in sums of determinants, and when the dimension of the space is the same as the vector, we have a pseudoscalar times the determinant of the components.

For example, in $\mathbb{R}^2$, let us take the wedge product of a pair of vectors. As preparation for the relativistic $\mathbb{R}^4$ case We will not require an orthonormal basis, but express the vector in terms of a reciprocal frame and the associated components

$$a = a^i e_i = a_j e^j \tag{3.245}$$

where

$$e^i \cdot e_j = {\delta^i}_j. \tag{3.246}$$

When we get to the relativistic case, we can pick (but do not have to) the standard basis

$$\begin{aligned} e_0 &= (1, 0, 0, 0) \\ e_1 &= (0, 1, 0, 0) \\ e_2 &= (0, 0, 1, 0) \\ e_3 &= (0, 0, 0, 1), \end{aligned} \tag{3.247}$$

for which our reciprocal frame is implicitly defined by the metric

$$\begin{aligned} e^0 &= (1, 0, 0, 0) \\ e^1 &= (0, -1, 0, 0) \\ e^2 &= (0, 0, -1, 0) \\ e^3 &= (0, 0, 0, -1). \end{aligned} \tag{3.248}$$

Anyways. Back to the problem. Let us examine the $\mathbb{R}^2$ case. Our wedge product in coordinates is

$$a \wedge b = a^i b^j (e_i \wedge e_j) \tag{3.249}$$

Since there are only two basis vectors we have

$$a \wedge b = (a^1 b^2 - a^2 b^1) e_1 \wedge e_2 = \det \left\| a^i b^j \right\| (e_1 \wedge e_2). \tag{3.250}$$

Our wedge product is a product of the determinant of the vector coordinates, times the $\mathbb{R}^2$ pseudoscalar $e_1 \wedge e_2$.

This does not look quite like the $\mathbb{R}^3$ relation that we want to prove, which had an antisymmetric tensor factor for the determinant. Observe that we get the determinant by picking off the $e_1 \wedge e_2$ component of the bivector result (the only component in this case), and we can do that by dotting with $e^2 \cdot e^1$. To get an antisymmetric tensor times the determinant, we have only to dot with a different pseudoscalar (one that differs by a possible sign due to permutation of the indices). That is

$$\begin{aligned}
(e^t \wedge e^s) \cdot (a \wedge b) &= a^i b^j (e^t \wedge e^s) \cdot (e_i \wedge e_j) \\
&= a^i b^j \left({\delta^s}_i {\delta^t}_j - {\delta^t}_i {\delta^s}_j \right) \\
&= a^i b^j {\delta^{[t}}_j {\delta^{s]}}_i \\
&= a^i b^j {\delta^t}_{[j} {\delta^s}_{i]} \\
&= a^{[i} b^{j]} {\delta^t}_j {\delta^s}_i \\
&= a^{[s} b^{t]}
\end{aligned} \tag{3.251}$$

Now, if we write $a^i = A^{1i}$ and $b^j = A^{2j}$ we have

$$(e^t \wedge e^s) \cdot (a \wedge b) = A^{1s} A^{2t} - A^{1t} A^{2s} \tag{3.252}$$

We can write this in two different ways. One of which is

$$A^{1s} A^{2t} - A^{1t} A^{2s} = \epsilon^{st} \det \left\| A^{ij} \right\| \tag{3.253}$$

and the other of which is by introducing free indices for 1 and 2, and summing antisymmetrically over these. That is

$$A^{1s} A^{2t} - A^{1t} A^{2s} = A^{as} A^{bt} \epsilon_{ab} \tag{3.254}$$

So, we have

$$\boxed{A^{as} A^{bt} \epsilon_{ab} = A^{1i} A^{2j} {\delta^{[t}}_j {\delta^{s]}}_i = \epsilon^{st} \det \left\| A^{ij} \right\|,} \tag{3.255}$$

This result hold regardless of the metric for the space, and does not require that we were using an orthonormal basis. When the metric is Euclidean and we have an orthonormal basis, then all the indices can be dropped.

The $\mathbb{R}^3$ and $\mathbb{R}^4$ cases follow in exactly the same way, we just need more vectors in the wedge products.

For the $\mathbb{R}^3$ case we have

$$\begin{aligned}(e^u \wedge e^t \wedge e^s) \cdot (a \wedge b \wedge c) &= a^i b^j c^k (e^u \wedge e^t \wedge e^s) \cdot (e_i \wedge e_j \wedge e_k) \\ &= a^i b^j c^k \delta^{[u}{}_k \delta^t{}_j \delta^{s]}{}_i \\ &= a^{[s} b^t c^{u]}\end{aligned} \tag{3.256}$$

Again, with $a^i = A^{1i}$ and $b^j = A^{2j}$, and $c^k = A^{3k}$ we have

$$(e^u \wedge e^t \wedge e^s) \cdot (a \wedge b \wedge c) = A^{1i} A^{2j} A^{3k} \delta^{[u}{}_k \delta^t{}_j \delta^{s]}{}_i \tag{3.257}$$

and we can choose to write this in either form, resulting in the identity

$$\boxed{\epsilon^{stu} \det \left\| A^{ij} \right\| = A^{1i} A^{2j} A^{3k} \delta^{[u}{}_k \delta^t{}_j \delta^{s]}{}_i = \epsilon_{abc} A^{as} A^{bt} A^{cu}.} \tag{3.258}$$

The $\mathbb{R}^4$ case follows exactly the same way, and we have

$$\begin{aligned}(e^v \wedge e^u \wedge e^t \wedge e^s) \cdot (a \wedge b \wedge c \wedge d) &= a^i b^j c^k d^l (e^v \wedge e^u \wedge e^t \wedge e^s) \cdot (e_i \wedge e_j \wedge e_k \wedge e_l) \\ &= a^i b^j c^k d^l \delta^{[v}{}_l \delta^u{}_k \delta^t{}_j \delta^{s]}{}_i \\ &= a^{[s} b^t c^u d^{v]}.\end{aligned} \tag{3.259}$$

This time with $a^i = A^{0i}$ and $b^j = A^{1j}$, and $c^k = A^{2k}$, and $d^l = A^{3l}$ we have

$$\boxed{\epsilon^{stuv} \det \left\| A^{ij} \right\| = A^{0i} A^{1j} A^{2k} A^{3l} \delta^{[v}{}_l \delta^u{}_k \delta^t{}_j \delta^{s]}{}_i = \epsilon_{abcd} A^{as} A^{bt} A^{cu} A^{dv}.} \tag{3.260}$$

This one is almost the identity to be established later in problem 1.4. We have only to raise and lower some indices to get that one. Note that in the Minkowski standard basis above, because s, t, u, v must be a permutation of $0, 1, 2, 3$ for a non-zero result, we must have

$$\epsilon^{stuv} = (-1)^3 (+1) \epsilon_{stuv}. \tag{3.261}$$

So raising and lowering the identity above gives us

$$-\epsilon_{stuv} \det \left\| A_{ij} \right\| = \epsilon^{abcd} A_{as} A_{bt} A_{cu} A_{du}. \tag{3.262}$$

No sign changes were required for the indices a, b, c, d, since they are paired.

Until we did the raising and lowering operations here, there was no specific metric required, so our first result eq. (3.260) is the more general one.

There is one more part to this problem, doing the antisymmetric sums over the indices $s, t, \cdots$. For the $\mathbb{R}^2$ case we have

$$\begin{aligned}\epsilon_{st}\epsilon_{ab}A^{as}A^{bt} &= \epsilon_{st}\epsilon^{st}\det\left\|A^{ij}\right\| \\ &= \left(\epsilon_{12}\epsilon^{12} + \epsilon_{21}\epsilon^{21}\right)\det\left\|A^{ij}\right\| \\ &= \left(1^2 + (-1)^2\right)\det\left\|A^{ij}\right\|\end{aligned} \tag{3.263}$$

We conclude that

$$\boxed{\epsilon_{st}\epsilon_{ab}A^{as}A^{bt} = 2!\det\left\|A^{ij}\right\|.} \tag{3.264}$$

For the $\mathbb{R}^3$ case we have the same operation

$$\begin{aligned}\epsilon_{stu}\epsilon_{abc}A^{as}A^{bt}A^{cu} &= \epsilon_{stu}\epsilon^{stu}\det\left\|A^{ij}\right\| \\ &= \left(\epsilon_{123}\epsilon^{123} + \epsilon_{132}\epsilon^{132} + \cdots\right)\det\left\|A^{ij}\right\| \\ &= (\pm 1)^2(3!\,)\det\left\|A^{ij}\right\|.\end{aligned} \tag{3.265}$$

So we conclude

$$\boxed{\epsilon_{stu}\epsilon_{abc}A^{as}A^{bt}A^{cu} = 3!\det\left\|A^{ij}\right\|.} \tag{3.266}$$

It is clear what the pattern is, and if we evaluate the sum of the antisymmetric tensor squares in $\mathbb{R}^4$ we have

$$\begin{aligned}\epsilon_{stuv}\epsilon_{stuv} &= \epsilon_{0123}\epsilon_{0123} + \epsilon_{0132}\epsilon_{0132} + \epsilon_{0213}\epsilon_{0213} + \cdots \\ &= (\pm 1)^2(4!\,),\end{aligned} \tag{3.267}$$

So, for our SR case we have

$$\boxed{\epsilon_{stuv}\epsilon_{abcd}A^{as}A^{bt}A^{cu}A^{dv} = 4!\det\left\|A^{ij}\right\|.} \tag{3.268}$$

This was part of question 1.4, albeit in lower index form. Here since all indices are matched, we have the same result without major change

$$\boxed{\epsilon^{stuv}\epsilon^{abcd}A_{as}A_{bt}A_{cu}A_{dv} = 4!\det\left\|A_{ij}\right\|.} \tag{3.269}$$

The main difference is that we are now taking the determinant of a lower index tensor.

Part c. We apply transformations to coordinates (and thus indices) of the form

$$x_\mu \to O_{\mu\nu} x_\nu \tag{3.270}$$

With our tensor transforming as its indices, we have

$$\epsilon_{\mu\nu\lambda} \to \epsilon_{\alpha\beta\sigma} O_{\mu\alpha} O_{\nu\beta} O_{\lambda\sigma}. \tag{3.271}$$

We have got eq. (3.258), which after dropping indices, because we are in a Euclidean space, we have

$$\epsilon_{\mu\nu\lambda} \det \left\| A_{ij} \right\| = \epsilon_{\alpha\beta\sigma} A_{\alpha\mu} A_{\beta\nu} A_{\sigma\lambda}. \tag{3.272}$$

Let $A_{ij} = O_{ji}$, which gives us

$$\epsilon_{\mu\nu\lambda} \to \epsilon_{\mu\nu\lambda} \det A^{\mathrm{T}} \tag{3.273}$$

but since $\det O = \det O^{\mathrm{T}}$, we have shown that $\epsilon_{\mu\nu\lambda}$ is invariant under rotation.

Part d. This follows the same way. We assume a transformation of coordinates of the following form

$$\begin{aligned} (x')^i &= O^i{}_j x^j \\ (x')_i &= O_i{}^j x_j, \end{aligned} \tag{3.274}$$

where the determinant of $O^i{}_j = 1$ (sanity check of sign: $O^i{}_j = \delta^i{}_j$).
Our antisymmetric tensor transforms as its coordinates individually

$$\begin{aligned} \epsilon_{ijkl} &\to \epsilon_{abcd} O_i{}^a O_j{}^b O_k{}^c O_l{}^d \\ &= \epsilon^{abcd} O_{ia} O_{jb} O_{kc} O_{ld} \end{aligned} \tag{3.275}$$

Let $P_{ij} = O_{ji}$, and raise and lower all the indices in eq. (3.276) for

$$-\epsilon_{stuv} \det \left\| P_{ij} \right\| = \epsilon^{abcd} P_{as} P_{bt} P_{cu} P_{dv}. \tag{3.276}$$

We have

$$\begin{aligned}
\epsilon_{ijkl} &= \epsilon^{abcd} P_{ai} P_{aj} P_{ak} P_{al} \\
&= -\epsilon_{ijkl} \det \left\| P_{ij} \right\| \\
&= -\epsilon_{ijkl} \det \left\| O_{ij} \right\| \\
&= -\epsilon_{ijkl} \det \left\| g_{im} O^m{}_j \right\| \\
&= -\epsilon_{ijkl} (-1)(1) \\
&= \epsilon_{ijkl}
\end{aligned} \tag{3.277}$$

Since $\epsilon_{ijkl} = -\epsilon^{ijkl}$ both are therefore invariant under Lorentz transformation.

Part e. We swap indices in B, switch dummy indices, then swap indices in A

$$\begin{aligned}
A^{ij} B_{ij} &= -A^{ij} B_{ji} \\
&= -A^{ji} B_{ij} \\
&= -A^{ij} B_{ij}
\end{aligned} \tag{3.278}$$

Our result is the negative of itself, so must be zero.

Part f.

The invariance of the determinant Let us consider how any lower index rank 2 tensor transforms. Given a transformation of coordinates

$$\begin{aligned}
(x^i)' &= O^i{}_j x^j \\
(x_i)' &= O_i{}^j x^j,
\end{aligned} \tag{3.279}$$

where $\det \left\| O^i{}_j \right\| = 1$, and $O_i{}^j = O^m{}_n g_{im} g^{jn}$. Let us reflect briefly on why this determinant is unit valued. We have

$$(x^i)'(x_i)' = O_i{}^a x^a O^i{}_b x^b = x^b x_b, \tag{3.280}$$

which implies that the transformation product is

$$O_i{}^a O^i{}_b = \delta^a{}_b, \tag{3.281}$$

the identity matrix. The identity matrix has unit determinant, so we must have

$$1 = (\det \hat{G})^2 (\det \left\| O^i{}_j \right\|)^2. \tag{3.282}$$

Since $\det \hat{G} = -1$ we have

$$\det \left\| O^i{}_j \right\| = \pm 1, \tag{3.283}$$

which is all that we can say about the determinant of this class of transformations by considering just invariance. If we restrict the transformations of coordinates to those of the same determinant sign as the identity matrix, we rule out reflections in time or space. This seems to be the essence of the $SO(1,3)$ labeling.

Why dwell on this? Well, I wanted to be clear on the conventions I had chosen, since parts of the course notes used $\hat{O} = \left\| O^{ij} \right\|$, and $X' = \hat{O}X$, and gave that matrix unit determinant. That O^{ij} looks like it is equivalent to my $O^i{}_j$, except that the one in the course notes is loose when it comes to lower and upper indices since it gives $(x')^i = O^{ij} x^j$.

I will write

$$\hat{O} = \left\| O^i{}_j \right\|, \tag{3.284}$$

and require this (not $\left\| O^{ij} \right\|$) to be the matrix with unit determinant. Having cleared the index upper and lower confusion I had trying to reconcile the class notes with the rules for index manipulation, let us now consider the Lorentz transformation of a lower index rank 2 tensor (not necessarily antisymmetric or symmetric)

We have, transforming in the same fashion as a lower index coordinate four vector (but twice, once for each index)

$$A_{ij} \rightarrow A_{km} O_i{}^k O_j{}^m. \tag{3.285}$$

The determinant of the transformation tensor $O_i{}^j$ is

$$\det \left\| O_i{}^j \right\| = \det \left\| g^{im} O^m{}_n g^{nj} \right\| = (\det \hat{G})(1)(\det \hat{G}) = (-1)^2(1) = 1. \tag{3.286}$$

We see that the determinant of a lower index rank 2 tensor is invariant under Lorentz transformation. This would include our characteristic polynomial $P(\lambda)$.

Expanding the determinant Utilizing eq. (3.269) we can now calculate the characteristic polynomial. This is

$$\begin{aligned} \det \left\| F_{ij} - \lambda g_{ij} \right\| &= \frac{1}{4!} \epsilon^{stuv} \epsilon^{abcd} (F_{as} - \lambda g_{as})(F_{bt} - \lambda g_{bt})(F_{cu} - \lambda g_{cu})(F_{dv} - \lambda g_{dv}) \\ &= \frac{1}{24} \epsilon^{stuv} \epsilon_{abcd} (F^a{}_s - \lambda g^a{}_s)(F^b{}_t - \lambda g^b{}_t)(F^c{}_u - \lambda g^c{}_u)(F^d{}_v - \lambda g^d{}_v) \end{aligned} \tag{3.287}$$

However, $g^a{}_b = g_{bc} g^{ac}$, or $\left\| g^a{}_b \right\| = \hat{G}^2 = I$. This means we have

$$g^a{}_b = \delta^a{}_b, \tag{3.288}$$

and our determinant is reduced to

$$\begin{aligned} P(\lambda) = \frac{1}{24} \epsilon^{stuv} \epsilon_{abcd} &\left(F^a{}_s F^b{}_t - \lambda (\delta^a{}_s F^b{}_t + \delta^b{}_t F^a{}_s) + \lambda^2 \delta^a{}_s \delta^b{}_t \right) \\ \times \qquad &\left(F^c{}_u F^d{}_v - \lambda (\delta^c{}_u F^d{}_v + \delta^d{}_v F^c{}_u) + \lambda^2 \delta^c{}_u \delta^d{}_v \right) \end{aligned} \tag{3.289}$$

If we expand this out we have our powers of λ coefficients are

$$\begin{aligned} \lambda^0 &: \frac{1}{24} \epsilon^{stuv} \epsilon_{abcd} F^a{}_s F^b{}_t F^c{}_u F^d{}_v \\ \lambda^1 &: \frac{1}{24} \epsilon^{stuv} \epsilon_{abcd} \left(-(\delta^c{}_u F^d{}_v + \delta^d{}_v F^c{}_u) F^a{}_s F^b{}_t - (\delta^a{}_s F^b{}_t + \delta^b{}_t F^a{}_s) F^c{}_u F^d{}_v \right) \\ \lambda^2 &: \frac{1}{24} \epsilon^{stuv} \epsilon_{abcd} \left(\delta^c{}_u \delta^d{}_v F^a{}_s F^b{}_t + (\delta^a{}_s F^b{}_t + \delta^b{}_t F^a{}_s)(\delta^c{}_u F^d{}_v + \delta^d{}_v F^c{}_u) + \delta^a{}_s \delta^b{}_t F^c{}_u F^d{}_v \right) \\ \lambda^3 &: \frac{1}{24} \epsilon^{stuv} \epsilon_{abcd} \left(-(\delta^a{}_s F^b{}_t + \delta^b{}_t F^a{}_s) \delta^c{}_u \delta^d{}_v - \delta^a{}_s \delta^b{}_t (\delta^c{}_u F^d{}_v + \delta^d{}_v F^c{}_u) \right) \\ \lambda^4 &: \frac{1}{24} \epsilon^{stuv} \epsilon_{abcd} \left(\delta^a{}_s \delta^b{}_t \delta^c{}_u \delta^d{}_v \right) \end{aligned} \tag{3.290}$$

By eq. (3.269) the λ^0 coefficient is just $\det \left\| F_{ij} \right\|$.
The λ^3 terms can be seen to be zero. For example, the first one is

$$\begin{aligned} -\frac{1}{24} \epsilon^{stuv} \epsilon_{abcd} \delta^a{}_s F^b{}_t \delta^c{}_u \delta^d{}_v &= -\frac{1}{24} \epsilon^{stuv} \epsilon_{sbuv} F^b{}_t \\ &= -\frac{1}{12} \delta^t_b F^b{}_t \\ &= -\frac{1}{12} F^b{}_b \\ &= -\frac{1}{12} F^{bu} g_{ub} \\ &= 0, \end{aligned} \tag{3.291}$$

where the final equality to zero comes from summing a symmetric and antisymmetric product.

Similarly the λ coefficients can be shown to be zero. Again the first as a sample is

$$\begin{aligned}
-\frac{1}{24}\epsilon^{stuv}\epsilon_{abcd}\delta^c{}_u F^d{}_v F^a{}_s F^b{}_t &= -\frac{1}{24}\epsilon^{ustv}\epsilon_{uabd} F^d{}_v F^a{}_s F^b{}_t \\
&= -\frac{1}{24}\delta_a^{[s}\delta_b^t\delta_d^{v]} F^d{}_v F^a{}_s F^b{}_t \\
&= -\frac{1}{24} F^a{}_{[s} F^b{}_t F^d{}_{v]}
\end{aligned} \tag{3.292}$$

Disregarding the $-1/24$ factor, let us just expand this antisymmetric sum

$$\begin{aligned}
F^a{}_{[a} F^b{}_b F^d{}_{d]} &= F^a{}_a F^b{}_b F^d{}_d + F^a{}_d F^b{}_a F^d{}_b + F^a{}_b F^b{}_d F^d{}_a \\
&\quad - F^a{}_a F^b{}_d F^d{}_b - F^a{}_d F^b{}_b F^d{}_a - F^a{}_b F^b{}_a F^d{}_d \\
&= F^a{}_d F^b{}_a F^d{}_b + F^a{}_b F^b{}_d F^d{}_a
\end{aligned} \tag{3.293}$$

Of the two terms above that were retained, they are the only ones without a zero $F^i{}_i$ factor. Consider the first part of this remaining part of the sum. Employing the metric tensor, to raise indices so that the antisymmetry of F^{ij} can be utilized, and then finally relabeling all the dummy indices we have

$$\begin{aligned}
F^a{}_d F^b{}_a F^d{}_b &= F^{au} F^{bv} F^{dw} g_{du} g_{av} g_{bw} \\
&= (-1)^3 F^{ua} F^{vb} F^{wd} g_{du} g_{av} g_{bw} \\
&= -(F^{ua} g_{av})(F^{vb} g_{bw})(F^{wd} g_{du}) \\
&= -F^u{}_v F^v{}_w F^w{}_u \\
&= -F^a{}_b F^b{}_d F^d{}_a
\end{aligned} \tag{3.294}$$

This is just the negative of the second term in the sum, leaving us with zero.

Finally, we have for the λ^2 coefficient ($\times 24$)

$$\begin{aligned}
&\epsilon^{stuv}\epsilon_{abcd}\big(\delta^c{}_u\delta^d{}_v F^a{}_s F^b{}_t + \delta^a{}_s F^b{}_t\delta^c{}_u F^d{}_v + \delta^b{}_t F^a{}_s\delta^d{}_v F^c{}_u \\
&\qquad + \delta^b{}_t F^a{}_s\delta^c{}_u F^d{}_v + \delta^a{}_s F^b{}_t\delta^d{}_v F^c{}_u + \delta^a{}_s\delta^b{}_t F^c{}_u F^d{}_v\big) \\
&= \epsilon^{stuv}\epsilon_{abuv}F^a{}_s F^b{}_t + \epsilon^{stuv}\epsilon_{sbud}F^b{}_t F^d{}_v + \epsilon^{stuv}\epsilon_{atcv}F^a{}_s F^c{}_u \\
&\qquad + \epsilon^{stuv}\epsilon_{atud}F^a{}_s F^d{}_v + \epsilon^{stuv}\epsilon_{sbcv}F^b{}_t F^c{}_u + \epsilon^{stuv}\epsilon_{stcd}F^c{}_u F^d{}_v \\
&= \epsilon^{stuv}\epsilon_{abuv}F^a{}_s F^b{}_t + \epsilon^{tvsu}\epsilon_{bdsu}F^b{}_t F^d{}_v + \epsilon^{sutv}\epsilon_{actv}F^a{}_s F^c{}_u \\
&\qquad + \epsilon^{svtu}\epsilon_{adtu}F^a{}_s F^d{}_v + \epsilon^{tusv}\epsilon_{bcsv}F^b{}_t F^c{}_u + \epsilon^{uvst}\epsilon_{cdst}F^c{}_u F^d{}_v \\
&= 6\epsilon^{stuv}\epsilon_{abuv}F^a{}_s F^b{}_t \\
&= 6(2)\delta^{[s}{}_a\delta^{t]}{}_b F^a{}_s F^b{}_t \\
&= 12F^a{}_{[a}F^b{}_{b]} \\
&= 12(F^a{}_a F^b{}_b - F^a{}_b F^b{}_a) \\
&= -12F^a{}_b F^b{}_a \\
&= -12F^{ab}F_{ba} \\
&= 12F^{ab}F_{ab}
\end{aligned} \tag{3.295}$$

Therefore, our characteristic polynomial is

$$\boxed{P(\lambda) = \det\left\|F_{ij}\right\| + \frac{\lambda^2}{2}F^{ab}F_{ab} + \lambda^4.} \tag{3.296}$$

Observe that in matrix form our strength tensors are

$$\begin{aligned}
\left\|F^{ij}\right\| &= \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \\
\left\|F_{ij}\right\| &= \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}.
\end{aligned} \tag{3.297}$$

From these we can compute $F^{ab}F_{ab}$ easily by inspection

$$F^{ab}F_{ab} = 2(\mathbf{B}^2 - \mathbf{E}^2). \tag{3.298}$$

Computing the determinant is not so easy. The dumb and simple way of expanding by cofactors takes two pages, and yields eventually

$$\det \left\| F^{ij} \right\| = (\mathbf{E} \cdot \mathbf{B})^2. \tag{3.299}$$

That supplies us with a relation for the characteristic polynomial in **E** and **B**

$$\boxed{P(\lambda) = (\mathbf{E} \cdot \mathbf{B})^2 + \lambda^2(\mathbf{B}^2 - \mathbf{E}^2) + \lambda^4.} \tag{3.300}$$

Observe that we found this for the special case where **E** and **B** were perpendicular in homework 2. Observe that when we have that perpendicularity, we can solve for the eigenvalues by inspection

$$\lambda \in \{0, 0, \pm\sqrt{\mathbf{E}^2 - \mathbf{B}^2}\}, \tag{3.301}$$

and were able to diagonalize the matrix ${F^i}_j$ to solve the Lorentz force equation in parametric form. When $|\mathbf{E}| > |\mathbf{B}|$ we had real eigenvalues and an orthogonal diagonalization when $\mathbf{B} = 0$. For the $|\mathbf{B}| > |\mathbf{E}|$, we had a two purely imaginary eigenvalues, and when $\mathbf{E} = 0$ this was a Hermitian diagonalization. For the general case, when one of **E**, or **B** was zero, things did not have the same nice closed form solution.

In general our eigenvalues are

$$\lambda = \pm\frac{1}{\sqrt{2}}\sqrt{\mathbf{E}^2 - \mathbf{B}^2 \pm \sqrt{(\mathbf{E}^2 - \mathbf{B}^2)^2 - 4(\mathbf{E} \cdot \mathbf{B})^2}}. \tag{3.302}$$

For the purposes of this problem we really only wish to show that $\mathbf{E} \cdot \mathbf{B}$ and $\mathbf{E}^2 - \mathbf{B}^2$ are Lorentz invariants. When $\lambda = 0$ we have $P(\lambda) = (\mathbf{E} \cdot \mathbf{B})^2$, a Lorentz invariant. This must mean that $\mathbf{E} \cdot \mathbf{B}$ is itself a Lorentz invariant. Since that is invariant, and we require $P(\lambda)$ to be invariant for any other possible values of λ, the difference $\mathbf{E}^2 - \mathbf{B}^2$ must also be Lorentz invariant.

Part g. This proceeds in a fairly straightforward fashion

$$\begin{aligned}
\int d^4 x \epsilon^{ijkl} F_{ij} F_{kl} &= \int d^4 x \epsilon^{ijkl} (\partial_i A_j - \partial_j A_i) F_{kl} \\
&= \int d^4 x \epsilon^{ijkl} (\partial_i A_j) F_{kl} - \epsilon^{jikl} (\partial_i A_j) F_{kl} \\
&= 2 \int d^4 x \epsilon^{ijkl} (\partial_i A_j) F_{kl} \\
&= 2 \int d^4 x \epsilon^{ijkl} \left(\frac{\partial}{\partial x^i} (A_j F_{kl} - A_j \frac{\partial F_{kl}}{\partial x^i} \right)
\end{aligned} \tag{3.303}$$

Now, observe that by the Bianchi identity, this second term is zero

$$\epsilon^{ijkl} \frac{\partial F_{kl}}{\partial x^i} = -\epsilon^{jikl} \partial_i F_{kl} = 0 \tag{3.304}$$

Now we have a set of perfect differentials, and can integrate

$$\begin{aligned} \int d^4 x \epsilon^{ijkl} F_{ij} F_{kl} &= 2 \int d^4 x \epsilon^{ijkl} \frac{\partial}{\partial x^i} (A_j F_{kl}) \\ &= 2 \int dx^j dx^k dx^l \epsilon^{ijkl} (A_j F_{kl}) \big|_{\Delta x^i} \end{aligned} \tag{3.305}$$

We are left with a only contributions to the integral from the boundary terms on the spacetime hypervolume, three-volume normals bounding the four-volume integration in the original integral.

Part h. Let us first consider the explanation of the name. First recall what the expansions are of F_{ij} and F^{ij} in terms of **E** and **E**. These are

$$\begin{aligned} F_{0\alpha} &= \partial_0 A_\alpha - \partial_\alpha A_0 \\ &= -\frac{1}{c} \frac{\partial A^\alpha}{\partial t} - \frac{\partial \phi}{\partial x^\alpha} \\ &= E_\alpha \end{aligned} \tag{3.306}$$

with $F^{0\alpha} = -E^\alpha$, and $E^\alpha = E_\alpha$.
The magnetic field components are

$$\begin{aligned} F_{\beta\alpha} &= \partial_\beta A_\alpha - \partial_\alpha A_\beta \\ &= -\partial_\beta A^\alpha + \partial_\alpha A^\beta \\ &= \epsilon_{\alpha\beta\sigma} B^\sigma \end{aligned} \tag{3.307}$$

with $F^{\beta\alpha} = \epsilon^{\alpha\beta\sigma} B_\sigma$ and $B_\sigma = B^\sigma$.

Now let us expand the dual tensors. These are

$$
\begin{aligned}
\tilde{F}_{0\alpha} &= \frac{1}{2}\epsilon_{0\alpha ij}F^{ij} \\
&= \frac{1}{2}\epsilon_{0\alpha\beta\sigma}F^{\beta\sigma} \\
&= \frac{1}{2}\epsilon_{0\alpha\beta\sigma}\epsilon^{\sigma\beta\mu}B_\mu \\
&= -\frac{1}{2}\epsilon_{0\alpha\beta\sigma}\epsilon^{\mu\beta\sigma}B_\mu \\
&= -\frac{1}{2}(2!)\delta_\alpha{}^\mu B_\mu \\
&= -B_\alpha
\end{aligned}
\tag{3.308}
$$

and

$$
\begin{aligned}
\tilde{F}_{\beta\alpha} &= \frac{1}{2}\epsilon_{\beta\alpha ij}F^{ij} \\
&= \frac{1}{2}\left(\epsilon_{\beta\alpha 0\sigma}F^{0\sigma} + \epsilon_{\beta\alpha\sigma 0}F^{\sigma 0}\right) \\
&= \epsilon_{0\beta\alpha\sigma}(-E^\sigma) \\
&= \epsilon_{\alpha\beta\sigma}E^\sigma
\end{aligned}
\tag{3.309}
$$

Summarizing we have

$$
\begin{aligned}
F_{0\alpha} &= E^\alpha \\
F^{0\alpha} &= -E^\alpha \\
F^{\beta\alpha} &= F_{\beta\alpha} = \epsilon_{\alpha\beta\sigma}B^\sigma \\
\tilde{F}_{0\alpha} &= -B_\alpha \\
\tilde{F}^{0\alpha} &= B_\alpha \\
\tilde{F}_{\beta\alpha} &= \tilde{F}^{\beta\alpha} = \epsilon_{\alpha\beta\sigma}E^\sigma
\end{aligned}
\tag{3.310}
$$

Is there a sign error in the $\tilde{F}_{0\alpha} = -B_\alpha$ result? Other than that we have the same sort of structure for the tensor with E and B switched around.

Let us write these in matrix form, to compare

$$\begin{aligned}
\left\|\tilde{F}_{ij}\right\| &= \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & E_x \\ B_z & -E_y & -E_x & 0 \end{bmatrix} \quad \left\|\tilde{F}^{ij}\right\| = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{bmatrix} \\
\left\|F^{ij}\right\| &= \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad \left\|F_{ij}\right\| = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}.
\end{aligned} \tag{3.311}$$

From these we can see by inspection that we have

$$\tilde{F}^{ij}F_{ij} = \tilde{F}_{ij}F^{ij} = 4(\mathbf{E}\cdot\mathbf{B}) \tag{3.312}$$

This is consistent with the stated result in [18] (except for a factor of c due to units differences), so it appears the signs above are all kosher.

Now, let us see if the if the dual tensor satisfies the vacuum equations.

$$\begin{aligned}
\partial_j \tilde{F}^{ij} &= \partial_j \frac{1}{2}\epsilon^{ijkl}F_{kl} \\
&= \frac{1}{2}\epsilon^{ijkl}\partial_j(\partial_k A_l - \partial_l A_k) \\
&= \frac{1}{2}\epsilon^{ijkl}\partial_j\partial_k A_l - \frac{1}{2}\epsilon^{ijlk}\partial_k A_l \\
&= \frac{1}{2}(\epsilon^{ijkl} - \epsilon^{ijkl}\partial_k A_l \\
&= 0. \qquad \square
\end{aligned} \tag{3.313}$$

So the first checks out, provided we have no sources. If we have sources, then we see here that Maxwell's equations do not hold since this would imply that the four current density must be zero.

How about the Bianchi identity? That gives us

$$\begin{aligned}
\epsilon^{ijkl}\partial_j\tilde{F}_{kl} &= \epsilon^{ijkl}\partial_j\frac{1}{2}\epsilon_{klab}F^{ab} \\
&= \frac{1}{2}\epsilon^{klij}\epsilon_{klab}\partial_j F^{ab} \\
&= \frac{1}{2}(2!)\delta^i{}_{[a}\delta^j{}_{b]}\partial_j F^{ab} \\
&= \partial_j(F^{ij} - F^{ji}) \\
&= 2\partial_j F^{ij}.
\end{aligned} \tag{3.314}$$

The factor of two is slightly curious. Is there a mistake above? If there is a mistake, it does not change the fact that Maxwell's equation

$$\partial_k F^{ki} = \frac{4\pi}{c} j^i \tag{3.315}$$

Gives us zero for the Bianchi identity under source free conditions of $j^i = 0$.

Exercise 3.9 Transformation properties of E, B again

a. Use the form of F^{ij} from page 82 in the class notes, the transformation law for $\left\|F^{ij}\right\|$ given further down that same page, and the explicit form of the $SO(1,3)$ matrix $\hat{O}$ (say, corresponding to motion in the positive x_1 direction with speed v) to derive the transformation law of the fields **E** and **B**. Use the transformation law to find the electromagnetic field of a charged particle moving with constant speed v in the positive x_1 direction and check that the result agrees with the one that you obtained in Homework 2.

b. A particle is moving with velocity **v** in perpendicular **E** and **B** fields, all given in some particular "stationary" frame of reference.

   1. Show that there exists a frame where the problem of finding the particle trajectory can be reduced to having either only an electric or only a magnetic field.
   2. Explain what determines which case takes place.
   3. Find the velocity $\mathbf{v}_0$ of that frame relative to the "stationary" frame.

Answer for Exercise 3.9

Part a. Given a transformation of coordinates

$$x'^i \to O^i{}_j x^j \tag{3.316}$$

our rank 2 tensor F^{ij} transforms as

$$F^{ij} \rightarrow O^i{}_a F^{ab} O^j{}_b. \tag{3.317}$$

Introducing matrices

$$\begin{aligned}\hat{O} &= \left\| O^i{}_j \right\| \\ \hat{F} &= \left\| F^{ij} \right\| = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}\end{aligned} \tag{3.318}$$

and noting that $\hat{O}^{\mathrm{T}} = \left\| O^j{}_i \right\|$, we can express the electromagnetic strength tensor transformation as

$$\hat{F} \rightarrow \hat{O}\hat{F}\hat{O}^{\mathrm{T}}. \tag{3.319}$$

The class notes use $x'^i \rightarrow O^{ij} x^j$, which violates our conventions on mixed upper and lower indices, but the end result eq. (3.319) is the same.

$$\left\| O^i{}_j \right\| = \begin{bmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.320}$$

Writing

$$\begin{aligned} C &= \cosh\alpha = \gamma \\ S &= -\sinh\alpha = -\gamma\beta, \end{aligned} \tag{3.321}$$

we can compute the transformed field strength tensor

$$
\begin{aligned}
\hat{F}' &= \begin{bmatrix} C & S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} C & S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C & S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -SE_x & -CE_x & -E_y & -E_z \\ CE_x & SE_x & -B_z & B_y \\ CE_y + SB_z & SE_y + CB_z & 0 & -B_x \\ CE_z - SB_y & SE_z - CB_y & B_x & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -E_x & -CE_y - SB_z & -CE_z + SB_y \\ E_x & 0 & -SE_y - CB_z & -SE_z + CB_y \\ CE_y + SB_z & SE_y + CB_z & 0 & -B_x \\ CE_z - SB_y & SE_z - CB_y & B_x & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -E_x & -\gamma(E_y - \beta B_z) & -\gamma(E_z + \beta B_y) \\ E_x & 0 & -\gamma(-\beta E_y + B_z) & \gamma(\beta E_z + B_y) \\ \gamma(E_y - \beta B_z) & \gamma(-\beta E_y + B_z) & 0 & -B_x \\ \gamma(E_z + \beta B_y) & -\gamma(\beta E_z + B_y) & B_x & 0 \end{bmatrix}.
\end{aligned}
\tag{3.322}
$$

As a check we have the antisymmetry that is expected. There is also a regularity to the end result that is aesthetically pleasing, hinting that things are hopefully error free. In coordinates for **E** and **B** this is

$$
\begin{aligned}
E_x &\rightarrow E_x \\
E_y &\rightarrow \gamma(E_y - \beta B_z) \\
E_z &\rightarrow \gamma(E_z + \beta B_y) \\
B_z &\rightarrow B_x \\
B_y &\rightarrow \gamma(B_y + \beta E_z) \\
B_z &\rightarrow \gamma(B_z - \beta E_y)
\end{aligned}
\tag{3.323}
$$

Writing $\boldsymbol{\beta} = \mathbf{e}_1\beta$, we have

$$\boldsymbol{\beta}\times\mathbf{B} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \beta & 0 & 0 \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{e}_2(-\beta B_z) + \mathbf{e}_3(\beta B_y), \tag{3.324}$$

which puts us enroute to a tidier vector form

$$\begin{aligned} E_x &\rightarrow E_x \\ E_y &\rightarrow \gamma(E_y + (\boldsymbol{\beta}\times\mathbf{B})_y) \\ E_z &\rightarrow \gamma(E_z + (\boldsymbol{\beta}\times\mathbf{B})_z) \\ B_z &\rightarrow B_x \\ B_y &\rightarrow \gamma(B_y - (\boldsymbol{\beta}\times\mathbf{E})_y) \\ B_z &\rightarrow \gamma(B_z - (\boldsymbol{\beta}\times\mathbf{E})_z). \end{aligned} \tag{3.325}$$

For a vector $\mathbf{A}$, write $\mathbf{A}_{\parallel} = (\mathbf{A}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$, $\mathbf{A}_{\perp} = \mathbf{A} - \mathbf{A}_{\parallel}$, allowing a compact description of the field transformation

$$\begin{aligned} \mathbf{E} &\rightarrow \mathbf{E}_{\parallel} + \gamma\mathbf{E}_{\perp} + \gamma(\boldsymbol{\beta}\times\mathbf{B})_{\perp} \\ \mathbf{B} &\rightarrow \mathbf{B}_{\parallel} + \gamma\mathbf{B}_{\perp} - \gamma(\boldsymbol{\beta}\times\mathbf{E})_{\perp}. \end{aligned} \tag{3.326}$$

Now, we want to consider the field of a moving particle. In the particle's (unprimed) rest frame the field due to its potential $\phi = q/r$ is

$$\begin{aligned} \mathbf{E} &= \frac{q}{r^2}\hat{\mathbf{r}} \\ \mathbf{B} &= 0. \end{aligned} \tag{3.327}$$

Coordinates for a "stationary" observer, who sees this particle moving along the x-axis at speed v are related by a boost in the $-v$ direction

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} \begin{bmatrix} \gamma & \gamma(v/c) & 0 & 0 \\ \gamma(v/c) & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}. \tag{3.328}$$

Therefore the fields in the observer frame will be

$$\begin{aligned}\mathbf{E}' &= \mathbf{E}_\parallel + \gamma \mathbf{E}_\perp - \gamma \frac{v}{c}(\mathbf{e}_1 \times \mathbf{B})_\perp = \mathbf{E}_\parallel + \gamma \mathbf{E}_\perp \\ \mathbf{B}' &= \mathbf{B}_\parallel + \gamma \mathbf{B}_\perp + \gamma \frac{v}{c}(\mathbf{e}_1 \times \mathbf{E})_\perp = \gamma \frac{v}{c}(\mathbf{e}_1 \times \mathbf{E})_\perp\end{aligned} \tag{3.329}$$

More explicitly with $\mathbf{E} = \frac{q}{r^3}(x, y, z)$ this is

$$\begin{aligned}\mathbf{E}' &= \frac{q}{r^3}(x, \gamma y, \gamma z) \\ \mathbf{B}' &= \gamma \frac{qv}{cr^3}(0, -z, y)\end{aligned} \tag{3.330}$$

Comparing to Problem 3 in Problem set 2, I see that this matches the result obtained by separately transforming the gradient, the time partial, and the scalar potential. Actually, if I am being honest, I see that I made a sign error in all the coordinates of $\mathbf{E}'$ when I initially did (this ungraded problem) in problem set 2. That sign error should have been obvious by considering the $v = 0$ case which would have mysteriously resulted in inversion of all the coordinates of the observed electric field.

Part b.

Part 1 and 2: Existence of the transformation.

In the single particle Lorentz trajectory problem we wish to solve

$$mc \frac{du^i}{ds} = \frac{e}{c} F^{ij} u_j, \tag{3.331}$$

which in matrix form we can write as

$$\frac{dU}{ds} = \frac{e}{mc^2} \hat{F} \hat{G} U. \tag{3.332}$$

where we write our column vector proper velocity as $U = \left\| u^i \right\|$. Under transformation of coordinates $u'^i = O^i{}_j x^j$, with $\hat{O} = \left\| O^i{}_j \right\|$, this becomes

$$\hat{O} \frac{dU}{ds} = \frac{e}{mc^2} \hat{O} \hat{F} \hat{O}^{\mathrm{T}} \hat{G} \hat{O} U. \tag{3.333}$$

Suppose we can find eigenvectors for the matrix $\hat{O} \hat{F} \hat{O}^{\mathrm{T}} \hat{G}$. That is for some eigenvalue λ, we can find an eigenvector Σ

$$\hat{O} \hat{F} \hat{O}^{\mathrm{T}} \hat{G} \Sigma = \lambda \Sigma. \tag{3.334}$$

Rearranging we have

$$(\hat{O}\hat{F}\hat{O}^{\mathrm{T}}\hat{G} - \lambda I)\Sigma = 0 \tag{3.335}$$

and conclude that Σ lies in the null space of the matrix $\hat{O}\hat{F}\hat{O}^{\mathrm{T}}\hat{G} - \lambda I$ and that this difference of matrices must have a zero determinant

$$\det(\hat{O}\hat{F}\hat{O}^{\mathrm{T}}\hat{G} - \lambda I) = -\det(\hat{O}\hat{F}\hat{O}^{\mathrm{T}} - \lambda\hat{G}) = 0. \tag{3.336}$$

Since $\hat{G} = \hat{O}\hat{G}\hat{O}^{\mathrm{T}}$ for any Lorentz transformation $\hat{O}$ in $SO(1,3)$, and $\det ABC = \det A \det B \det C$ we have

$$\det(\hat{O}\hat{F}\hat{O}^{\mathrm{T}} - \lambda G) = \det(\hat{F} - \lambda\hat{G}). \tag{3.337}$$

In problem 1.6, we called this our characteristic equation $P(\lambda) = \det(\hat{F} - \lambda\hat{G})$. Observe that the characteristic equation is Lorentz invariant for any λ, which requires that the eigenvalues λ are also Lorentz invariants.

In problem 1.6 of this problem set we computed that this characteristic equation expands to

$$P(\lambda) = \det(\hat{F} - \lambda\hat{G}) = (\mathbf{E}\cdot\mathbf{B})^2 + \lambda^2(\mathbf{B}^2 - \mathbf{E}^2) + \lambda^4. \tag{3.338}$$

The eigenvalues for the system, also each necessarily Lorentz invariants, are

$$\lambda = \pm\frac{1}{\sqrt{2}}\sqrt{\mathbf{E}^2 - \mathbf{B}^2 \pm \sqrt{(\mathbf{E}^2 - \mathbf{B}^2)^2 - 4(\mathbf{E}\cdot\mathbf{B})^2}}. \tag{3.339}$$

Observe that in the specific case where $\mathbf{E}\cdot\mathbf{B} = 0$, as in this problem, we must have $\mathbf{E}'\cdot\mathbf{B}'$ in all frames, and the two non-zero eigenvalues of our characteristic polynomial are simply

$$\lambda = \pm\sqrt{\mathbf{E}^2 - \mathbf{B}^2}. \tag{3.340}$$

These and $\mathbf{E}\cdot\mathbf{B} = 0$ are the invariants for this system. If we have $\mathbf{E}^2 > \mathbf{B}^2$ in one frame, we must also have $\mathbf{E}'^2 > \mathbf{B}'^2$ in another frame, still maintaining perpendicular fields. In particular if $\mathbf{B}' = 0$ we maintain real eigenvalues. Similarly if $\mathbf{B}^2 > \mathbf{E}^2$ in some frame, we must always have imaginary eigenvalues, and this is also true in the $\mathbf{E}' = 0$ case.

While the problem can be posed as a pure diagonalization problem (and even solved numerically this way for the general constant fields case), we can also work symbolically, thinking of the trajectories problem as simply seeking a transformation of frames that reduce the scope of the problem to one that is more tractable. That does not have to be the linear transformation that diagonalizes the system. Instead we are free to transform to a frame where one of the two fields $\mathbf{E}'$ or $\mathbf{B}'$ is zero, provided the invariants discussed are maintained.

Part 3: Finding the boost velocity that wipes out one of the fields.

Let us now consider a Lorentz boost $\hat{O}$, and seek to solve for the boost velocity that wipes out one of the fields, given the invariants that must be maintained for the system

To make things concrete, suppose that our perpendicular fields are given by $\mathbf{E} = E\mathbf{e}_2$ and $\mathbf{B} = B\mathbf{e}_3$.

Let also assume that we can find the velocity $\mathbf{v}_0$ for which one or more of the transformed fields is zero. Suppose that velocity is

$$\mathbf{v}_0 = v_0(\alpha_1, \alpha_2, \alpha_3) = v_0\hat{\mathbf{v}}_0, \tag{3.341}$$

where α_i are the direction cosines of $\mathbf{v}_0$ so that $\sum_i \alpha_i^2 = 1$. We will want to compute the components of $\mathbf{E}$ and $\mathbf{B}$ parallel and perpendicular to this velocity.

Those are

$$\begin{aligned}\mathbf{E}_\parallel &= E\mathbf{e}_2 \cdot (\alpha_1, \alpha_2, \alpha_3)(\alpha_1, \alpha_2, \alpha_3) \\ &= E\alpha_2(\alpha_1, \alpha_2, \alpha_3)\end{aligned} \tag{3.342}$$

$$\begin{aligned}\mathbf{E}_\perp &= E\mathbf{e}_2 - \mathbf{E}_\parallel \\ &= E(-\alpha_1\alpha_2, 1 - \alpha_2^2, -\alpha_2\alpha_3) \\ &= E(-\alpha_1\alpha_2, \alpha_1^2 + \alpha_3^2, -\alpha_2\alpha_3)\end{aligned} \tag{3.343}$$

For the magnetic field we have

$$\mathbf{B}_\parallel = B\alpha_3(\alpha_1, \alpha_2, \alpha_3), \tag{3.344}$$

and

$$\begin{aligned}\mathbf{B}_\perp &= B\mathbf{e}_3 - \mathbf{B}_\parallel \\ &= B(-\alpha_1\alpha_3, -\alpha_2\alpha_3, \alpha_1^2 + \alpha_2^2)\end{aligned} \tag{3.345}$$

Now, observe that $(\boldsymbol{\beta} \times \mathbf{B})_\parallel \sim ((\mathbf{v}_0 \times \mathbf{B}) \cdot \mathbf{v}_0)\mathbf{v}_0$, but this is just zero. So we have $(\boldsymbol{\beta} \times \mathbf{B})_\parallel = \boldsymbol{\beta} \times \mathbf{B}$. So our cross products terms are just

$$\begin{aligned}\hat{\mathbf{v}}_0 \times \mathbf{B} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & B \end{vmatrix} = B(\alpha_2, -\alpha_1, 0) \\ \hat{\mathbf{v}}_0 \times \mathbf{E} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & E & 0 \end{vmatrix} = E(-\alpha_3, 0, \alpha_1)\end{aligned} \tag{3.346}$$

We can now express how the fields transform, given this arbitrary boost velocity. From eq. (3.326), this is

$$\begin{aligned}\mathbf{E} &\rightarrow E\alpha_2(\alpha_1, \alpha_2, \alpha_3) + \gamma E(-\alpha_1\alpha_2, \alpha_1^2 + \alpha_3^2, -\alpha_2\alpha_3) + \gamma \frac{v_0^2}{c^2} B(\alpha_2, -\alpha_1, 0) \\ \mathbf{B} &\rightarrow B\alpha_3(\alpha_1, \alpha_2, \alpha_3) + \gamma B(-\alpha_1\alpha_3, -\alpha_2\alpha_3, \alpha_1^2 + \alpha_2^2) - \gamma \frac{v_0^2}{c^2} E(-\alpha_3, 0, \alpha_1)\end{aligned} \tag{3.347}$$

Zero Electric field case Let us tackle the two cases separately. First when $|\mathbf{B}| > |\mathbf{E}|$, we can transform to a frame where $\mathbf{E}' = 0$. In coordinates from eq. (3.347) this supplies us three sets of equations. These are

$$\begin{aligned}0 &= E\alpha_2\alpha_1(1 - \gamma) + \gamma \frac{v_0^2}{c^2} B\alpha_2 \\ 0 &= E\alpha_2^2 + \gamma E(\alpha_1^2 + \alpha_3^2) - \gamma \frac{v_0^2}{c^2} B\alpha_1 \\ 0 &= E\alpha_2\alpha_3(1 - \gamma).\end{aligned} \tag{3.348}$$

With an assumed solution the $\mathbf{e}_3$ coordinate equation implies that one of α_2 or α_3 is zero. Perhaps there are solutions with $\alpha_3 = 0$ too, but inspection shows that $\alpha_2 = 0$ nicely kills off the first equation. Since $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$, that also implies that we are left with

$$0 = E - \frac{v_0^2}{c^2} B\alpha_1 \tag{3.349}$$

Or

$$\begin{aligned}
\alpha_1 &= \frac{E}{B}\frac{c^2}{v_0^2} \\
\alpha_2 &= 0 \\
\alpha_3 &= \sqrt{1 - \frac{E^2}{B^2}\frac{c^4}{v_0^4}}
\end{aligned} \tag{3.350}$$

Our velocity was $\mathbf{v}_0 = v_0(\alpha_1, \alpha_2, \alpha_3)$ solving the problem for the $|\mathbf{B}|^2 > |\mathbf{E}|^2$ case up to an adjustable constant v_0. That constant comes with constraints however, since we must also have our cosine $\alpha_1 \le 1$. Expressed another way, the magnitude of the boost velocity is constrained by the relation

$$\frac{\mathbf{v}_0^2}{c^2} \ge \left|\frac{E}{B}\right|. \tag{3.351}$$

It appears we may also pick the equality case, so one velocity (not unique) that should transform away the electric field is

$$\boxed{\mathbf{v}_0 = c\sqrt{\left|\frac{E}{B}\right|}\mathbf{e}_1 = \pm c\sqrt{\left|\frac{E}{B}\right|}\frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{E}||\mathbf{B}|}.} \tag{3.352}$$

This particular boost direction is perpendicular to both fields. Observe that this highlights the invariance condition $\left|\frac{E}{B}\right| < 1$ since we see this is required for a physically realizable velocity. Boosting in this direction will reduce our problem to one that has only the magnetic field component.

Zero Magnetic field case Now, let us consider the case where we transform the magnetic field away, the case when our characteristic polynomial has strictly real eigenvalues $\lambda = \pm\sqrt{\mathbf{E}^2 - \mathbf{B}^2}$. In this case, if we write out our equations for the transformed magnetic field and require these to separately equal zero, we have

$$\begin{aligned}
0 &= B\alpha_3\alpha_1(1-\gamma) + \gamma\frac{v_0^2}{c^2}E\alpha_3 \\
0 &= B\alpha_2\alpha_3(1-\gamma) \\
0 &= B(\alpha_3^2 + \gamma(\alpha_1^2 + \alpha_2^2)) - \gamma\frac{v_0^2}{c^2}E\alpha_1.
\end{aligned} \tag{3.353}$$

Similar to before we see that $\alpha_3 = 0$ kills off the first and second equations, leaving just

$$0 = B - \frac{v_0^2}{c^2} E\alpha_1. \tag{3.354}$$

We now have a solution for the family of direction vectors that kill the magnetic field off

$$\begin{aligned}
\alpha_1 &= \frac{B}{E}\frac{c^2}{v_0^2} \\
\alpha_2 &= \sqrt{1 - \frac{B^2}{E^2}\frac{c^4}{v_0^4}} \\
\alpha_3 &= 0.
\end{aligned} \tag{3.355}$$

In addition to the initial constraint that $\left|\frac{B}{E}\right| < 1$, we have as before, constraints on the allowable values of v_0

$$\frac{\mathbf{v}_0^2}{c^2} \geq \left|\frac{B}{E}\right|. \tag{3.356}$$

Like before we can pick the equality $\alpha_1^2 = 1$, yielding a boost direction of

$$\boxed{\mathbf{v}_0 = c\sqrt{\left|\frac{B}{E}\right|}\mathbf{e}_1 = \pm c\sqrt{\left|\frac{B}{E}\right|}\frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{E}||\mathbf{B}|}.} \tag{3.357}$$

Again, we see that the invariance condition $|\mathbf{B}| < |\mathbf{E}|$ is required for a physically realizable velocity if that velocity is entirely perpendicular to the fields.

Notes on grading of my solution I lost two marks on this problem. One for eq. (3.330) where he wanted primes on the variables

$$\begin{aligned}
\mathbf{E}' &= \frac{q}{r'^3}(x', \gamma y', \gamma z') \\
\mathbf{B}' &= \gamma\frac{qv}{cr'^3}(0, -z', y'),
\end{aligned} \tag{3.358}$$

however, I do not think that is correct. Compare to problem set 2, problem 3, where this exactly matches the expected result, yet is only correct when the variables are the unprimed ones?

FIXME: Talk to Simon to see what he means.

Also, immediately before eq. (3.352) he underlined "one velocity (not unique)", and put an X beside it.

FIXME: is all that logic before eq. (3.352) wrong? (because that shows the boost velocity is not unique). If I try the very simplest boost applied to the $\mathbf{E} = E\mathbf{e}_2$ and $\mathbf{B} = B\mathbf{e}_3$ I find a very different result (with no square root). I think I am guilty of trying to be too general and not going back and checking for the simplest case. Even so, where are my errors?

Exercise 3.10 Continuity equation for delta function current distributions

Show explicitly that the electromagnetic 4-current j^i for a particle moving with constant velocity (considered in class, p. 100-101 of notes) is conserved $\partial_i j^i = 0$. Give a physical interpretation of this conservation law, for example by integrating $\partial_i j^i$ over some spacetime region and giving an integral form to the conservation law ($\partial_i j^i = 0$ is known as the "continuity equation").

Answer for Exercise 3.10

First lets review. Our four current was defined as

$$j^i(x) = \sum_A c e_A \int_{x(\tau)} dx_A^i(\tau)\delta^4(x - x_A(\tau)). \tag{3.359}$$

If each of the trajectories $x_A(\tau)$ represents constant motion we have

$$x_A(\tau) = x_A(0) + \gamma_A \tau(c, \mathbf{v}_A). \tag{3.360}$$

The spacetime split of this four vector is

$$\begin{aligned} x_A^0(\tau) &= x_A^0(0) + \gamma_A \tau c \\ \mathbf{x}_A(\tau) &= \mathbf{x}_A(0) + \gamma_A \tau \mathbf{v}, \end{aligned} \tag{3.361}$$

with differentials

$$\begin{aligned} dx_A^0(\tau) &= \gamma_A d\tau c \\ d\mathbf{x}_A(\tau) &= \gamma_A d\tau \mathbf{v}_A. \end{aligned} \tag{3.362}$$

Writing out the delta functions explicitly we have

$$\begin{aligned} j^i(x) = \sum_A c e_A \int_{x(\tau)} & dx_A^i(\tau)\delta(x^0 - x_A^0(0) - \gamma_A c\tau)\delta(x^1 - x_A^1(0) - \gamma_A v_A^1 \tau) \\ & \delta(x^2 - x_A^2(0) - \gamma_A v_A^2 \tau)\delta(x^3 - x_A^3(0) - \gamma_A v_A^3 \tau) \end{aligned} \tag{3.363}$$

So our time and space components of the current can be written

$$
\begin{aligned}
j^0(x) &= \sum_A c^2 e_A \gamma_A \int_{x(\tau)} d\tau \delta(x^0 - x^0_A(0) - \gamma_A c\tau)\delta^3(\mathbf{x} - \mathbf{x}_A(0) - \gamma_A \mathbf{v}_A \tau) \\
\mathbf{j}(x) &= \sum_A c e_A \mathbf{v}_A \gamma_A \int_{x(\tau)} d\tau \delta(x^0 - x^0_A(0) - \gamma_A c\tau)\delta^3(\mathbf{x} - \mathbf{x}_A(0) - \gamma_A \mathbf{v}_A \tau).
\end{aligned}
\tag{3.364}
$$

Each of these integrals can be evaluated with respect to the time coordinate delta function leaving the distribution

$$
\begin{aligned}
j^0(x) &= \sum_A c e_A \delta^3(\mathbf{x} - \mathbf{x}_A(0) - \frac{\mathbf{v}_A}{c}(x^0 - x^0_A(0))) \\
\mathbf{j}(x) &= \sum_A e_A \mathbf{v}_A \delta^3(\mathbf{x} - \mathbf{x}_A(0) - \frac{\mathbf{v}_A}{c}(x^0 - x^0_A(0)))
\end{aligned}
\tag{3.365}
$$

With this more general expression (multi-particle case) it should be possible to show that the four divergence is zero, however, the problem only asks for one particle. For the one particle case, we can make things really easy by taking the initial point in space and time as the origin, and aligning our velocity with one of the coordinates (say x).

Doing so we have the result derived in class

$$
j = e \begin{bmatrix} c \\ v \\ 0 \\ 0 \end{bmatrix} \delta(x - v x^0/c)\delta(y)\delta(z).
\tag{3.366}
$$

Our divergence then has only two portions

$$
\begin{aligned}
\frac{\partial j^0}{\partial x^0} &= ec(-v/c)\delta'(x - v x^0/c)\delta(y)\delta(z) \\
\frac{\partial j^1}{\partial x} &= ev\delta'(x - v x^0/c)\delta(y)\delta(z).
\end{aligned}
\tag{3.367}
$$

and these cancel out when summed. Note that this requires us to be loose with our delta functions, treating them like regular functions that are differentiable.

For the more general multiparticle case, we can treat the sum one particle at a time, and in each case, rotate coordinates so that the four divergence only picks up one term.

As for physical interpretation via integral, we have using the four dimensional divergence theorem

$$\int d^4 x \partial_i j^i = \int j^i dS_i \tag{3.368}$$

where dS_i is the three-volume element perpendicular to a x^i = constant plane. These volume elements are detailed generally in the text [11], however, they do note that one special case specifically $dS_0 = dxdydz$, the element of the three-dimensional (spatial) volume "normal" to hyperplanes ct = constant.

Without actually computing the determinants, we have something that is roughly of the form

$$0 = \int j^i dS_i = \int c\rho dxdydz + \int \mathbf{j} \cdot (\mathbf{n}_x cdtdydz + \mathbf{n}_y cdtdxdz + \mathbf{n}_z cdtdxdy). \tag{3.369}$$

This is cheating a bit to just write $\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z$. Are there specific orientations required by the metric? One way to be precise about this would be calculate the determinants detailed in the text, and then do the duality transformations.

Per unit time, we can write instead

$$\frac{\partial}{\partial t} \int \rho dV = -\int \mathbf{j} \cdot (\mathbf{n}_x dydz + \mathbf{n}_y dxdz + \mathbf{n}_z dxdy) \tag{3.370}$$

Rather loosely this appears to roughly describe that the rate of change of charge in a volume must be matched with the "flow" of current through the surface within that amount of time.

Exercise 3.11 Collision of photon and electron

Determine the velocity of an electron, initially at rest, after absorbing a photon.

Answer for Exercise 3.11

I made a dumb error on the exam on this one. I setup the four momentum conservation statement, but then did not multiply out the cross terms properly. This led me to incorrectly assume that I had to try doing this the hard way (something akin to what I did on the midterm). Simon later told us in the tutorial the simple way, and that is all we needed here too. Here is the setup.

An electron at rest initially has four momentum

$$(mc, 0) \tag{3.371}$$

where the incoming photon has four momentum

$$\left(\hbar \frac{\omega}{c}, \hbar \mathbf{k}\right) \tag{3.372}$$

After the collision our electron has some velocity so its four momentum becomes (say)

$$\gamma(mc, m\mathbf{v}), \tag{3.373}$$

and our new photon, going off on an angle θ relative to $\mathbf{k}$ has four momentum

$$\left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right) \tag{3.374}$$

Our conservation relationship is thus

$$(mc, 0) + \left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) = \gamma(mc, m\mathbf{v}) + \left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right) \tag{3.375}$$

I squared both sides, but dropped my cross terms, which was just plain wrong, and costly for both time and effort on the exam. What I should have done was just

$$\gamma(mc, m\mathbf{v}) = (mc, 0) + \left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) - \left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right), \tag{3.376}$$

and then square this (really making contractions of the form $p_i p^i$). That gives (and this time keeping my cross terms)

$$\begin{aligned}
(\gamma(mc, m\mathbf{v}))^2 &= \gamma^2 m^2 (c^2 - \mathbf{v}^2) \\
&= m^2 c^2 \\
&= m^2 c^2 + 0 + 0 + 2(mc, 0) \cdot \left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) \\
&\quad - 2(mc, 0) \cdot \left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right) - 2\left(\hbar\frac{\omega}{c}, \hbar\mathbf{k}\right) \cdot \left(\hbar\frac{\omega'}{c}, \hbar\mathbf{k}'\right) \\
&= m^2 c^2 + 2mc\,\hbar\frac{\omega}{c} - 2mc\,\hbar\frac{\omega'}{c} - 2\,\hbar^2\left(\frac{\omega}{c}\frac{\omega'}{c} - \mathbf{k} \cdot \mathbf{k}'\right) \\
&= m^2 c^2 + 2mc\,\hbar\frac{\omega}{c} - 2mc\,\hbar\frac{\omega'}{c} - 2\,\hbar^2\frac{\omega}{c}\frac{\omega'}{c}(1 - \cos\theta)
\end{aligned} \tag{3.377}$$

Rearranging a bit we have

$$\omega'\left(m + \frac{\hbar\omega}{c^2}(1 - \cos\theta)\right) = m\omega, \tag{3.378}$$

or

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)} \tag{3.379}$$

Exercise 3.12 Pion decay

FIXME: What was the exact question? Looks like calculating the muon energy was desired, but this write up is confused, with discussion of multiple problems.

Answer for Exercise 3.12

The problem above is very much like a midterm problem we had, so there was no justifiable excuse for messing up on it. That midterm problem was to consider the split of a pion at rest into a neutrino (massless) and a muon, and to calculate the energy of the muon. That one also follows the same pattern, a calculation of four momentum conservation, say

$$(m_\pi c, 0) = \hbar \frac{\omega}{c}(1, \hat{\mathbf{k}}) + (\mathcal{E}_\mu / c, \mathbf{p}_\mu). \tag{3.380}$$

Here ω is the frequency of the massless neutrino. The massless nature is encoded by a four momentum that squares to zero, which follows from $(1, \hat{\mathbf{k}}) \cdot (1, \hat{\mathbf{k}}) = 1^2 - \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 0$.

When I did this problem on the midterm, I perversely put in a scattering angle, instead of recognizing that the particles must scatter at 180 degree directions since spatial momentum components must also be preserved. This and the combination of trying to work in spatial quantities led to a mess and I did not get the end result in anything that could be considered tidy.

The simple way to do this is to just rearrange to put the null vector on one side, and then square. This gives us

$$\begin{aligned}
0 &= \left(\hbar \frac{\omega}{c}(1, \hat{\mathbf{k}})\right) \cdot \left(\hbar \frac{\omega}{c}(1, \hat{\mathbf{k}})\right) \\
&= \left((m_\pi c, 0) - (\mathcal{E}_\mu / c, \mathbf{p}_\mu)\right) \cdot \left((m_\pi c, 0) - (\mathcal{E}_\mu / c, \mathbf{p}_\mu)\right) \\
&= {m_\pi}^2 c^2 + {m_\nu}^2 c^2 - 2 (m_\pi c, 0) \cdot (\mathcal{E}_\mu / c, \mathbf{p}_\mu) \\
&= {m_\pi}^2 c^2 + {m_\nu}^2 c^2 - 2 m_\pi \mathcal{E}_\mu
\end{aligned} \tag{3.381}$$

A final re-arrangement gives us the muon energy

$$\mathcal{E}_\mu = \frac{1}{2} \frac{{m_\pi}^2 + {m_\nu}^2}{m_\pi} c^2 \tag{3.382}$$

4 PARTICLE ACTION AND RELATIVISTIC DYNAMICS

4.1 DYNAMICS

In Newtonian dynamics we have

$$m\ddot{\mathbf{r}} = \mathbf{f} \tag{4.1}$$

An equation of motion should be expressed in terms of vectors. This equation is written in a way that shows that the law of physics is independent of the choice of coordinates. We can do this in the context of tensor algebra as well. Ironically, this will require us to explicitly work with the coordinate representation, but this work will be augmented by the fact that we require our tensors to transform in specific ways.

In Newtonian mechanics we can look to symmetries and the invariance of the action with respect to those symmetries to express the equations of motion. Our symmetries in Newtonian mechanics leave the action invariant with respect to spatial translation and with respect to rotation.

We want to express relativistic dynamics in a similar way, and will have to express the action as a Lorentz scalar. We are going to impose the symmetries of the Poincare group to determine the relativistic laws of dynamics, and the next task will be to consider the possibilities for our relativistic action, and see what that action implies for dynamics in a relativistic context.

Reading Covering chapter 2 material from the text [11], and lecture notes RelEMpp52-56.pdf, and RelEMp53.1.pdf.

4.2 THE RELATIVITY PRINCIPLE

The relativity principle implies that the EOM should be expressed in 4-vector form, just like Newton's EOM are expressed in 3-vector form

$$m\ddot{\mathbf{r}} = \mathbf{f} \tag{4.2}$$

Observe that in coordinate form this is

$$m\ddot{r}^i = f^i, \qquad i = 1, 2, 3 \tag{4.3}$$

or for a rotated frame O'

$$m\ddot{r}'^i = f'^i, \qquad i = 1, 2, 3 \tag{4.4}$$

Need to generalize to 4 vectors, so we need 4-velocity and 4-acceleration.

Later we will study action and Lagrangian, and then relativity will require that the action be a Lorentz scalar. The analogy for a Newtonian point particle is a scalar under rotations.

Four vector velocity

Definition: Velocity s the rate of change of position in $(ct, \mathbf{x})$-space. Position means specifying both ct and $\mathbf{x}$ for a point in spacetime.

PICTURE: $x^0 = ct$ axis up, and x^1, x^2, x^3 axis over, with worldline $x = x(\tau)$. Here τ is a parameter for the worldline, and provides a mapping for the curve in spacetime.

PICTURE: 3D vectors, $\mathbf{r}(t)$, $\mathbf{r}(t + \Delta t)$, and the difference vector $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$.

We write

$$\mathbf{v}(t) \equiv \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \tag{4.5}$$

For four vectors we will parametrize the worldline by its "length", with O taken from some arbitrary point on it. We can also take τ to be the proper time, and the only difference will be the factor of c (which becomes especially easy with the choice $c = 1$ that is avoided in this class).

$$\frac{x^i(\tau + \Delta\tau) - x^i(\tau)}{\Delta\tau} \tag{4.6}$$

We will take the limit

$$\frac{dx^i}{d\tau} = \lim_{\Delta\tau \to 0} \frac{x^i(\tau + \Delta\tau) - x^i(\tau)}{\Delta\tau} \tag{4.7}$$

and then define a dimensionless "proper velocity"

$$u^i \equiv \frac{1}{c}\frac{dx^i}{d\tau} = \frac{dx^i}{ds}. \tag{4.8}$$

This is a nice quantity, we are dividing a vector by a Lorentz scalar, and thus get a four vector as a result (i.e. the result transforms as a four vector).

PICTURE: small fragment of a worldline with constant slope over the infinitesimal interval. dx^0 up and dx^1 over.

$$\begin{aligned} ds^2 &= (dx^0)^2 - (dx^1)^2 \\ &= c^2 \left((dt)^2 - \frac{1}{c^2}(dx^1)^2 \right) \\ &= c^2 (dt)^2 \left(1 - \frac{1}{c^2} \frac{dx^1}{dt^2} \right) \end{aligned} \tag{4.9}$$

Or

$$ds = cdt \sqrt{1 - \frac{1}{c^2} \frac{dx^1}{dt^2}} \tag{4.10}$$

NOTE: Prof admits pulling a fast one, since he has aligned the worldline along the x^1 axis, however this is always possible by rotating the coordinate system.

$$\begin{aligned} u^0 &= \frac{dx^0}{ds} \\ &= \frac{cdt}{cdt \sqrt{1 - \mathbf{v}^2/c^2}} \\ &= \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}} \\ &= \gamma \end{aligned} \tag{4.11}$$

$$\begin{aligned} u^1 &= \frac{dx^1}{ds} \\ &= \frac{dx^1}{cdt \sqrt{1 - \mathbf{v}^2/c^2}} \\ &= \frac{v^1/c}{\sqrt{1 - \mathbf{v}^2/c^2}} \\ &= \gamma \frac{v^1}{c} \end{aligned} \tag{4.12}$$

Similarly

$$\begin{aligned} u^2 &= \gamma \frac{v^2}{c} \\ u^3 &= \gamma \frac{v^2}{c} \end{aligned} \tag{4.13}$$

We have now unpacked the four velocity, and have

$$u^i = \left(\gamma, \frac{\mathbf{v}}{c}\gamma\right) \tag{4.14}$$

Length of the four velocity vector Recall that this length is

$$\begin{aligned} u^i g_{ij} u^j &= u^i u_i \\ &= u_i u^i \\ &= (u^0)^2 - (u_i)^2 \\ &= \gamma^2 - \gamma^2 \frac{\mathbf{v}}{c} \cdot \frac{\mathbf{v}}{c} \\ &= \gamma^2 \left(1 - \frac{\mathbf{v}^2}{c^2}\right) \end{aligned} \tag{4.15}$$

The four velocity in physics is

$$u^i = \left(\gamma, \frac{\mathbf{v}}{c}\gamma\right) \tag{4.16}$$

but in mathematics the meaning of $u^i u_i = 1$ means that this quantity is the unit tangent vector to the worldline.

Four acceleration In Newtonian physics we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{4.17}$$

Our relativistic mapping of this, with $v \rightarrow u^i$ and $t \rightarrow s$, gives

$$w^i = \frac{du^i}{ds} \tag{4.18}$$

Geometrically w^i is the normal to the worldline. This follows from $u^i g_{ij} u^j = 1$, so

$$
\begin{aligned}
\frac{d}{ds}\left(u^i g_{ij} u^j\right) &= \frac{du^i}{ds} g_{ij} u^j + u^i g_{ij} \frac{du^j}{ds} \\
&= \frac{du^i}{ds} g_{ij} u^j + u^j \overbrace{\boxed{g_{ji}}}^{=g_{ij}} \frac{du^i}{ds} \\
&= \frac{du^i}{ds} g_{ij} u^j + u^j g_{ji} \frac{du^i}{ds} \\
&= 2 \frac{du^i}{ds} g_{ij} u^j
\end{aligned}
\tag{4.19}
$$

Note that we have utilized the fact above that the dummy summation indices can be swapped (or changed to anything else we feel inclined to use).

The conclusion is that the dot product of the acceleration and the velocity is zero

$$
w_i u^i = 0. \tag{4.20}
$$

4.3 RELATIVISTIC ACTION

$$
S_{ab} = ? \tag{4.21}
$$

What is the action for a worldline from $a \rightarrow b$.

We want something that has velocity dependence (u^i not $\mathbf{v}$), but that is Lorentz invariant and has only first derivatives.

The relativistic length is the simplest so we could form

$$
\int ds u^i u_i \tag{4.22}
$$

but that is not interesting since $u^i u_i = 1$. We could form

$$
\int ds u^i \frac{u_i}{ds} = \int ds w^i u_i \tag{4.23}
$$

but then this is just zero.

We could form something like

$$
\int ds \frac{w^i}{ds} u_i \tag{4.24}
$$

This is non zero and non-constant, but evaluating the EOM for such an action would produce a result that has higher than second order derivatives.

We are left with

$$S_{ab} = \text{constant} \int_a^b ds \tag{4.25}$$

To fix this constant we note that if we want to minimize the action over the infinitesimal interval, then we need a minus sign. Since the Lagrangian has dimensions of energy, and the dimensions of energy times time are momentum, our action must then have dimensions of momentum. So one possible constant that fixes up our dimensions is mc. Construct an action with the following form

$$S_{ab} = -mc \int_a^b ds, \tag{4.26}$$

does the job we want. Here "m" is a characteristic of the particle, which is a Lorentz scalar. It also happens to have dimensions of mass. With $ds = cdt\sqrt{1 - \mathbf{v}^2/c^2}$, we have

$$S_{ab} = -mc^2 \int_{t_a}^{t_b} dt \sqrt{1 - \frac{1}{c^2}\left(\frac{d\mathbf{x}(t)}{dt}\right)^2} \tag{4.27}$$

Now everything looks like it was in classical mechanics.

$$S_{ab} = \int_{t_a}^{t_b} \mathcal{L}(\dot{\mathbf{x}}(t))dt \tag{4.28}$$

$$\mathcal{L}(\dot{\mathbf{x}}(t)) = -mc^2 \tag{4.29}$$

Now find the extremum of S. That problem is really to compute the variation in the action that results from varying the coordinates around the stationary point, and equate that variation to zero to find the extremum

$$\delta S = S[\mathbf{x}(t) + \delta\mathbf{x}(t)] - S[\mathbf{x}(t)] = 0 \tag{4.30}$$

The usual condition is imposed where we have zero variation of the coordinates at the boundaries of the action integral

$$0 = \delta\mathbf{x}(t_a) = \delta\mathbf{x}(t_b) \tag{4.31}$$

Returning to our action we have

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = 0 \tag{4.32}$$

This last is zero because it is a free particle with no position dependence.

$$\begin{aligned} 0 &= -mc^2 \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{x}}}\sqrt{1 - \dot{\mathbf{x}}^2} \\ &= -mc^2 \frac{d}{dt}\frac{-\dot{\mathbf{x}}}{\sqrt{1 - \dot{\mathbf{x}}^2}} \\ &= mc^2 \frac{d}{dt}\gamma \dot{\mathbf{x}} \end{aligned} \tag{4.33}$$

So we have

$$\frac{d}{dt}(\gamma \dot{\mathbf{x}}) = 0 \tag{4.34}$$

By evaluating this, we can eventually show that we can construct a four vector equation. Doing this we have

$$\begin{aligned} \frac{d}{dt}(\gamma \mathbf{v}) &= \frac{d}{dt}\left(\left(1 - \mathbf{v}^2/c^2\right)^{-1/2} \mathbf{v}\right) \\ &= -2(-1/2)\mathbf{v}(\mathbf{v} \cdot \dot{\mathbf{v}})/c^2 \left(1 - \mathbf{v}^2/c^2\right)^{-3/2} + \left(1 - \mathbf{v}^2/c^2\right)^{-1/2} \dot{\mathbf{v}} \\ &= \gamma \left(\frac{\mathbf{v}(\mathbf{v} \cdot \dot{\mathbf{v}})}{c^2 - \mathbf{v}^2} + \dot{\mathbf{v}}\right) \end{aligned} \tag{4.35}$$

Or

$$\frac{\mathbf{v}(\mathbf{v} \cdot \dot{\mathbf{v}})}{c^2 - \mathbf{v}^2} + \dot{\mathbf{v}} = 0 \tag{4.36}$$

Clearly $\dot{\mathbf{v}} = 0$ is a solution, but is it the only solution?
By dotting this with $\mathbf{v}$ we have

$$\begin{aligned} 0 &= \frac{\mathbf{v}^2(\mathbf{v} \cdot \dot{\mathbf{v}})}{c^2 - \mathbf{v}^2} + \dot{\mathbf{v}} \cdot \mathbf{v} \\ &= (\mathbf{v} \cdot \dot{\mathbf{v}})\left(1 + \frac{\mathbf{v}^2}{c^2 - \mathbf{v}^2}\right) \\ &= (\mathbf{v} \cdot \dot{\mathbf{v}})\frac{c^2}{c^2 - \mathbf{v}^2} \end{aligned} \tag{4.37}$$

This implies that $\dot{\mathbf{v}} = 0$ (a contraction) or that $\mathbf{v} \cdot \dot{\mathbf{v}} = 0$. To examine the perpendicularity question, let us take cross products. This gives

$$0 = \frac{(\mathbf{v} \times \mathbf{v})(\mathbf{v} \cdot \dot{\mathbf{v}})}{c^2 - \mathbf{v}^2} + \dot{\mathbf{v}} \times \mathbf{v} \tag{4.38}$$

We have found that $\mathbf{v} \cdot \dot{\mathbf{v}} = 0$ and $\mathbf{v} \times \dot{\mathbf{v}} = 0$. This can only mean that $\dot{\mathbf{v}} = 0$, contradicting the assumption that is non-zero. We conclude that $\dot{\mathbf{v}} = 0$ is the only solution to eq. (4.36).

4.4 NEXT TIME

We want to finish up and show how this results in a four velocity equation. We have

$$\frac{d}{dt}(\gamma \mathbf{v}) = 0 \tag{4.39}$$

which is

$$\frac{d}{dt}(u^\alpha) = 0, \qquad \text{for } u^\alpha = u^1, u^2, u^3 \tag{4.40}$$

eventually, we will show that we also have

$$\frac{d}{dt}(u^i) = 0 \tag{4.41}$$

Reading Covering chapter 2 material from the text [11], and lecture notes RelEMpp52-56.pdf, and lecture notes RelEMpp56.1-73.pdf.

4.5 FINISHING PREVIOUS ARGUMENTS ON ACTION AND PROPER VELOCITY

For a free particle, our action is

$$\begin{aligned} S &= -mc \int ds \\ &= -mc^2 \int dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \end{aligned} \tag{4.42}$$

Our Lagrangian is

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}. \tag{4.43}$$

We can also make a non-relativistic velocity approximation

$$\begin{aligned}
\mathcal{L} &= -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \\
&= -mc^2 \left(1 - \frac{1}{2}\frac{\mathbf{v}^2}{c^2}\right) + O((\mathbf{v}^2/c^2)^2) \\
&\approx \underbrace{\boxed{-mc^2}}_{\text{constant}} + \underbrace{\boxed{\frac{1}{2}m\mathbf{v}^2}}_{\text{Classical Lagrangian for free particle}}.
\end{aligned} \tag{4.44}$$

It is good to know that we recover the familiar Newtonian case when our velocities are small enough.

Our job is to vary the action between a pair of spacetime points

$$(t_a, \mathbf{x}_a) \rightarrow (t_b, \mathbf{x}_b) \tag{4.45}$$

The equations of motion that result from this variation, or from the Euler-Lagrange equations that one can obtain from this variation, are

$$\frac{d}{dt}(\gamma \mathbf{v}) = 0 \tag{4.46}$$

We argued last time, by evaluating the derivatives of eq. (4.46), and taking dot and cross products with $\mathbf{v}$ that we also have

$$\frac{d\mathbf{v}}{dt} = 0 \tag{4.47}$$

Observe that since $d\mathbf{v}/dt = 0$, we also have $d\gamma/dt = 0$

$$\begin{aligned} \frac{d\gamma}{dt} &= \frac{d}{dt}\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \\ &= \frac{d}{dt}\frac{1}{\left(1-\frac{\mathbf{v}^2}{c^2}\right)^{3/2}}(-1/2)(2)(-\mathbf{v}\cdot\dot{\mathbf{v}})/c^2 \\ &= 0. \end{aligned} \tag{4.48}$$

We can therefore combine the pair of equations (after adjusting both to have dimensions of velocity)

$$\begin{aligned} \frac{d}{dt}(\gamma\mathbf{v}) &= 0 \\ \frac{d}{dt}(\gamma c) &= 0, \end{aligned} \tag{4.49}$$

into

$$u^i = (u^0, \mathbf{u}). \tag{4.50}$$

Here

$$\begin{aligned} u^0 &= \gamma \\ \mathbf{u} &= \gamma\frac{\mathbf{v}}{c}. \end{aligned} \tag{4.51}$$

Since we have $du^i/dt = 0$, pre-multiplying this by γ/c does not change the equation, and we have

$$0 = \frac{1}{c\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\frac{du^i}{dt}. \tag{4.52}$$

This now puts things in a nice invariant form, with no bias towards any specific observer's time coordinates, and we have for the free particle

$$\frac{du^i}{ds} = 0. \tag{4.53}$$

4.6 SYMMETRIES OF SPACETIME TRANSLATION INVARIANCE

The symmetries of S imply conservation laws. Our action has $SO(1,3) \times T^4$ = Lorentz x space-time translation ≡ Poincaré group of symmetries.

Consider quantities conserved due to T^4 factor

$$\begin{aligned} &\mathbf{x} \rightarrow \mathbf{x} + \mathbf{a} \qquad \text{where } \mathbf{a} \text{ is constant} \\ &t \rightarrow t + \text{constant} \end{aligned} \tag{4.54}$$

Observe that the Lagrangian is not a function of $\mathbf{x}$, or t explicitly

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = -mc\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} = \mathcal{L}(\mathbf{v}). \tag{4.55}$$

A consequence from this, utilizing the Euler-Lagrange equations is that we have a zero for the time derivative of the generalized momentum $\partial\mathcal{L}/\partial\mathbf{v}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = 0, \tag{4.56}$$

Let us calculate that generalized momentum

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{v}} &= \frac{\partial}{\partial \mathbf{v}}\left(-mc^2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\right) \\ &= \frac{\partial}{\partial \mathbf{v}}\left(-mc^2\frac{(1/2)(-2)\mathbf{v}/c^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}\right) \\ &= m\frac{\mathbf{v}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \end{aligned} \tag{4.57}$$

So our generalized momentum is

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = m\mathbf{v}\gamma. \tag{4.58}$$

Evaluating the Euler-Lagrange equations above we find

$$\begin{aligned} 0 &= \frac{d}{dt}\left(m\gamma\mathbf{v}\right) \\ &= \frac{d}{dt}\left(mcu^{1,2,3}\right) \end{aligned} \tag{4.59}$$

Recall that $u^0 = \gamma$, and that $d\gamma/dt = 0$, so we also have

$$\frac{d}{dt}\left(mcu^i\right) = 0 \tag{4.60}$$

and again with multiplication by γ/c we have a Lorentz invariant relation, mostly a consequence of spacetime translation invariance

$$\frac{d}{ds}\left(mcu^i\right) = 0. \tag{4.61}$$

We define this quantity, the invariant quantity (a four vector), as the relativistic momentum

$$p^i = mcu^i. \tag{4.62}$$

A relativistic particle is characterizes by a conserved 4 vector quantity p^i with

$$\begin{aligned} p^0 &= mc\gamma \\ \mathbf{p} &= m\gamma\mathbf{v} \\ p^i &= (p^0, \mathbf{p}) \end{aligned} \tag{4.63}$$

4.7 TIME TRANSLATION INVARIANCE

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = \mathcal{L}(\mathbf{v}) \tag{4.64}$$

However, it helps to consider the more general case

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = \mathcal{L}(\mathbf{x}, \mathbf{v}) \tag{4.65}$$

since we have no explicit time dependence.

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(\mathbf{v}) &= \frac{\partial \mathcal{L}}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \dot{\mathbf{v}} \\ &= \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{v}}\right) \cdot \mathbf{v} + \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \mathbf{v}\right) \end{aligned} \tag{4.66}$$

Regrouping, to pull all the derivative terms together provides the conservation identity

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \mathbf{v} - \mathcal{L}\right) = 0. \tag{4.67}$$

This quantity $\frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \mathbf{v} - \mathcal{L}$ is usually identified as the Hamiltonian H, the energy, but we will call it E here.

In our case, with the relativistic free particle Lagrangian

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}, \tag{4.68}$$

we have

$$\begin{aligned}
E &= \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \mathbf{v} - \mathcal{L} \\
&= \mathbf{v} \cdot \left(m \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \mathbf{v} \right) + mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \\
&= \frac{m\mathbf{v}^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} + mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \\
&= \frac{\mathbf{v}^2 + mc^2 \left(1 - \frac{\mathbf{v}^2}{c^2}\right)}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \\
&= \frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}
\end{aligned} \tag{4.69}$$

So we define, for the energy, a conserved quantity under time translation, we have

$$\boxed{E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}} \tag{4.70}$$

It is only with the $\mathbf{v} \rightarrow 0$ that we recover the famous tee-shirt expression

$$E = mc^2. \tag{4.71}$$

Since we also know (from the spacetime translation) that $p^0 = mc\gamma = E/c$, we get another conserved quantity for free since $(p^0, \mathbf{p})$ then is also a symmetry (i.e. thus a conserved quantity)

$$\begin{aligned} p^0 &= m\gamma c = \frac{E}{c} \\ \mathbf{p} &= m\gamma\mathbf{v} \end{aligned} \tag{4.72}$$

$$p^i = (p^0, \mathbf{p}) \tag{4.73}$$

Note that the only "mass" you ever want to talk about is "m". This is a Lorentz scalar, and we will not use the old notions that mass changes with velocity or "relativistic mass".

4.8 SOME PROPERTIES OF THE FOUR MOMENTUM

We have

$$\begin{aligned} p^i p_i &= (p^0)^2 - \mathbf{p}^2 \\ &= mc^2\gamma^2 - m^2\gamma^2\mathbf{v}^2 \\ &= mc^2\gamma^2\left(1 - \frac{\mathbf{v}^2}{c^2}\right) \\ &= m^2c^2 \end{aligned} \tag{4.74}$$

So we have

$$\boxed{p^i p_i = m^2c^2} \tag{4.75}$$

We say that the 4-vector p^i represents a particle with mass m.

Since four momentum is a conserved quantity we can use this conservation property to study relativistic collisions

PICTURE: two particles colliding with two particles resulting (particles trajectories as arrows)

four momentum before

$$\underbrace{p_1^i + p_2^i} = \underbrace{p_3^i + p_4^i} \tag{4.76}$$

four momentum after

$$\begin{aligned} \mathbf{p} &= \frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \rightarrow 0 \qquad \text{when } m \rightarrow 0 \\ E &= \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \rightarrow 0 \qquad \text{when } m \rightarrow 0 \end{aligned} \tag{4.77}$$

except when $|\mathbf{v}| = c$, where if you take $m \rightarrow 0$ and $|\mathbf{v}| = c$ you can get anything (any values) in such a limit (limit does not exist).

However, because

$$\frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2 = 0 \tag{4.78}$$

when $m \rightarrow 0$, E and $\mathbf{p}$ for a massless particle must obey $E = c|\mathbf{p}|$.

Massless particles like photons (and gravitons if/when eventually measured) have lightlike 4 momentum vectors

$$p^i p_i = 0 \tag{4.79}$$

Gravity waves have not been seen yet, but the LIGO and LISA (extremely large infraferometers) experiments are expected to get some results on this in the near future.

4.9 WHERE ARE WE?

In the notes there is a review (see that on one's own). We will also want to eventually deal with the conservation laws in four vector form, since it will illustrate how the electric and magnetic fields have to be transformed. We will get to that eventually.

4.10 INTERACTIONS

In classical mechanics we have

$$\mathcal{L}_{\text{kinetic}} = \frac{1}{2}m\mathbf{v}^2 \tag{4.80}$$

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}^2 - U(\mathbf{r}) \tag{4.81}$$

Here $U(\mathbf{r})$ is an external potential.

$$S = S_{\text{free}} + S_{\text{interaction}} = \int dt \frac{1}{2} m\mathbf{v}^2 + \int dt(-U(\mathbf{r}, t)) \tag{4.82}$$

The quantity $U(\mathbf{r}, t)$ is what we call a potential field.

What is the simplest invariant field we can have? The simplest possibility is to have a relativistic particle which interacts with an external Lorentz scalar field. We would imagine that this is due to some other particle or some distribution of other fields.

Recall that the scalar field under rotations (reminder)

PICTURE: a point with coordinates in a fixed and a rotated coordinate system

That point is

$$P = (x, y) = (x', y') \tag{4.83}$$

Similarly we can define a scalar quantity (like temperature or the Coulomb potential) is then assigned a value at each point

$$\phi(x, y) = \phi'(x', y') \tag{4.84}$$

The value of this scalar in the x, y coordinates system at point P equals the value of this scalar in the x', y' coordinates system at the same point P.

A Lorentz scalar field is like this, but for an event $P = (ct, x) = (ct', x')$ is the same.

So, we would have

$$\phi(ct, x) = \phi'(ct', x') \tag{4.85}$$

The value of this scalar in the x, ct coordinates system at event P equals the value of this scalar in the x', ct' coordinates system at the same event P in the primed frame.

Our action would then be

$$S = -mc \int ds + g \int ds \phi(x^i) \tag{4.86}$$

Here g is a coupling constant, also called the "charge" of a particle under that scalar field.

Note that unfortunately nature has not provided us with scalar fields that are stable enough to observe in classical interactions

We do however have some scalar particles

$$\pi^0, \pi^\pm, k^0, k^\pm \tag{4.87}$$

These are unstable and short ranged.

The LHC is looking for another unstable short lived scalar field (the Higgs). So we have to unfortunately study a more complicated field, a vector field. We will do that next time.

Reading Covering chapter 2 material from the text [11], and lecture notes RelEMpp56.1-73.pdf.

4.11 MORE ON THE ACTION

Action for a relativistic particle in an external 4-scalar field

$$S = -mc \int ds - g \int ds \phi(x) \tag{4.88}$$

Unfortunately we have no 4-vector scalar fields (at least for particles that are long lived and stable).

PICTURE: 3-vector field, some arrows in various directions.

PICTURE: A vector $\mathbf{A}$ in an x, y frame, and a rotated (counterclockwise by angle α) x', y' frame with the components in each shown pictorially.

We have

$$\begin{aligned} A'_x(x', y') &= \cos\alpha A_x(x, y) + \sin\alpha A_y(x, y) \\ A'_y(x', y') &= -\sin\alpha A_x(x, y) + \cos\alpha A_y(x, y) \end{aligned} \tag{4.89}$$

$$\begin{bmatrix} A'_x(x', y') \\ A'_y(x', y') \end{bmatrix} = \begin{bmatrix} \cos\alpha A_x(x, y) & \sin\alpha A_y(x, y) \\ -\sin\alpha A_x(x, y) & \cos\alpha A_y(x, y) \end{bmatrix} \begin{bmatrix} A_x(x, y) \\ A_y(x, y) \end{bmatrix} \tag{4.90}$$

More generally we have

$$\begin{bmatrix} A'_x(x', y', z') \\ A'_y(x', y', z') \\ A'_z(x', y', z') \end{bmatrix} = \hat{O} \begin{bmatrix} A_x(x, y, z) \\ A_y(x, y, z) \\ A_z(x, y, z) \end{bmatrix} \tag{4.91}$$

Here $\hat{O}$ is an $SO(3)$ matrix rotating $x \to x'$

$$\mathbf{A}(\mathbf{x}) \cdot \mathbf{y} = \mathbf{A}'(\mathbf{x}') \cdot \mathbf{y}' \tag{4.92}$$

$$\mathbf{A} \cdot \mathbf{B} = \text{invariant} \tag{4.93}$$

A four vector field is $A^i(x)$, with $x = x^i$, $i = 0, 1, 2, 3$ and we would write

$$\begin{bmatrix} (x^0)' \\ (x^1)' \\ (x^2)' \\ (x^3)' \end{bmatrix} = \hat{O} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{4.94}$$

Now $\hat{O}$ is an $SO(1, 3)$ matrix. Our four vector field is then

$$\begin{bmatrix} (A^0)' \\ (A^1)' \\ (A^2)' \\ (A^3)' \end{bmatrix} = \hat{O} \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} \tag{4.95}$$

We have

$$A^i g_{ij} x^i = \text{invariant} = A'^i g_{ij} x'^i \tag{4.96}$$

From electrodynamics we know that we have a scalar field, the electrostatic potential, and a vector field

What is a plausible action?

How about

$$\int ds x^i g_{ij} A^j \tag{4.97}$$

This is not translation invariant.

$$\int ds x^i g_{ij} A^j \tag{4.98}$$

Next simplest is

$$\int ds u^i g_{ij} A^j \tag{4.99}$$

Could also do

$$\int ds A^i g_{ij} A^j \tag{4.100}$$

but it turns out that this is not gauge invariant (to be defined and discussed in detail).

An aside. Dimensions of proper velocity. Note that the convention for this course is to write

$$u^i = \left(\gamma, \gamma \frac{\mathbf{v}}{c}\right) = \frac{dx^i}{ds} \tag{4.101}$$

Where u^i is dimensionless ($u^i u_i = 1$). Some authors use

$$u^i = (\gamma c, \gamma \mathbf{v}) = \frac{dx^i}{d\tau}, \tag{4.102}$$

where $u^i u_i = c^2$, and u^i has dimensions of velocity.

Return to the problem The simplest action for a four vector field A^i is then

$$S = -mc \int ds - \frac{e}{c} \int ds u^i A_i \tag{4.103}$$

(Recall that $u^i A_i = u^i g_{ij} A^j$).

In this action e is nothing but a Lorentz scalar, a property of the particle that describes how it "couples" (or "feels") the electrodynamics field.

Similarly mc is a Lorentz scalar which is a property of the particle (inertia).

It turns out that all the electric charges in nature are quantized, and there are some deep reasons (in magnetic monopoles exist) for this.

Another reason for charge quantization apparently has to do with gauge invariance and associated compact groups. Poppitz is amusing himself a bit here, hinting at some stuff that we can eventually learn.

Returning to our discussion, we have

$$S = -mc \int ds - \frac{e}{c} \int ds u^i g_{ij} A^j \tag{4.104}$$

with the electrodynamics four vector potential

$$\begin{aligned} A^i &= (\phi, \mathbf{A}) \\ u^i &= \left(\gamma, \gamma \frac{\mathbf{v}}{c}\right) \\ u^i g_{ij} A^j &= \gamma\phi - \gamma \frac{\mathbf{v} \cdot \mathbf{A}}{c} \end{aligned} \tag{4.105}$$

$$\begin{aligned} S &= -mc^2 \int dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - \frac{e}{c} \int c dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \left(\gamma\phi - \gamma \frac{\mathbf{v}}{c} \cdot \mathbf{A}\right) \\ &= \int dt \left(-mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - e\phi(\mathbf{x}, t) + \frac{e}{c} \mathbf{v} \cdot \mathbf{A}(\mathbf{x}, t)\right) \end{aligned} \tag{4.106}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \frac{\mathbf{v}}{c^2} + \frac{e}{c} \mathbf{A}(\mathbf{x}, t) \tag{4.107}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \mathbf{v}} = m \frac{d}{dt} (\gamma \mathbf{v}) + \frac{e}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{e}{c} \frac{\partial \mathbf{A}}{\partial x^\alpha} v^\alpha \tag{4.108}$$

Here $\alpha, \beta = 1, 2, 3$ and are summed over.
For the other half of the Euler-Lagrange equations we have

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = -e \frac{\partial \phi}{\partial x^\alpha} + \frac{e}{c} v^\beta \frac{\partial A^\beta}{\partial x^\alpha} \tag{4.109}$$

Equating these, and switching to coordinates for eq. (4.108), we have

$$m \frac{d}{dt} (\gamma v^\alpha) + \frac{e}{c} \frac{\partial A^\alpha}{\partial t} + \frac{e}{c} \frac{\partial A^\alpha}{\partial x^\beta} v^\beta = -e \frac{\partial \phi}{\partial x^\alpha} + \frac{e}{c} v^\beta \frac{\partial A^\beta}{\partial x^\alpha} \tag{4.110}$$

A final rearrangement yields

$$\frac{d}{dt} m\gamma v^\alpha = e \overbrace{\boxed{\left(-\frac{1}{c} \frac{\partial A^\alpha}{\partial t} - \frac{\partial \phi}{\partial x^\alpha}\right)}}^{E^\alpha} + \frac{e}{c} v^\beta \left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta}\right) \tag{4.111}$$

We can identity the second term with the magnetic field but first have to introduce antisymmetric matrices.

4.12 ANTISYMMETRIC MATRICES

$$\begin{aligned} M_{\mu\nu} &= \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \\ &= \epsilon_{\mu\nu\lambda} B_\lambda, \end{aligned} \tag{4.112}$$

where

$$\epsilon_{\mu\nu\lambda} = \begin{matrix} 0 & \text{if any two indices coincide} \\ 1 & \text{for even permutations of } \mu\nu\lambda \\ -1 & \text{for odd permutations of } \mu\nu\lambda \end{matrix} \tag{4.113}$$

Example:

$$\begin{aligned} \epsilon_{123} &= 1 \\ \epsilon_{213} &= -1 \\ \epsilon_{231} &= 1. \end{aligned} \tag{4.114}$$

We can show that

$$B_\lambda = \frac{1}{2}\epsilon_{\lambda\mu\nu} M_{\mu\nu} \tag{4.115}$$

$$\begin{aligned} B_1 &= \frac{1}{2}(\epsilon_{123} M_{23} + \epsilon_{132} M_{32}) \\ &= \frac{1}{2}(M_{23} - M_{32}) \\ &= \partial_2 A_3 - \partial_3 A_2. \end{aligned} \tag{4.116}$$

Using

$$\epsilon_{\mu\nu\alpha}\epsilon_{\sigma\kappa\alpha} = \delta_{\mu\sigma}\delta_{\nu\kappa} - \delta_{\nu\sigma}\delta_{\mu\kappa}, \tag{4.117}$$

we can verify the identity eq. (4.115) by expanding

$$\begin{aligned}
\epsilon_{\mu\nu\lambda} B_\lambda &= \frac{1}{2} \epsilon_{\mu\nu\lambda} \epsilon_{\lambda\alpha\beta} M_{\alpha\beta} \\
&= \frac{1}{2} (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta}) M_{\alpha\beta} \\
&= \frac{1}{2} (M_{\mu\nu} - M_{\nu\mu}) \\
&= M_{\mu\nu}
\end{aligned} \tag{4.118}$$

Returning to the action evaluation we have

$$\frac{d}{dt}(m\gamma v^\alpha) = eE^\alpha + \frac{e}{c}\epsilon_{\alpha\beta\gamma} v^\beta B_\gamma, \tag{4.119}$$

but

$$\epsilon_{\alpha\beta\gamma} B_\gamma = (\mathbf{v} \times \mathbf{B})_\alpha. \tag{4.120}$$

So

$$\frac{d}{dt}(m\gamma \mathbf{v}) = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{4.121}$$

or

$$\frac{d}{dt}(\mathbf{p}) = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{4.122}$$

What is the energy component of the Lorentz force equation I asked this, not because I do not know (I could answer this myself from $dp/d\tau = F \cdot v/c$, in the geometric algebra formalism, but I was curious if he had a way of determining this from what we have derived so far (intuitively I had expect this to be possible). Answer was:

Observe that this is almost a relativistic equation, but we are not going to get to the full equation yet. The energy component can be obtained from

$$\frac{du^0}{ds} = eF^{0j} u_j \tag{4.123}$$

Since the full equation is

$$\frac{du^i}{ds} = eF^{ij} u_j \tag{4.124}$$

"take with a grain of salt, may be off by sign, or factors of c".

Also curious is that he claimed the energy component of this equation was not very important. Why would that be?

4.13 GAUGE TRANSFORMATIONS

Claim

$$S_{\text{interaction}} = -\frac{e}{c} \int ds u^i A_i \tag{4.125}$$

changes by boundary terms only under
"gauge transformation" :

$$A_i = A'_i + \frac{\partial \chi}{\partial x^i} \tag{4.126}$$

where χ is a Lorentz scalar. This $\partial/\partial x^i$ is the four gradient. Let us see this

Therefore the equations of motion are the same in an external A^i and A'^i.

Recall that the **E** and **B** fields do not change under such transformations. Let us see how the action transforms

$$\begin{aligned} S &= -\frac{e}{c} \int ds u^i A_i \\ &= -\frac{e}{c} \int ds u^i \left(A'_i + \frac{\partial \chi}{\partial x^i} \right) \\ &= -\frac{e}{c} \int ds u^i A'_i - \frac{e}{c} \int ds \frac{dx^i}{ds} \frac{\partial \chi}{\partial x^i} \end{aligned} \tag{4.127}$$

Observe that this last bit is just a chain rule expansion

$$\begin{aligned} \frac{d}{ds} \chi(x^0, x^1, x^2, x^3) &= \frac{\partial \chi}{\partial x^0} \frac{dx^0}{ds} + \frac{\partial \chi}{\partial x^1} \frac{dx^1}{ds} + \frac{\partial \chi}{\partial x^2} \frac{dx^2}{ds} + \frac{\partial \chi}{\partial x^3} \frac{dx^3}{ds} \\ &= \frac{\partial \chi}{\partial x^i} \frac{dx^i}{ds}, \end{aligned} \tag{4.128}$$

so we have

$$S = -\frac{e}{c} \int ds u^i A'_i - \frac{e}{c} \int ds \frac{d\chi}{ds}. \tag{4.129}$$

This allows the line integral to be evaluated, and we find that it only depends on the end points of the interval

$$S = -\frac{e}{c}\int ds u^i {A'}_i - \frac{e}{c}(\chi(x_b) - \chi(x_a)), \tag{4.130}$$

which completes the proof of the claim that this gauge transformation results in an action difference that only depends on the end points of the interval.

Gauge invariance of $A \cdot A$ action Now that we know what gauge invariance means, let us look at the portion of the potential action eq. (4.100) discarded because it was not gauge invariant. Under gauge transformation this becomes

$$\begin{aligned}\int ds {A'}^i {A'}_i &= \int ds \left(A_i + \frac{\partial \chi}{\partial x^i}\right)\left(A^i + \frac{\partial \chi}{\partial x_i}\right) \\ &= \int ds A^i A_i + A^i \frac{\partial \chi}{\partial x^i} + A_i \frac{\partial \chi}{\partial x_i} + \frac{\partial \chi}{\partial x^i}\frac{\partial \chi}{\partial x_i} \\ &= \int ds A^i A_i + 2A^i \frac{\partial \chi}{\partial x^i} + \frac{\partial \chi}{\partial x^i}\frac{\partial \chi}{\partial x_i}\end{aligned} \tag{4.131}$$

Without the proper velocity term we do not have a way to simply re-pack the chain rule expansion and eliminate the last two terms as we did with the Lorentz force action.

Reading Covering chapter 3 material from the text [11], and lecture notes RelEMpp74-83.pdf.

4.14 WHAT IS THE SIGNIFICANCE TO THE GAUGE INVARIANCE OF THE ACTION?

We had argued that under a gauge transformation

$$A_i \rightarrow A_i + \frac{\partial \chi}{\partial x^i}, \tag{4.132}$$

the action for a particle changes by a boundary term

$$-\frac{e}{c}(\chi(x_b) - \chi(x_a)). \tag{4.133}$$

Because S changes by a boundary term only, variation problem is not affected. The extremal trajectories are then the same, hence the EOM are the same.

A less high brow demonstration With our four potential split into space and time components

$$A^i = (\phi, \mathbf{A}), \tag{4.134}$$

the lower index representation of the same vector is

$$A_i = (\phi, -\mathbf{A}). \tag{4.135}$$

Our gauge transformation is then

$$\begin{aligned} A_0 &\rightarrow A_0 + \frac{\partial \chi}{\partial x^0} \\ -\mathbf{A} &\rightarrow -\mathbf{A} + \frac{\partial \chi}{\partial \mathbf{x}} \end{aligned} \tag{4.136}$$

or

$$\begin{aligned} \phi &\rightarrow \phi + \frac{1}{c}\frac{\partial \chi}{\partial t} \\ \mathbf{A} &\rightarrow \mathbf{A} - \boldsymbol{\nabla}\chi. \end{aligned} \tag{4.137}$$

Now observe how the electric and magnetic fields are transformed

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ &\rightarrow -\boldsymbol{\nabla}\left(\phi + \frac{1}{c}\frac{\partial \chi}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{A} - \boldsymbol{\nabla}\chi\right) \end{aligned} \tag{4.138}$$

Sufficient continuity of χ is assumed, allowing commutation of the space and time derivatives, and we are left with just $\mathbf{E}$

For the magnetic field we have

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\ &\rightarrow \boldsymbol{\nabla} \times (\mathbf{A} - \boldsymbol{\nabla}\chi) \end{aligned} \tag{4.139}$$

Again with continuity assumptions, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\chi) = 0$, and we are left with just $\mathbf{B}$. The electromagnetic fields (as opposed to potentials) do not change under gauge transformations.

We conclude that the $\{A_i\}$ description is hugely redundant, but despite that, local $\mathcal{L}$ and H can only be written in terms of the potentials A_i.

Energy term of the Lorentz force. Three vector approach With the Lagrangian for the particle given by

$$\mathcal{L} = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} + \frac{e}{c}\mathbf{A} \cdot \mathbf{v} - e\phi, \tag{4.140}$$

we define the energy as

$$\mathcal{E} = \mathbf{v} \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \mathcal{L} \tag{4.141}$$

This is not necessarily a conserved quantity, but we define it as the energy anyways (we do not really have a Hamiltonian when the fields are time dependent). Associated with this quantity is the general relationship

$$\frac{d\mathcal{E}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}, \tag{4.142}$$

and when the Lagrangian is invariant with respect to time translation the energy $\mathcal{E}$ will be a conserved quantity (and also the Hamiltonian).

Our canonical momentum is

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \gamma m \mathbf{v} + \frac{e}{c}\mathbf{A} \tag{4.143}$$

So our energy is

$$\mathcal{E} = \gamma m \mathbf{v}^2 + \frac{e}{c}\mathbf{A} \cdot \mathbf{v} - \left(-mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} + \frac{e}{c}\mathbf{A} \cdot \mathbf{v} - e\phi \right). \tag{4.144}$$

Or

$$\mathcal{E} = \underbrace{\boxed{\frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}}}_{(*)} + e\phi. \tag{4.145}$$

The contribution of $(*)$ to the energy $\mathcal{E}$ comes from the free (kinetic) particle portion of the Lagrangian $\mathcal{L} = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}$, and we identify the remainder as a potential energy

$$\mathcal{E} = \frac{mc^2}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} + \underbrace{\boxed{e\phi}}_{\text{"potential"}}. \tag{4.146}$$

For the kinetic portion we can also show that we have

$$\frac{d}{dt}\mathcal{E}_{\text{kinetic}} = \frac{mc^2}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} = e\mathbf{E}\cdot\mathbf{v}. \tag{4.147}$$

To show this observe that we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_{\text{kinetic}} &= mc^2\frac{d\gamma}{dt} \\
&= mc^2\frac{d}{dt}\frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \\
&= mc^2\frac{\frac{\mathbf{v}}{c^2}\cdot\frac{d\mathbf{v}}{dt}}{\left(1-\frac{\mathbf{v}^2}{c^2}\right)^{3/2}} \\
&= \frac{m\gamma\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}}{1-\frac{\mathbf{v}^2}{c^2}}
\end{aligned} \tag{4.148}$$

We also have

$$\begin{aligned}
\mathbf{v}\cdot\frac{d\mathbf{p}}{dt} &= \mathbf{v}\cdot\frac{d}{dt}\frac{m\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \\
&= m\mathbf{v}^2\frac{d\gamma}{dt} + m\gamma\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \\
&= m\mathbf{v}^2\frac{d\gamma}{dt} + mc^2\frac{d\gamma}{dt}\left(1-\frac{\mathbf{v}^2}{c^2}\right) \\
&= mc^2\frac{d\gamma}{dt}.
\end{aligned} \tag{4.149}$$

Utilizing the Lorentz force equation, we have

$$\mathbf{v}\cdot\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\mathbf{v} = e\mathbf{E}\cdot\mathbf{v} \tag{4.150}$$

and are able to assemble the above, and find that we have

$$\frac{d(mc^2\gamma)}{dt} = e\mathbf{E}\cdot\mathbf{v} \tag{4.151}$$

4.15 FOUR VECTOR LORENTZ FORCE

Using $ds = \sqrt{dx^i dx_i}$ our action can be rewritten

$$\begin{aligned} S &= \int \left(-mcds - \frac{e}{c} u^i A_i ds \right) \\ &= \int \left(-mcds - \frac{e}{c} dx^i A_i \right) \\ &= \int \left(-mc\sqrt{dx^i dx_i} - \frac{e}{c} dx^i A_i \right) \end{aligned} \tag{4.152}$$

$x^i(\tau)$ is a worldline $x^i(0) = a^i$, $x^i(1) = b^i$,
We want $\delta S = S[x + \delta x] - S[x] = 0$ (to linear order in δx)
The variation of our proper length is

$$\begin{aligned} \delta ds &= \delta \sqrt{dx^i dx_i} \\ &= \frac{1}{2\sqrt{dx^i dx_i}} \delta(dx^j dx_j) \end{aligned} \tag{4.153}$$

Observe that for the numerator we have

$$\begin{aligned} \delta(dx^j dx_j) &= \delta(dx^j g_{jk} dx^k) \\ &= \delta(dx^j) g_{jk} dx^k + dx^j g_{jk} \delta(dx^k) \\ &= \delta(dx^j) g_{jk} dx^k + dx^k g_{kj} \delta(dx^j) \\ &= 2\delta(dx^j) g_{jk} dx^k \\ &= 2\delta(dx^j) dx_j \end{aligned} \tag{4.154}$$

TIP: If this goes too quick, or there is any disbelief, write these all out explicitly as $dx^j dx_j = dx^0 dx_0 + dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3$ and compute it that way.
For the four vector potential our variation is

$$\delta A_i = A_i(x + \delta x) - A_i = \frac{\partial A_i}{\partial x^j} \delta x^j = \partial_j A_i \delta x^j \tag{4.155}$$

(i.e. By chain rule)

Completing the proper length variations above we have

$$\begin{aligned}\delta\sqrt{dx^i dx_i} &= \frac{1}{\sqrt{dx^i dx_i}}\delta(dx^j)dx_j \\ &= \delta(dx^j)\frac{dx_j}{ds} \\ &= \delta(dx^j)u_j \\ &= d\delta x^j u_j\end{aligned} \tag{4.156}$$

We are now ready to assemble results and do the integration by parts

$$\begin{aligned}\delta S &= \int \left(-mcd(\delta x^j)u_j - \frac{e}{c}d(\delta x^i)A_i - \frac{e}{c}dx^i\partial_j A_i \delta x^j\right) \\ &= \left(-mc(\delta x^j)u_j - \frac{e}{c}(\delta x^i)A_i\right)\Big|_a^b + \int \left(mc\delta x^j du_j + \frac{e}{c}(\delta x^i)dA_i - \frac{e}{c}dx^i\partial_j A_i \delta x^j\right)\end{aligned} \tag{4.157}$$

Our variation at the endpoints is zero $\delta x^i\big|_a = \delta x^i\big|_b = 0$, killing the non-integral terms

$$\delta S = \int \delta x^j \left(mcdu_j + \frac{e}{c}dA_j - \frac{e}{c}dx^i\partial_j A_i\right). \tag{4.158}$$

Observe that our differential can also be expanded by chain rule

$$dA_j = \frac{\partial A_j}{\partial x^i}dx^i = \partial_i A_j dx^i, \tag{4.159}$$

which simplifies the variation further

$$\begin{aligned}\delta S &= \int \delta x^j \left(mcdu_j + \frac{e}{c}dx^i(\partial_i A_j - \partial_j A_i)\right) \\ &= \int \delta x^j ds \left(mc\frac{du_j}{ds} + \frac{e}{c}u^i(\partial_i A_j - \partial_j A_i)\right)\end{aligned} \tag{4.160}$$

Since this is true for all variations δx^j, which is arbitrary, the interior part is zero everywhere in the trajectory. The antisymmetric portion, a rank 2 4-tensor is called the electromagnetic field strength tensor, and written

$$\boxed{F_{ij} = \partial_i A_j - \partial_j A_i.} \tag{4.161}$$

In matrix form this is

$$\left\|F_{ij}\right\| = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{4.162}$$

In terms of the field strength tensor our Lorentz force equation takes the form

$$\boxed{\frac{d(mcu_i)}{ds} = \frac{e}{c} F_{ij} u^j.} \tag{4.163}$$

Reading Covering chapter 3 material from the text [11], lecture notes RelEMpp74-83.pdf, and lecture notes RelEMpp84-102.pdf.

4.16 CHEWING ON THE FOUR VECTOR FORM OF THE LORENTZ FORCE EQUATION

After much effort, we arrived at

$$\frac{d(mcu_l)}{ds} = \frac{e}{c}\left(\partial_l A_i - \partial_i A_l\right) u^i \tag{4.164}$$

or

$$\frac{dp_l}{ds} = \frac{e}{c} F_{li} u^i \tag{4.165}$$

Elements of the strength tensor

Claim : there are only 6 independent elements of this matrix (tensor)

$$\begin{bmatrix} 0 & . & . & . \\ & 0 & . & . \\ & & 0 & . \\ & & & 0 \end{bmatrix} \tag{4.166}$$

This is a no-brainer, for we just have to mechanically plug in the elements of the field strength tensor

Recall

$$\begin{aligned} A^i &= (\phi, \mathbf{A}) \\ A_i &= (\phi, -\mathbf{A}) \end{aligned} \tag{4.167}$$

$$\begin{aligned} F_{0\alpha} &= \partial_0 A_\alpha - \partial_\alpha A_0 \\ &= -\partial_0 (\mathbf{A})_\alpha - \partial_\alpha \phi \end{aligned} \tag{4.168}$$

$$F_{0\alpha} = E_\alpha \tag{4.169}$$

For the purely spatial index combinations we have

$$\begin{aligned} F_{\alpha\beta} &= \partial_\alpha A_\beta - \partial_\beta A_\alpha \\ &= -\partial_\alpha (\mathbf{A})_\beta + \partial_\beta (\mathbf{A})_\alpha \end{aligned} \tag{4.170}$$

Written out explicitly, these are

$$\begin{aligned} F_{12} &= \partial_2 (\mathbf{A})_1 - \partial_1 (\mathbf{A})_2 \\ F_{23} &= \partial_3 (\mathbf{A})_2 - \partial_2 (\mathbf{A})_3 \\ F_{31} &= \partial_1 (\mathbf{A})_3 - \partial_3 (\mathbf{A})_1. \end{aligned} \tag{4.171}$$

We can compare this to the elements of **B**

$$\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_1 & \partial_2 & \partial_3 \\ A_x & A_y & A_z \end{vmatrix} \tag{4.172}$$

We see that

$$\begin{aligned} (\mathbf{B})_z &= \partial_1 A_y - \partial_2 A_x \\ (\mathbf{B})_x &= \partial_2 A_z - \partial_3 A_y \\ (\mathbf{B})_y &= \partial_3 A_x - \partial_1 A_z \end{aligned} \tag{4.173}$$

So we have

$$\begin{aligned} F_{12} &= -(\mathbf{B})_3 \\ F_{23} &= -(\mathbf{B})_1 \\ F_{31} &= -(\mathbf{B})_2. \end{aligned} \tag{4.174}$$

These can be summarized as simply

$$F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} B_\gamma. \tag{4.175}$$

This provides all the info needed to fill in the matrix above

$$\left\| F_{ij} \right\| = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0. \end{bmatrix}. \tag{4.176}$$

Index raising of rank 2 tensor To raise indices we compute

$$F^{ij} = g^{il} g^{jk} F_{lk}. \tag{4.177}$$

Justifying the raising operation To justify this consider raising one index at a time by applying the metric tensor to our definition of F_{lk}. That is

$$\begin{aligned} g^{al} F_{lk} &= g^{al} (\partial_l A_k - \partial_k A_l) \\ &= \partial^a A_k - \partial_k A^a. \end{aligned} \tag{4.178}$$

Now apply the metric tensor once more

$$\begin{aligned} g^{bk} g^{al} F_{lk} &= g^{bk} (\partial^a A_k - \partial_k A^a) \\ &= \partial^a A^b - \partial^b A^a. \end{aligned} \tag{4.179}$$

This is, by definition F^{ab}. Since a rank 2 tensor has been defined as an object that transforms like the product of two pairs of coordinates, it makes sense that this particular tensor raises in the same fashion as would a product of two vector coordinates (in this case, it happens to be an antisymmetric product of two vectors, and one of which is an operator, but we have the same idea).

Consider the components of the raised F_{ij} tensor

$$\begin{aligned} F^{0\alpha} &= -F_{0\alpha} \\ F^{\alpha\beta} &= F_{\alpha\beta}. \end{aligned} \tag{4.180}$$

$$\left\| F^{ij} \right\| = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{4.181}$$

Back to chewing on the Lorentz force equation

$$mc\frac{du_i}{ds} = \frac{e}{c}F_{ij}u^j \tag{4.182}$$

$$\begin{aligned} u^i &= \gamma\left(1, \frac{\mathbf{v}}{c}\right) \\ u_i &= \gamma\left(1, -\frac{\mathbf{v}}{c}\right) \end{aligned} \tag{4.183}$$

For the spatial components of the Lorentz force equation we have

$$\begin{aligned} mc\frac{du_\alpha}{ds} &= \frac{e}{c}F_{\alpha j}u^j \\ &= \frac{e}{c}F_{\alpha 0}u^0 + \frac{e}{c}F_{\alpha\beta}u^\beta \\ &= \frac{e}{c}(-E_\alpha)\gamma + \frac{e}{c}(-\epsilon_{\alpha\beta\gamma}B_\gamma)\frac{v^\beta}{c}\gamma \end{aligned} \tag{4.184}$$

But

$$\begin{aligned} mc\frac{du_\alpha}{ds} &= -m\frac{d(\gamma\mathbf{v}_\alpha)}{ds} \\ &= -m\frac{d(\gamma\mathbf{v}_\alpha)}{c\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}dt} \\ &= -\gamma\frac{d(m\gamma\mathbf{v}_\alpha)}{cdt}. \end{aligned} \tag{4.185}$$

Canceling the common $-\gamma/c$ terms, and switching to vector notation, we are left with

$$\frac{d(m\gamma\mathbf{v}_\alpha)}{dt} = e\left(E_\alpha + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_\alpha\right). \tag{4.186}$$

Now for the energy term. We have

$$\begin{aligned} mc\frac{du_0}{ds} &= \frac{e}{c}F_{0\alpha}u^\alpha \\ &= \frac{e}{c}E_\alpha \gamma \frac{v^\alpha}{c} \\ \frac{dmc\gamma}{ds} &= \end{aligned} \tag{4.187}$$

Putting the final two lines into vector form we have

$$\frac{d(mc^2\gamma)}{dt} = e\mathbf{E}\cdot\mathbf{v}, \tag{4.188}$$

or

$$\frac{d\mathcal{E}}{dt} = e\mathbf{E}\cdot\mathbf{v} \tag{4.189}$$

4.17 TRANSFORMATION OF RANK TWO TENSORS IN MATRIX AND INDEX FORM

Transformation of the metric tensor, and some identities With

$$\hat{G} = \left\|g_{ij}\right\| = \left\|g^{ij}\right\| \tag{4.190}$$

We claim: The rank two tensor $\hat{G}$ transforms in the following sort of sandwich operation, and this leaves it invariant

$$\hat{G} \rightarrow \hat{O}\hat{G}\hat{O}^{\mathrm{T}} = \hat{G}. \tag{4.191}$$

To demonstrate this let us consider a transformed vector in coordinate form as follows

$$\begin{aligned} x'^i &= O^{ij}x_j = O^i{}_j x^j \\ x'_i &= O_{ij}x^j = O_i{}^j x_j. \end{aligned} \tag{4.192}$$

We can thus write the equation in matrix form with

$$\begin{aligned} X &= \left\|x^i\right\| \\ X' &= \left\|x'^i\right\| \\ \hat{O} &= \left\|O^i{}_j\right\| \\ X' &= \hat{O}X \end{aligned} \tag{4.193}$$

Our invariant for the vector square, which is required to remain unchanged is

$$\begin{aligned} x'^i x'_i &= (O^{ij} x_j)(O_{ik} x^k) \\ &= x^k (O^{ij} O_{ik}) x_j. \end{aligned} \tag{4.194}$$

This shows that we have a delta function relationship for the Lorentz transform matrix, when we sum over the first index

$$O^{ai} O_{aj} = \delta^i{}_j. \tag{4.195}$$

It appears we can put eq. (4.195) into matrix form as

$$\hat{G}\hat{O}^{\mathrm{T}}\hat{G}\hat{O} = I \tag{4.196}$$

Now, if one considers that the transpose of a rotation is an inverse rotation, and the transpose of a boost leaves it unchanged, the transpose of a general Lorentz transformation, a composition of an arbitrary sequence of boosts and rotations, must also be a Lorentz transformation, and must then also leave the norm unchanged. For the transpose of our Lorentz transformation $\hat{O}$ lets write

$$\hat{P} = \hat{O}^{\mathrm{T}} \tag{4.197}$$

For the action of this on our position vector let us write

$$\begin{aligned} x''^i &= P^{ij} x_j = O^{ji} x_j \\ x''_i &= P_{ij} x^j = O_{ji} x^j \end{aligned} \tag{4.198}$$

so that our norm is

$$\begin{aligned} x''^a x''_a &= (O_{ka} x^k)(O^{ja} x_j) \\ &= x^k (O_{ka} O^{ja}) x_j \\ &= x^j x_j \end{aligned} \tag{4.199}$$

We must then also have an identity when summing over the second index

$$\delta_k{}^j = O_{ka} O^{ja} \tag{4.200}$$

Armed with these facts on the products of O_{ij} and O^{ij} we can now consider the transformation of the metric tensor.

The rule (definition) supplied to us for the transformation of an arbitrary rank two tensor, is that this transforms as its indices transform individually. Very much as if it was the product of two coordinate vectors and we transform those coordinates separately. Doing so for the metric tensor we have

$$\begin{aligned} g^{ij} &\to O^i{}_k g^{km} O^j{}_m \\ &= (O^i{}_k g^{km}) O^j{}_m \\ &= O^{im} O^j{}_m \\ &= O^{im} (O_{am} g^{aj}) \\ &= (O^{im} O_{am}) g^{aj} \end{aligned} \tag{4.201}$$

However, by eq. (4.200), we have $O_{am} O^{im} = \delta_a{}^i$, and we prove that

$$g^{ij} \to g^{ij}. \tag{4.202}$$

Finally, we wish to put the above transformation in matrix form, look more carefully at the very first line

$$g^{ij} \to O^i{}_k g^{km} O^j{}_m \tag{4.203}$$

which is

$$\hat{G} \to \hat{O}\hat{G}\hat{O}^{\mathrm{T}} = \hat{G} \tag{4.204}$$

We see that this particular form of transformation, a sandwich between $\hat{O}$ and $\hat{O}^{\mathrm{T}}$, leaves the metric tensor invariant.

Lorentz transformation of the electrodynamic tensor Having identified a composition of Lorentz transformation matrices, when acting on the metric tensor, leaves it invariant, it is a reasonable question to ask how this form of transformation acts on our electrodynamic tensor F^{ij}?

Claim: A transformation of the following form is required to maintain the norm of the Lorentz force equation

$$\hat{F} \to \hat{O}\hat{F}\hat{O}^{\mathrm{T}}, \tag{4.205}$$

where $\hat{F} = \left\| F^{ij} \right\|$. Observe that our Lorentz force equation can be written exclusively in upper index quantities as

$$mc\frac{du^i}{ds} = \frac{e}{c}F^{ij}g_{jl}u^l \tag{4.206}$$

Because we have a vector on one side of the equation, and it transforms by multiplication with by a Lorentz matrix in SO(1,3)

$$\frac{du^i}{ds} \rightarrow \hat{O}\frac{du^i}{ds} \tag{4.207}$$

The LHS of the Lorentz force equation provides us with one invariant

$$(mc)^2\frac{du^i}{ds}\frac{du_i}{ds} \tag{4.208}$$

so the RHS must also provide one

$$\frac{e^2}{c^2}F^{ij}g_{jl}u^l F_{ik}g^{km}u_m = \frac{e^2}{c^2}F^{ij}u_j F_{ik}u^k. \tag{4.209}$$

Let us look at the RHS in matrix form. Writing

$$U = \left\| u^i \right\|, \tag{4.210}$$

we can rewrite the Lorentz force equation as

$$mc\dot{U} = \frac{e}{c}\hat{F}\hat{G}U. \tag{4.211}$$

In this matrix formalism our invariant eq. (4.209) is

$$\frac{e^2}{c^2}(\hat{F}\hat{G}U)^{\mathrm{T}}\hat{G}\hat{F}\hat{G}U = \frac{e^2}{c^2}U^{\mathrm{T}}\hat{G}\hat{F}^{\mathrm{T}}\hat{G}\hat{F}\hat{G}U. \tag{4.212}$$

If we compare this to the transformed Lorentz force equation we have

$$mc\hat{O}\dot{U} = \frac{e}{c}\hat{F}'\hat{G}\hat{O}U. \tag{4.213}$$

Our invariant for the transformed equation is

$$\frac{e^2}{c^2}(\hat{F}'\hat{G}\hat{O}U)^{\mathrm{T}}\hat{G}\hat{F}'\hat{G}\hat{O}U = \frac{e^2}{c^2}U^{\mathrm{T}}\hat{O}^{\mathrm{T}}\hat{G}\hat{F}'^{\mathrm{T}}\hat{G}\hat{F}'\hat{G}\hat{O}U \tag{4.214}$$

Thus the transformed electrodynamic tensor $\hat{F}'$ must satisfy the identity

$$\hat{O}^{\mathrm{T}}\hat{G}\hat{F}'^{\mathrm{T}}\hat{G}\hat{F}'\hat{G}\hat{O} = \hat{G}\hat{F}^{\mathrm{T}}\hat{G}\hat{F}\hat{G} \tag{4.215}$$

With the substitution $\hat{F}' = \hat{O}\hat{F}\hat{O}^{\mathrm{T}}$ the LHS is

$$\begin{aligned}\hat{O}^{\mathrm{T}}\hat{G}\hat{F}'^{\mathrm{T}}\hat{G}\hat{F}'\hat{G}\hat{O} &= \hat{O}^{\mathrm{T}}\hat{G}(\hat{O}\hat{F}\hat{O}^{\mathrm{T}})^{\mathrm{T}}\hat{G}(\hat{O}\hat{F}\hat{O}^{\mathrm{T}})\hat{G}\hat{O} \\ &= (\hat{O}^{\mathrm{T}}\hat{G}\hat{O})\hat{F}^{\mathrm{T}}(\hat{O}^{\mathrm{T}}\hat{G}\hat{O})\hat{F}(\hat{O}^{\mathrm{T}}\hat{G}\hat{O})\end{aligned} \tag{4.216}$$

We have argued that $\hat{P} = \hat{O}^{\mathrm{T}}$ is also a Lorentz transformation, thus

$$\begin{aligned}\hat{O}^{\mathrm{T}}\hat{G}\hat{O} &= \hat{P}\hat{G}\hat{O}^{\mathrm{T}} \\ &= \hat{G}\end{aligned} \tag{4.217}$$

This is enough to make both sides of eq. (4.215) match, verifying that this transformation does provide the invariant properties desired.

Direct computation of the Lorentz transformation of the electrodynamic tensor We can construct the transformed field tensor more directly, by simply transforming the coordinates of the four gradient and the four potential directly. That is

$$\begin{aligned}F^{ij} = \partial^i A^j - \partial^j A^i &\rightarrow O^i{}_a O^j{}_b \left(\partial^a A^b - \partial^b A^a\right) \\ &= O^i{}_a F^{ab} O^j{}_b\end{aligned} \tag{4.218}$$

By inspection we can see that this can be represented in matrix form as

$$\hat{F} \rightarrow \hat{O}\hat{F}\hat{O}^{\mathrm{T}} \tag{4.219}$$

Reading Covering chapter 3 material from the text [11], and lecture notes RelEMpp84-102.pdf.

4.18 WHERE WE ARE

$$F_{ij} = \partial_i A_j - \partial_j A_i \tag{4.220}$$

We learned that one half of Maxwell's equations comes from the Bianchi identity

$$\epsilon^{ijkl}\partial_j F_{kl} = 0 \tag{4.221}$$

the other half (for vacuum) is

$$\partial_j F_{ji} = 0 \tag{4.222}$$

To get here we have to consider the action for the field.

4.19 GENERALIZING THE ACTION TO MULTIPLE PARTICLES

We have learned that the action for a single particle is

$$\begin{aligned} S &= S_{\text{matter}} + S_{\text{interaction}} \\ &= -mc \int ds - \frac{e}{c}\int ds^i A_i \end{aligned} \tag{4.223}$$

This generalizes to more particles

$$S_{\text{“particles in field”}} = -\sum_A m_A c \int_{x^A(\tau)} ds - \sum_A \frac{e_A}{c}\int dx^i_A A_i(x_A(\tau)) \tag{4.224}$$

A labels the particles, and $x^A(\tau)$, $\{x^A(\tau), A = 1 \cdots N\}$ is the worldline of particle A.

4.20 PROBLEMS

Exercise 4.1 Energy term of the Lorentz force equation

In class this week, the Lorentz force was derived from an action (the simplest Lorentz invariant, gauge invariant, action that could be constructed)

$$S = -mc \int ds - \frac{e}{c}\int ds A^i u_i. \tag{4.225}$$

We end up with the familiar equation, with the exception that the momentum includes the relativistically required gamma factor

$$\frac{d(\gamma m \mathbf{v})}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{4.226}$$

I asked what the energy term of this equation would be and was answered that we would get to it, and This can be obtained by a four vector minimization of the action which produces the Lorentz force equation of the following form

$$\frac{du^i}{d\tau} \sim e F^{ij} u_j. \tag{4.227}$$

Let us see if we can work this out without the four-vector approach, using the action expressed with an explicit space time split, then also work it out in the four vector form and compare as a consistency check.

a. Lorentz force equations.
Derive the Lorentz force equation from the action eq. (4.225).

b. The power (energy) term.
When we start with an action explicitly constructed with Lorentz invariance as a requirement, it might seem somewhat odd to end up with a result that has only the spatial vector portion of what should logically be a four vector result. We have an equation for the particle momentum, but not one for the energy. We have also calculated the Hamiltonian, the generalization of energy, for the free particle, but have not yet done so for the Lorentz force, or for an action containing potentials. Generalized this by calculating the Hamiltonian for the Lorentz force.

c. Proper time action.
Express the action using a proper time parameterization, and evaluate the Euler-Lagrange equations. Leave the results in four vector notation.

d. Power term.
From the four vector expression derived, extract the power term found earlier using a time parameterized action.

e. The Lorentz force terms.
Now do the same, extracting the Lorentz force terms, and compare to the results found using the time parameterized action.

Answer for Exercise 4.1

Part a. Parameterizing the action by time we have

$$
\begin{aligned}
S &= -mc^2 \int dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - e \int dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \gamma \left(1, \frac{1}{c}\mathbf{v}\right) \cdot (\phi, \mathbf{A}) \\
&= -mc^2 \int dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - e \int dt \left(\phi - \frac{1}{c}\mathbf{A} \cdot \mathbf{v}\right)
\end{aligned}
\tag{4.228}
$$

Our Lagrangian is therefore

$$
\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = -mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - e\phi(\mathbf{x}, t) + \frac{e}{c}\mathbf{A}(\mathbf{x}, t) \cdot \mathbf{v}
\tag{4.229}
$$

We can calculate our conjugate momentum easily enough

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \gamma m \mathbf{v} + \frac{e}{c}\mathbf{A},
\tag{4.230}
$$

and for the gradient portion of the Euler-Lagrange equations we have

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = -e\boldsymbol{\nabla}\phi + e\boldsymbol{\nabla}\left(\frac{\mathbf{v}}{c} \cdot \mathbf{A}\right).
\tag{4.231}
$$

Utilizing the convective derivative (i.e. chain rule in fancy clothes)

$$
\frac{d}{dt} = \mathbf{v} \cdot \boldsymbol{\nabla} + \frac{\partial}{\partial t}.
\tag{4.232}
$$

This gives us

$$
-e\boldsymbol{\nabla}\phi + e\boldsymbol{\nabla}\left(\frac{\mathbf{v}}{c} \cdot \mathbf{A}\right) = \frac{d(\gamma m \mathbf{v})}{dt} + \frac{e}{c}(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t},
\tag{4.233}
$$

and a final bit of rearranging gives us

$$
\frac{d(\gamma m \mathbf{v})}{dt} = e\left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) + \frac{e}{c}\left(\boldsymbol{\nabla}\left(\mathbf{v} \cdot \mathbf{A}\right) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}\right).
\tag{4.234}
$$

The first set of derivatives we identify with the electric field $\mathbf{E}$. For the second, utilizing the vector triple product identity from [19], gives

$$
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},
\tag{4.235}
$$

which we recognize as related to the magnetic field $\mathbf{v} \times \mathbf{B} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})$.

Part b. We can only actually calculate a Hamiltonian for the case where $\phi(\mathbf{x}, t) = \phi(\mathbf{x})$ and $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x})$, because when the potentials have any sort of time dependence we do not have a Lagrangian that is invariant under time translation. Returning to the derivation of the Hamiltonian conservation equation, we see that we must modify the argument slightly when there is a time dependence and get instead

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}} \cdot \mathbf{v} - \mathcal{L}\right) + \frac{\partial \mathcal{L}}{\partial t} = 0. \tag{4.236}$$

Only when there is no time dependence in the Lagrangian, do we have our conserved quantity, what we label as energy, or Hamiltonian.

From eq. (4.230), we have

$$0 = \frac{d}{dt}\left(\left(\gamma m\mathbf{v} + \frac{e}{c}\mathbf{A}\right) \cdot \mathbf{v} + mc^2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} + e\phi - \frac{e}{c}\mathbf{A} \cdot \mathbf{v}\right) - e\frac{\partial \phi}{\partial t} + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{v} \tag{4.237}$$

Our $\mathbf{A} \cdot \mathbf{v}$ terms cancel, and we can combine the γ and γ^{-1} terms, then apply the convective derivative again

$$\begin{aligned}
\frac{d}{dt}\left(\gamma mc^2\right) &= -e\left(\mathbf{v} \cdot \boldsymbol{\nabla} + \frac{\partial}{\partial t}\right)\phi + e\frac{\partial \phi}{\partial t} - \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{v} \\
&= -e\mathbf{v} \cdot \boldsymbol{\nabla}\phi - \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{v} \\
&= +e\mathbf{v} \cdot \left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right).
\end{aligned} \tag{4.238}$$

This is just

$$\frac{d}{dt}\left(\gamma mc^2\right) = e\mathbf{v} \cdot \mathbf{E}, \tag{4.239}$$

and we find the rate of change of energy term of our four momentum equation

$$\frac{d}{dt}\left(\frac{E}{c}, \mathbf{p}\right) = e\left(\frac{\mathbf{v}}{c} \cdot \mathbf{E}, \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{4.240}$$

Specified explicitly, this is

$$\frac{d}{dt}\left(\gamma m\left(c, \mathbf{v}\right)\right) = e\left(\frac{\mathbf{v}}{c} \cdot \mathbf{E}, \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{4.241}$$

While this was the result I was looking for, once written it now stands out as incomplete relativistically. We have an equation that specifies the time derivative of a four vector. What about the spatial derivatives? We really ought to have a rank two tensor result, and not a four vector result relating the fields and the energy and momentum of the particle. The Lorentz force equation, even when expanded to four vector form, does not seem complete relativistically.

With $u^i = dx^i/ds$, we can rewrite eq. (4.241) as

$$\partial_0(\gamma m u^i) = e\left(\frac{\mathbf{v}}{c} \cdot \mathbf{E}, \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{4.242}$$

If we were to vary the action with respect to a spatial coordinate instead of time, we should end up with a similar equation of the form $\partial_\alpha(\gamma m u^i) = ?$. Having been pointed at the explicitly invariant result, I wonder if those equations are independent. Let us defer exploring this, until at least after calculating the result using a four vector form of the action.

Part c. We can rewrite our action, parameterizing with proper time. This is

$$S = -mc^2 \int d\tau \sqrt{\frac{dx^i}{d\tau}\frac{dx_i}{d\tau}} - \frac{e}{c}\int d\tau A_i \frac{dx^i}{d\tau} \tag{4.243}$$

Writing $\dot{x}^i = dx^i/d\tau$, our Lagrangian is then

$$\mathcal{L}(x^i, \dot{x}^i, \tau) = -mc^2\sqrt{\dot{x}^i \dot{x}_i} - \frac{e}{c} A_i \dot{x}^i \tag{4.244}$$

The Euler-Lagrange equations take the form

$$\frac{\partial \mathcal{L}}{\partial x^i} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^i}. \tag{4.245}$$

Our gradient and conjugate momentum are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x^i} &= -\frac{e}{c}\frac{\partial A_j}{\partial x^i}\dot{x}^j \\ \frac{\partial \mathcal{L}}{\partial \dot{x}^i} &= -m\dot{x}_i - \frac{e}{c}A_i.\end{aligned} \tag{4.246}$$

With our convective derivative taking the form

$$\frac{d}{d\tau} = \dot{x}^i \frac{\partial}{\partial x^i}, \tag{4.247}$$

we have

$$\begin{aligned} m\frac{d^2 x_i}{d\tau^2} &= \frac{e}{c}\frac{\partial A_j}{\partial x^i}\dot{x}^j - \frac{e}{c}\dot{x}^j \frac{\partial A_i}{\partial x^j} \\ &= \frac{e}{c}\dot{x}^j \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) \\ &= \frac{e}{c}\dot{x}^j \left(\partial_i A_j - \partial_j A_i \right) \\ &= \frac{e}{c}\dot{x}^j F_{ij} \end{aligned} \tag{4.248}$$

Our Prof wrote this with indices raised and lowered respectively

$$m\frac{d^2 x^i}{d\tau^2} = \frac{e}{c}F^{ij}\dot{x}_j. \tag{4.249}$$

Following the text [11] he also writes $u^i = dx^i/ds = (1/c)dx^i/d\tau$, and in that form we have

$$\frac{d(mcu^i)}{ds} = \frac{e}{c}F^{ij}u_j. \tag{4.250}$$

Part d. From eq. (4.250), lets extract the $i = 0$ term, relating the rate of change of energy to the field and particle velocity. With

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma\frac{d}{dt}, \tag{4.251}$$

we have

$$\frac{d(m\gamma \frac{dx^i}{dt})}{dt} = \frac{e}{c}F^{ij}\frac{dx_j}{dt}. \tag{4.252}$$

For $i = 0$ we have

$$F^{0j}\frac{dx_j}{dt} = -F^{0\alpha}\frac{dx^\alpha}{dt} \tag{4.253}$$

That component of the field is

$$\begin{aligned} F^{\alpha 0} &= \partial^\alpha A^0 - \partial^0 A^\alpha \\ &= -\frac{\partial \phi}{\partial x^\alpha} - \frac{1}{c}\frac{\partial A^\alpha}{\partial t} \\ &= \left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right)^\alpha . \end{aligned} \tag{4.254}$$

This verifies the result obtained with considerably more difficulty, using the Hamiltonian like conservation relation obtained for a time translation of a time dependent Lagrangian

$$\frac{d(m\gamma c^2)}{dt} = e\mathbf{E} \cdot \mathbf{v}. \tag{4.255}$$

Part e. Let us also verify the signs for the $i > 0$ terms. For those we have

$$\begin{aligned}\frac{d(m\gamma \frac{dx^\alpha}{dt})}{dt} &= \frac{e}{c} F^{\alpha j} \frac{dx_j}{dt} \\ &= \frac{e}{c} F^{\alpha 0} \frac{dx_0}{dt} + \frac{e}{c} F^{\alpha\beta} \frac{dx_\beta}{dt} \\ &= eE^\alpha - \sum_{\alpha\beta} \frac{e}{c} \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha \right) v^\beta\end{aligned} \tag{4.256}$$

Since we have only spatial indices left, lets be sloppy and imply summation over all repeated indices, even if unmatched upper and lower. This leaves us with

$$\begin{aligned}-\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha \right) v^\beta &= \left(\partial_\alpha A^\beta - \partial_\beta A^\alpha \right) v^\beta \\ &= \epsilon_{\alpha\beta\gamma} B^\gamma\end{aligned} \tag{4.257}$$

With the v^β contraction we have

$$\epsilon_{\alpha\beta\gamma} B^\gamma v^\beta = (\mathbf{v} \times \mathbf{B})^\alpha, \tag{4.258}$$

leaving our first result obtained by the time parametrization of the Lagrangian

$$\frac{d(m\gamma \mathbf{v})}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right). \tag{4.259}$$

This now has a nice symmetrical form. It is slightly disappointing not to have a rank two tensor on the LHS like we have with the symmetric stress tensor with Poynting Vector and energy and other similar terms that relates field energy and momentum with $\mathbf{E} \cdot \mathbf{J}$ and the charge density equivalents of the Lorentz force equation. Is there such a symmetric relationship for particles too?

5 ACTION FOR THE FIELD

5.1 ACTION FOR THE FIELD

However, **E** and **B** are created by charged particles and can "move" or "propagate" on their own. EM field is its own dynamical system. The variables are $A^i(\mathbf{x}, t)$. These are the "$q_a(t)$".

The values of $\{A^i(\mathbf{x}, t), \forall \mathbf{x}\}$ is the dynamical degrees of freedom. This is a system with a continuum of dynamical degrees of freedom.

We need to write an action for this continuous field system $A^i(\mathbf{x}, t)$, and need some principles to guide the construction of this action.

When we have an action with many degrees of freedom, we sum over all the particles. The action for the electromagnetic field

$$S_{\text{EM field}} = \int dt \int d^3\mathbf{x} \mathcal{L}(A^i(\mathbf{x}, t)) \tag{5.1}$$

The quantity

$$\mathcal{L}(A^i(\mathbf{x}, t)) \tag{5.2}$$

is called the Lagrangian density, since the quantity

$$\int d^3\mathbf{x} \mathcal{L}(A^i(\mathbf{x}, t)) \tag{5.3}$$

is actually the Lagrangian.

While this may seem non-relativistic, with both t and $\mathbf{x}$ in the integration range, because we have both, it is actually relativistic. We are integrating over all of spacetime, or the region where the EM fields are non-zero.

We write

$$\int d^4x = c \int dt \int d^3\mathbf{x}, \tag{5.4}$$

which is a Lorentz scalar.

We write our action as

$$S_{\text{EM field}} = \int d^4 x \mathcal{L}(A^i(\mathbf{x}, t)) \tag{5.5}$$

and demand that the Lagrangian density $\mathcal{L}$ must also be an invariant (Lorentz) scalar in $SO(1,3)$.

Analogy : 3D rotations

$$\int d^3\mathbf{x}\phi(\mathbf{x}) \tag{5.6}$$

Here ϕ is a 3-scalar, invariant under rotations.

Principles for the action

1. Relativity.
2. Gauge invariance. Whatever $\mathcal{L}$ we write, it must be gauge invariant, implying that it be a function of F_{ij} only. Recall that we can adjust A^i by a four-gradient of any scalar, but the quantities **E** and **B** were gauge invariant, and so F^{ij} must also be.

   If we do not impose gauge invariance, then the resulting dynamical system will contain more than just **E** and **B**. i.e. It will not be electromagnetism.
3. Superposition principle. The sum of two solutions is a solution. This implies linearity of the equations for A^i.
4. Locality. Could write

   $$\int d^4 x \mathcal{L}_1(A) \int d^4 y \mathcal{L}_2(A) \tag{5.7}$$

   This would allow for fields that have aspects that effect the result from disjoint positions or times. This would probably result in non-causal results as well as the possibility of non-local results.

Principle 1 means we must have

$$\mathcal{L}(A(\mathbf{x}, t)) \tag{5.8}$$

and principle 2

$$\mathcal{L}(F^{ij}(\mathbf{x}, t)) \tag{5.9}$$

and principle 1, means we must have a four scalar.

Without principle 3, we could have products of these, but we rule this out due to violation of non-linearity.

Example 5.1: Lagrangian for the Harmonic oscillator

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \tag{5.10}$$

This gives

$$\ddot{q} \sim q \tag{5.11}$$

However, if we have

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 - \lambda q^3 \tag{5.12}$$

we get

$$\ddot{q} \sim q + q^3 \tag{5.13}$$

In HW3, you will show that

$$\int dt dx \mathbf{E} \cdot \mathbf{B} \tag{5.14}$$

only depends on A^i at ∞ (the boundary). Because this depends only on A^i spatial or time infinities, it can not affect the variational principle.

This is very much like in classical mechanics where we can add any total derivative to the Lagrangian. This does not change the Euler-Lagrange equation evaluation in any way. The $\mathbf{E} \cdot \mathbf{B}$ invariant has the same effect.

The invariants possible are $\mathbf{E}^2 - \mathbf{B}^2$, $(\mathbf{E} \cdot \mathbf{B})^2$, ..., but we are now done, and know what is required. Our action must depend on F squared.

Written in full with the constants in the right places we have

$$S_{\text{“particles in field”}} = \sum_A \left(-m_A c \int_{x^A(\tau)} ds - \frac{e_A}{c} \int dx_A^i A_i(x_A(\tau)) \right) - \frac{1}{16\pi c} \int d^4 x F_{ij} F^{ij} \tag{5.15}$$

To get the equation of motion for $A^i(\mathbf{x}, t)$ we need to vary $S_{\text{int}} + S_{\text{EM field}}$.

5.2 CURRENT DENSITY DISTRIBUTION

Before we do the variation, we want to show that

$$\begin{aligned} S_{\text{int}} &= -\sum_A \frac{e_A}{c} \int_{x_A(\tau)} dx_A^i A_i(x_A(\tau)) \\ &= -\frac{1}{c^2} \int d^4 x A_i(x) j^i(x) \end{aligned} \tag{5.16}$$

where

$$j^i(x) = \sum_A c e_A \int_{x(\tau)} d s u_A^i \delta(x^0 - x_A^0(\tau)) \delta(x^1 - x_A^1(\tau)) \delta(x^2 - x_A^2(\tau)) \delta(x^3 - x_A^3(\tau)). \tag{5.17}$$

We substitute in the integral

$$\begin{aligned} &\sum_A \int d^4 x A_i(x) j^i(x) \\ &= c e_A \sum_A \int d^4 x A_i(x) \\ &\quad \int_{x(\tau)} d s u_A^i \delta(x^0 - x_A^0(\tau)) \delta(x^1 - x_A^1(\tau)) \delta(x^2 - x_A^2(\tau)) \delta(x^3 - x_A^3(\tau)) \\ &= c e_A \sum_A \int d^4 x \\ &\quad \int_{x(\tau)} d x_A^i A_i(x) \delta(x^0 - x_A^0(\tau)) \delta(x^1 - x_A^1(\tau)) \delta(x^2 - x_A^2(\tau)) \delta(x^3 - x_A^3(\tau)) \\ &= c e_A \sum_A \\ &\quad \int_{x_A(\tau)} d x_A^i A_i(x_A(\tau)) \end{aligned} \tag{5.18}$$

From this we see that we have

$$S_{\text{int}} = -\frac{1}{c^2} \int d^4 x A_i(x) j^i(x) \tag{5.19}$$

Physical meaning of j^i
Minkowski diagram at angle $\arctan(v/c)$, with x^0 axis up and x^1 axis on horizontal.

$$\begin{aligned} x^0(\lambda) &= c\lambda \\ x^1(\lambda) &= v\lambda \\ x^2(\lambda) &= 0 \\ x^3(\lambda) &= 0 \end{aligned} \tag{5.20}$$

Note that λ here is just a parameter. τ was used in the lecture, but that makes it appear that we missing a factor of γ above (if one did the end result would be the same since the delta evaluation would bring down a factor of $1/\gamma$ to cancel it out).

$$j^i(x) = ec \int dx^i(\lambda) \delta^4(x - x(\lambda)) \tag{5.21}$$

$$\begin{aligned} j^0(x) &= ec^2 \int_{-\infty}^{\infty} d\lambda \delta(x^0 - c\lambda)\delta(x^1 - v\lambda)\delta(x^2)\delta(x^3) \\ j^1(x) &= ecv \int_{-\infty}^{\infty} d\lambda \delta(x^0 - c\lambda)\delta(x^1 - v\lambda)\delta(x^2)\delta(x^3) \\ j^2(x) &= 0 \\ j^3(x) &= 0 \end{aligned} \tag{5.22}$$

To evaluate the j^0 integral, we have only the contribution from $\lambda = x^0/c$. Recall that

$$\int dx \delta(bx - a) f(x) = \frac{1}{|b|} f\left(\frac{a}{b}\right) \tag{5.23}$$

This $-c\lambda$ scaling of the delta function, kills a factor of c above, and leaves us with

$$\begin{aligned} j^0(x) &= ec\delta(x^1 - vx^0/c)\delta(x^2)\delta(x^3) \\ j^1(x) &= ev\delta(x^1 - vx^0/c)\delta(x^2)\delta(x^3) \\ j^2(x) &= 0 \\ j^3(x) &= 0 \end{aligned} \tag{5.24}$$

The current is non-zero only on the worldline of the particle. We identify

$$\rho(ct, x^1, x^2, x^3) = e\delta(x^1 - vx^0/c)\delta(x^2)\delta(x^3) \tag{5.25}$$

so that our current can be interpreted as the charge and current density

$$\begin{aligned} j^0 &= c\rho(x) \\ j^\alpha(x) &= (\mathbf{v})^\alpha \rho(x) \end{aligned} \tag{5.26}$$

Except for the delta functions these are just the quantities that we are familiar with from the RHS of Maxwell's equations.

Reading Covering chapter 4 material from the text [11], and lecture notes RelEMpp103-113.pdf.

5.3 REVIEW. OUR ACTION

$$\begin{aligned} S &= S_{\text{particles}} + S_{\text{interaction}} + S_{\text{EM field}} \\ &= \sum_A \int_{x_A^i(\tau)} ds(-m_A c) - \sum_A \frac{e_A}{c} \int dx_A^i A_i(x_A) - \frac{1}{16\pi c} \int d^4 x F^{ij} F_{ij}. \end{aligned} \tag{5.27}$$

Our dynamics variables are

$$\begin{cases} x_A^i(\tau) & A = 1, \cdots, N \\ A^i(x) & A = 1, \cdots, N \end{cases} \tag{5.28}$$

We saw that the interaction term could also be written in terms of a delta function current, with

$$S_{\text{interaction}} = -\frac{1}{c^2} \int d^4 x j^i(x) A_i(x), \tag{5.29}$$

and

$$j^i(x) = \sum_A c e_A \int_{x(\tau)} dx_A^i \delta^4(x - x_A(\tau)). \tag{5.30}$$

Variation with respect to $x_A^i(\tau)$ gave us

$$mc \frac{du_A^i}{ds} = \frac{e}{c} u_A^j F_{ij}. \tag{5.31}$$

Note that it is easy to get the sign mixed up here. With our $(+,-,-,-)$ metric tensor, if the second index is the summation index, we have a positive sign.

Only the $S_{\text{particles}}$ and $S_{\text{interaction}}$ depend on $x_A^i(\tau)$.

5.4 THE FIELD ACTION VARIATION

Today: We will find the EOM for $A^i(x)$. The dynamical degrees of freedom are $A^i(\mathbf{x}, t)$

$$S[A^i(\mathbf{x}, t)] = -\frac{1}{16\pi c}\int d^4 x F_{ij}F^{ij} - \frac{1}{c^2}\int d^4 x A^i j_i. \tag{5.32}$$

Here j^i are treated as "sources".

We demand that

$$\delta S = S[A^i(\mathbf{x}, t) + \delta A^i(\mathbf{x}, t)] - S[A^i(\mathbf{x}, t)] = 0 + O(\delta A)^2. \tag{5.33}$$

We need to impose two conditions.

- At spatial ∞, i.e. at $|\mathbf{x}| \to \infty, \forall t$, we will impose the condition

$$A^i(\mathbf{x}, t)\Big|_{|\mathbf{x}|\to\infty} \to 0. \tag{5.34}$$

  This is sensible, because fields are created by charges, and charges are assumed to be localized in a bounded region. The field outside charges will $\to 0$ at $|\mathbf{x}| \to \infty$. Later we will treat the integration range as finite, and bounded, then later allow the boundary to go to infinity.

- at $t = -T$ and $t = T$ we will imagine that the values of $A^i(\mathbf{x}, \pm T)$ are fixed.

  This is analogous to $x(t_i) = x_1$ and $x(t_f) = x_2$ in particle mechanics.

  Since $A^i(\mathbf{x}, \pm T)$ is given, and equivalent to the initial and final field configurations, our extremes at the boundary is zero

$$\delta A^i(\mathbf{x}, \pm T) = 0. \tag{5.35}$$

PICTURE: a cylinder in spacetime, with an attempt to depict the boundary.

5.5 COMPUTING THE VARIATION

$$\delta S[A^i(\mathbf{x},t)] = -\frac{1}{16\pi c}\int d^4x\delta(F_{ij}F^{ij}) - \frac{1}{c^2}\int d^4x\delta(A^i)j_i. \tag{5.36}$$

Looking first at the variation of just the F^2 bit we have

$$\begin{aligned}\delta(F_{ij}F^{ij}) &= \delta(F_{ij})F^{ij} + F_{ij}\delta(F^{ij})\\ &= 2\delta(F^{ij})F_{ij}\\ &= 2\delta(\partial^i A^j - \partial^j A^i)F_{ij}\\ &= 2\delta(\partial^i A^j)F_{ij} - 2\delta(\partial^j A^i)F_{ij}\\ &= 2\delta(\partial^i A^j)F_{ij} - 2\delta(\partial^i A^j)F_{ji}\\ &= 4\delta(\partial^i A^j)F_{ij}\\ &= 4F_{ij}\partial^i\delta(A^j).\end{aligned} \tag{5.37}$$

Our variation is now reduced to

$$\begin{aligned}\delta S[A^i(\mathbf{x},t)] &= -\frac{1}{4\pi c}\int d^4xF_{ij}\partial^i\delta(A^j) - \frac{1}{c^2}\int d^4xj^i\delta(A_i)\\ &= -\frac{1}{4\pi c}\int d^4xF^{ij}\frac{\partial}{\partial x^i}\delta(A_j) - \frac{1}{c^2}\int d^4xj^i\delta(A_i).\end{aligned} \tag{5.38}$$

We can integrate this first term by parts

$$\int d^4xF^{ij}\frac{\partial}{\partial x^i}\delta(A_j) = \int d^4x\frac{\partial}{\partial x^i}\left(F^{ij}\delta(A_j)\right) - \int d^4x\left(\frac{\partial}{\partial x^i}F^{ij}\right)\delta(A_j) \tag{5.39}$$

The first term is a four dimensional divergence, with the contraction of the four gradient ∂_i with a four vector $B^i = F^{ij}\delta(A_j)$.

Prof. Poppitz chose $dx^0d^3\mathbf{x}$ split of d^4x to illustrate that this can be viewed as regular old spatial three vector divergences. It is probably more rigorous to mandate that the four volume element is oriented $d^4x = (1/4!)\epsilon_{ijkl}dx^idx^jdx^kdx^l$, and then utilize the 4D version of the divergence theorem (or its Stokes Theorem equivalent). The completely antisymmetric tensor should do most of the work required to express the oriented boundary volume.

Because we have specified that A^i is zero on the boundary, so is F^{ij}, so these boundary terms are killed off. We are left with

$$\begin{aligned}\delta S[A^i(\mathbf{x},t)] &= -\frac{1}{4\pi c}\int d^4x\delta(A_j)\partial_iF^{ij} - \frac{1}{c^2}\int d^4xj^i\delta(A_i)\\ &= \int d^4x\delta A_j(x)\left(-\frac{1}{4\pi c}\partial_iF^{ij}(x) - \frac{1}{c^2}j^i\right)\\ &= 0.\end{aligned} \tag{5.40}$$

This gives us

$$\boxed{\partial_i F^{ij} = \frac{4\pi}{c} j^j} \tag{5.41}$$

5.6 UNPACKING THESE

Recall that the Bianchi identity

$$\epsilon^{ijkl}\partial_j F_{kl} = 0, \tag{5.42}$$

gave us

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \end{aligned} \tag{5.43}$$

How about the EOM that we have found by varying the action? One of those equations is

$$\partial_\alpha F^{\alpha 0} = \frac{4\pi}{c} j^0 = 4\pi\rho, \tag{5.44}$$

since $j^0 = c\rho$.
Because

$$F^{\alpha 0} = (\mathbf{E})^\alpha, \tag{5.45}$$

we have

$$\nabla \cdot \mathbf{E} = 4\pi\rho. \tag{5.46}$$

The messier one to deal with is

$$\partial_i F^{i\alpha} = \frac{4\pi}{c} j^\alpha. \tag{5.47}$$

Splitting out the spatial and time indices for the four gradient we have

$$\begin{aligned} \partial_i F^{i\alpha} &= \partial_\beta F^{\beta\alpha} + \partial_0 F^{0\alpha} \\ &= \partial_\beta F^{\beta\alpha} - \frac{1}{c}\frac{\partial (\mathbf{E})^\alpha}{\partial t} \end{aligned} \tag{5.48}$$

The spatial index tensor element is

$$F^{\beta\alpha} = \partial^\beta A^\alpha - \partial^\alpha A^\beta$$
$$= -\frac{\partial A^\alpha}{\partial x^\beta} + \frac{\partial A^\beta}{\partial x^\alpha} \tag{5.49}$$
$$= \epsilon^{\alpha\beta\gamma} B^\gamma,$$

so the sum becomes

$$\partial_i F^{i\alpha} = \partial_\beta(\epsilon^{\alpha\beta\gamma} B^\gamma) - \frac{1}{c}\frac{\partial(\mathbf{E})^\alpha}{\partial t}$$
$$= \epsilon^{\beta\gamma\alpha}\partial_\beta B^\gamma - \frac{1}{c}\frac{\partial(\mathbf{E})^\alpha}{\partial t} \tag{5.50}$$
$$= (\boldsymbol{\nabla} \times \mathbf{B})^\alpha - \frac{1}{c}\frac{\partial(\mathbf{E})^\alpha}{\partial t}.$$

This gives us

$$\frac{4\pi}{c} j^\alpha = (\boldsymbol{\nabla} \times \mathbf{B})^\alpha - \frac{1}{c}\frac{\partial(\mathbf{E})^\alpha}{\partial t}, \tag{5.51}$$

or in vector form

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}. \tag{5.52}$$

Summarizing what we know so far, we have

$$\boxed{\begin{aligned} \partial_i F^{ij} &= \frac{4\pi}{c} j^j \\ \epsilon^{ijkl}\partial_j F_{kl} &= 0 \end{aligned}} \tag{5.53}$$

or in vector form

$$\boxed{\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= 4\pi\rho \\ \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{j} \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned}} \tag{5.54}$$

5.7 SPEED OF LIGHT

Claim : "c" is the speed of EM waves in vacuum.
Study equations in vacuum (no sources, so $j^i = 0$) for $A^i = (\phi, \mathbf{A})$.

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{B} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{5.55}$$

where

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \end{aligned} \tag{5.56}$$

In terms of potentials

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) \\ &= \boldsymbol{\nabla} \times \mathbf{B} \\ &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\ &= \frac{1}{c}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) \\ &= -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\phi - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \end{aligned} \tag{5.57}$$

Since we also have

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A}, \tag{5.58}$$

some rearrangement gives

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) = \boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\phi - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{5.59}$$

The remaining equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, in terms of potentials is

$$\boldsymbol{\nabla} \cdot \mathbf{E} = -\boldsymbol{\nabla}^2 \phi - \frac{1}{c}\frac{\partial \boldsymbol{\nabla} \cdot \mathbf{A}}{\partial t} \tag{5.60}$$

We can make a gauge transformation that completely eliminates eq. (5.60), and reduces eq. (5.59) to a wave equation.

$$(\phi, \mathbf{A}) \rightarrow (\phi', \mathbf{A}') \tag{5.61}$$

with

$$\begin{aligned}\phi &= \phi' - \frac{1}{c}\frac{\partial \chi}{\partial t} \\ \mathbf{A} &= \mathbf{A}' + \boldsymbol{\nabla}\chi\end{aligned} \tag{5.62}$$

Can choose $\chi(\mathbf{x}, t)$ to make $\phi' = 0$ ($\forall \phi \exists \chi, \phi' = 0$)

$$\frac{1}{c}\frac{\partial}{\partial t}\chi(\mathbf{x}, t) = \phi(\mathbf{x}, t) \tag{5.63}$$

$$\chi(\mathbf{x}, t) = c\int_{-\infty}^{t} dt' \phi(\mathbf{x}, t') \tag{5.64}$$

Can also find a transformation that also allows $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$

Q: What would that second transformation be explicitly?

A: To be revisited next lecture, when this is covered in full detail.
This is the Coulomb gauge

$$\begin{aligned}\phi &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{A} &= 0\end{aligned} \tag{5.65}$$

From eq. (5.59), we then have

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}'}{\partial t^2} - \boldsymbol{\nabla}^2 \mathbf{A}' = 0 \tag{5.66}$$

which is the wave equation for the propagation of the vector potential $\mathbf{A}'(\mathbf{x}, t)$ through space at velocity c, confirming that c is the speed of electromagnetic propagation (the speed of light).

Reading Covering chapter 4 material from the text [11], and lecture notes RelEMpp114-127.pdf.

5.8 TRYING TO UNDERSTAND "C"

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{5.67}$$

Maxwell's equations in a vacuum were

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{A}) &= \nabla^2 \mathbf{A} - \frac{1}{c}\frac{\partial}{\partial t}\nabla\phi - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} \\ \nabla \cdot \mathbf{E} &= -\nabla^2\phi - \frac{1}{c}\frac{\partial \nabla \cdot \mathbf{A}}{\partial t} \end{aligned} \tag{5.68}$$

There is a redundancy here since we can change ϕ and $\mathbf{A}$ without changing the EOM

$$(\phi, \mathbf{A}) \to (\phi', \mathbf{A}') \tag{5.69}$$

with

$$\begin{aligned} \phi &= \phi' + \frac{1}{c}\frac{\partial \chi}{\partial t} \\ \mathbf{A} &= \mathbf{A}' - \nabla\chi \end{aligned} \tag{5.70}$$

$$\chi(\mathbf{x}, t) = c\int dt\phi(\mathbf{x}, t) \tag{5.71}$$

which gives

$$\phi' = 0 \tag{5.72}$$

$$(\phi, \mathbf{A}) \sim (\phi = 0, \mathbf{A}') \tag{5.73}$$

Maxwell's equations are now

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{A}') &= \nabla^2 \mathbf{A}' - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}'}{\partial t^2} \\ \frac{\partial \nabla \cdot \mathbf{A}'}{\partial t} &= 0 \end{aligned} \tag{5.74}$$

Can we make $\boldsymbol{\nabla} \cdot \mathbf{A}'' = 0$, while $\phi'' = 0$.

$$\underbrace{\phi}_{=0} = \underbrace{\phi'}_{=0} + \frac{1}{c}\frac{\partial \chi'}{\partial t} \tag{5.75}$$

We need

$$\frac{\partial \chi'}{\partial t} = 0 \tag{5.76}$$

How about $\mathbf{A}'$

$$\mathbf{A}' = \mathbf{A}'' - \boldsymbol{\nabla}\chi' \tag{5.77}$$

We want the divergence of $\mathbf{A}'$ to be zero, which means

$$\boldsymbol{\nabla} \cdot \mathbf{A}' = \underbrace{\boldsymbol{\nabla} \cdot \mathbf{A}''}_{=0} - \boldsymbol{\nabla}^2\chi' \tag{5.78}$$

So we want

$$\boldsymbol{\nabla}^2\chi' = \boldsymbol{\nabla} \cdot \mathbf{A}' \tag{5.79}$$

This has the solution

$$\chi'(\mathbf{x}) = -\frac{1}{4\pi}\int d^3\mathbf{x}' \frac{\boldsymbol{\nabla}' \cdot \mathbf{A}'(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{5.80}$$

Green's function for the Laplacian

Laplacian!Green's function Recall that in electrostatics we have

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho \tag{5.81}$$

and

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \tag{5.82}$$

which meant that we had

$$\boldsymbol{\nabla}^2\phi = -4\pi\rho \tag{5.83}$$

This has the identical form to the equation in χ' that we wanted to solve (with $\phi \sim \chi$, and $4\pi\rho \sim \boldsymbol{\nabla}\cdot\mathbf{A}'$).

Without resorting to electrostatics another way to look at this problem is that it is just a Laplace equation, and we could utilize a Green's function solution if desired. This would generate the same result for χ' above, and also works for the electrostatics case.

Recall that the Green's function for the Laplacian was

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \tag{5.84}$$

with the property

$$\boldsymbol{\nabla}^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{5.85}$$

Our LDE to solve by Green's method is

$$\boldsymbol{\nabla}^2\phi = 4\pi\rho, \tag{5.86}$$

We let this equation (after switching to primed coordinates) operate on the Green's function

$$\int d^3\mathbf{x}' \boldsymbol{\nabla}'^2\phi(\mathbf{x}')G(\mathbf{x}, \mathbf{x}') = -\int d^3\mathbf{x}' 4\pi\rho(\mathbf{x}')G(\mathbf{x}, \mathbf{x}'). \tag{5.87}$$

Assuming that the left action of the Green's function on the test function $\phi(\mathbf{x}')$ is the same as the right action (i.e. $\phi(\mathbf{x}')$ and $G(\mathbf{x}, \mathbf{x}')$ commute), we have for the LHS

$$\begin{aligned}
\int d^3\mathbf{x}' \boldsymbol{\nabla}'^2\phi(\mathbf{x}')G(\mathbf{x}, \mathbf{x}') &= \int d^3\mathbf{x}' \boldsymbol{\nabla}'^2 G(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}') \\
&= \int d^3\mathbf{x}' \delta(\mathbf{x} - \mathbf{x}')\phi(\mathbf{x}') \\
&= \phi(\mathbf{x}).
\end{aligned} \tag{5.88}$$

Substitution of $G(\mathbf{x}, \mathbf{x}') = -1/4\pi|\mathbf{x} - \mathbf{x}'|$ on the RHS then gives us the general solution

$$\phi(\mathbf{x}) = \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{5.89}$$

Back to Maxwell's equations in vacuum What are the Maxwell's vacuum equations now? With the second gauge substitution we have

$$\begin{aligned} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}'') &= \boldsymbol{\nabla}^2 \mathbf{A}'' - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}''}{\partial t^2} \\ \frac{\partial \boldsymbol{\nabla} \cdot \mathbf{A}''}{\partial t} &= 0 \end{aligned} \tag{5.90}$$

but we can utilize

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A}, \tag{5.91}$$

to reduce Maxwell's equations (after dropping primes) to just

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}''}{\partial t^2} - \Delta \mathbf{A} = 0 \tag{5.92}$$

where

$$\Delta = \boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2} \tag{5.93}$$

Note that for this to be correct we have to also explicitly include the gauge condition used. This particular gauge is called the Coulomb gauge.

$$\begin{aligned} \phi &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{A}'' &= 0 \end{aligned} \tag{5.94}$$

5.9 CLAIM: EM WAVES PROPAGATE WITH SPEED c AND ARE TRANSVERSE

Note: Is the Coulomb gauge Lorentz invariant?

No. We can boost which will introduce a non-zero ϕ.
The gauge that is Lorentz Invariant is the "Lorentz gauge". This one uses

$$\partial_i A^i = 0 \tag{5.95}$$

Recall that Maxwell's equations are

$$\partial_i F^{ij} = j^j = 0 \tag{5.96}$$

where

$$\begin{aligned} \partial_i &= \frac{\partial}{\partial x^i} \\ \partial^i &= \frac{\partial}{\partial x_i} \end{aligned} \tag{5.97}$$

Writing out the equations in terms of potentials we have

$$\begin{aligned} 0 &= \partial_i(\partial^i A^j - \partial^j A^i) \\ &= \partial_i \partial^i A^j - \partial_i \partial^j A^i \\ &= \partial_i \partial^i A^j - \partial^j \partial_i A^i \end{aligned} \tag{5.98}$$

So, if we pick the gauge condition $\partial_i A^i = 0$, we are left with just

$$0 = \partial_i \partial^i A^j \tag{5.99}$$

Can we choose A'^i such that $\partial_i A^i = 0$?
Our gauge condition is

$$A^i = A'^i + \partial^i \chi \tag{5.100}$$

Hit it with a derivative for

$$\partial_i A^i = \partial_i A'^i + \partial_i \partial^i \chi \tag{5.101}$$

If we want $\partial_i A^i = 0$, then we have

$$-\partial_i A'^i = \partial_i \partial^i \chi = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \chi \tag{5.102}$$

This is the physicist proof. Yes, it can be solved. To really solve this, we would want to use Green's functions. I seem to recall the Green's function is a retarded time version of the Laplacian Green's function, and we can figure that exact form out by switching to a Fourier frequency domain representation.

Anyways. Returning to Maxwell's equations we have

$$\begin{aligned} 0 &= \partial_i \partial^i A^j \\ 0 &= \partial_i A^i, \end{aligned} \tag{5.103}$$

where the first is Maxwell's equation, and the second is our gauge condition.

Observe that the gauge condition is now a Lorentz scalar.

$$\partial^i A_i \rightarrow \partial^j {O_j}^i {O_i}^k A_k \tag{5.104}$$

But the Lorentz transform matrices multiply out to identity, in the same way that they do for the transformation of a plain old four vector dot product $x^i y_i$.

5.10 WHAT HAPPENS WITH A MASSIVE VECTOR FIELD?

$$S = \int d^4 x \left(\frac{1}{4} F^{ij} F_{ij} + \frac{m^2}{2} A^i A_i \right) \tag{5.105}$$

An aside on units "Note that this action is expressed in dimensions where $\hbar = c = 1$, making the action is unit-less (energy and time are inverse units of each other). The $d^4 x$ has units of m^{-4} (since $[x] = \hbar/mc$), so F has units of m^2, and then A has units of mass. Therefore $d^4 x A A$ has units of m^{-2} and therefore you need something that has units of m^2 to make the action unit-less. When you do not take $c = 1$, then you have got to worry about those factors, but I think you will see it works out fine."

For what it is worth, I can adjust the units of this action to those that we have used in class with,

$$S = \int d^4 x \left(-\frac{1}{16 \pi c} F^{ij} F_{ij} - \frac{m^2 c^2}{8 \hbar^2} A^i A_i \right) \tag{5.106}$$

Back to the problem The variation of the field invariant is

$$\begin{aligned}\delta(F_{ij}F^{ij}) &= 2(\delta F_{ij})F^{ij}) \\ &= 2(\delta(\partial_i A_j - \partial_j A_i))F^{ij}) \\ &= 2(\partial_i \delta(A_j) - \partial_j \delta(A_i))F^{ij}) \\ &= 4F^{ij}\partial_i \delta(A_j) \\ &= 4\partial_i(F^{ij}\delta(A_j)) - 4(\partial_i F^{ij})\delta(A_j).\end{aligned} \tag{5.107}$$

Variation of the A^2 term gives us

$$\delta(A^j A_j) = 2A^j \delta(A_j), \tag{5.108}$$

so we have

$$\begin{aligned}0 &= \delta S \\ &= \int d^4x \delta(A_j)\left(-\partial_i F^{ij} + m^2 A^j\right) + \int d^4x \partial_i(F^{ij}\delta(A_j))\end{aligned} \tag{5.109}$$

The last integral vanishes on the boundary with the assumption that $\delta(A_j) = 0$ on that boundary.

Since this must be true for all variations, this leaves us with

$$\partial_i F^{ij} = m^2 A^j \tag{5.110}$$

The RHS can be expanded into wave equation and divergence parts

$$\begin{aligned}\partial_i F^{ij} &= \partial_i(\partial^i A^j - \partial^j A^i) \\ &= (\partial_i \partial^i)A^j - \partial^j(\partial_i A^i)\end{aligned} \tag{5.111}$$

With $\Box$ for the wave equation operator

$$\Box = \partial_i \partial^i = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta, \tag{5.112}$$

we can manipulate the EOM to pull out an A_i factor

$$\begin{aligned}0 &= \left(\Box - m^2\right)A^j - \partial^j(\partial_i A^i) \\ &= \left(\Box - m^2\right)g^{ij}A_i - \partial^j(\partial^i A_i) \\ &= \left(\left(\Box - m^2\right)g^{ij} - \partial^j \partial^i\right)A_i.\end{aligned} \tag{5.113}$$

If we hit this with a derivative we get

$$\begin{aligned} 0 &= \partial_j \left(\left(\Box - m^2 \right) g^{ij} - \partial^j \partial^i \right) A_i \\ &= \left(\left(\Box - m^2 \right) \partial^i - \partial_j \partial^j \partial^i \right) A_i \\ &= \left(\left(\Box - m^2 \right) \partial^i - \Box \partial^i \right) A_i \\ &= \left(\Box - m^2 - \Box \right) \partial^i A_i \\ &= -m^2 \partial^i A_i \end{aligned} \tag{5.114}$$

Since m is presumed to be non-zero here, this means that the Lorentz gauge is already chosen for us by the equations of motion.

Reading Covering chapter 6 material from the text [11], and lecture notes RelEMpp114-127.pdf.

5.11 REVIEW OF WAVE EQUATION RESULTS OBTAINED

Maxwell's equations in vacuum lead to Coulomb gauge and the Lorentz gauge.

Coulomb gauge

$$\begin{aligned} A^0 &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{A} &= 0 \\ \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \mathbf{A} &= 0 \end{aligned} \tag{5.115}$$

Lorentz gauge

$$\begin{aligned} \partial_i A^i &= 0 \\ \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) A^i &= 0 \end{aligned} \tag{5.116}$$

Note that $\partial_i A^i = 0$ is invariant under gauge transformations

$$A^i \rightarrow A^i + \partial^i \chi \tag{5.117}$$

where

$$\partial_i \partial^i \chi = 0, \tag{5.118}$$

So if one uses the Lorentz gauge, this has to be fixed.

However, in both cases we have

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) f = 0 \tag{5.119}$$

where

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \tag{5.120}$$

is the wave operator.

Consider

$$\Delta = \frac{\partial^2}{\partial x^2} \tag{5.121}$$

where we are looking for a solution that is independent of y, z. Recall that the general solution for this equation has the form

$$f(t, x) = F_1\left(t - \frac{x}{c}\right) + F_2\left(t + \frac{x}{c}\right) \tag{5.122}$$

PICTURE: superposition of two waves with F_1 moving along the x-axis in the positive direction, and F_2 in the negative x direction.

It is notable that the text derives eq. (5.122) in a particularly slick way. It is still black magic, since one has to know the solution to find it, but very very cool.

5.12 REVIEW OF FOURIER METHODS

It is often convenient to impose periodic boundary conditions

$$\mathbf{A}(\mathbf{x} + \mathbf{e}_i L) = \mathbf{A}(\mathbf{x}), i = 1, 2, 3 \tag{5.123}$$

In one dimension

$$f(x + L) = f(x) \tag{5.124}$$

$$f(x) = \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{L}x} \tilde{f}_n \tag{5.125}$$

When $f(x)$ is real we also have

$$f^*(x) = \sum_{n=-\infty}^{\infty} e^{-i\frac{2\pi n}{L}x}(\tilde{f}_n)^* \tag{5.126}$$

which implies

$$\tilde{f}_n^* = \tilde{f}_{-n}. \tag{5.127}$$

We introduce a wave number

$$k_n = \frac{2\pi n}{L}, \tag{5.128}$$

allowing a slightly simpler expression of the Fourier decomposition

$$f(x) = \sum_{n=-\infty}^{\infty} e^{ik_n x}\tilde{f}_{k_n}. \tag{5.129}$$

The inverse transform is obtained by integration over some length L interval

$$\tilde{f}_{k_n} = \frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ik_n x} f(x) \tag{5.130}$$

Verify: We should be able to recover the Fourier coefficient by utilizing the above

$$\begin{aligned}\frac{1}{L}\int_{-L/2}^{L/2} dx e^{-ik_n x} \sum_{m=-\infty}^{\infty} e^{ik_m x}\tilde{f}_{k_m} \\ = \sum_{m=-\infty}^{\infty} \tilde{f}_{k_m}\delta_{mn} = \tilde{f}_{k_n},\end{aligned} \tag{5.131}$$

where we use the easily verifiable fact that

$$\frac{1}{L}\int_{-L/2}^{L/2} dx e^{i(k_m-k_n)x} = \begin{matrix} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{matrix}. \tag{5.132}$$

It is conventional to absorb $\tilde{f}_{k_n} = \tilde{f}(k_n)$ for

$$\begin{aligned} f(x) &= \frac{1}{L}\sum_n \tilde{f}(k_n) e^{ik_n x} \\ \tilde{f}(k_n) &= \int_{-L/2}^{L/2} dx f(x) e^{-ik_n x} \end{aligned} \tag{5.133}$$

To take $L \to \infty$ notice

$$k_n = \frac{2\pi}{L} n \tag{5.134}$$

when n changes by $\Delta n = 1$, k_n changes by $\Delta k_n = \frac{2\pi}{L}\Delta n$
Using this

$$f(x) = \frac{1}{2\pi}\sum_n \left(\frac{2\pi}{L}\Delta n\right)\tilde{f}(k_n) e^{ik_n x} \tag{5.135}$$

With $L \to \infty$, and $\Delta k_n \to 0$

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\tilde{f}(k) e^{ikx} \\ \tilde{f}(k) &= \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \end{aligned} \tag{5.136}$$

Verify: A loose verification of the inversion relationship (the most important bit) is possible by substitution

$$\begin{aligned} \int \frac{dk}{2\pi} e^{ikx}\tilde{f}(k) &= \iint \frac{dk}{2\pi} e^{ikx} dx' f(x') e^{-ikx'} \\ &= \int dx' f(x') \frac{1}{2\pi}\int dk e^{ik(x-x')} \end{aligned} \tag{5.137}$$

Now we employ the old physics ploy where we identify

$$\frac{1}{2\pi}\int dk e^{ik(x-x')} = \delta(x - x'). \tag{5.138}$$

With that we see that we recover the function $f(x)$ above as desired.

In three dimensions

$$\begin{aligned} \mathbf{A}(\mathbf{x},t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{\mathbf{A}}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}} \\ \tilde{\mathbf{A}}(\mathbf{x},t) &= \int d^3\mathbf{x}\mathbf{A}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}} \end{aligned} \tag{5.139}$$

Application to the wave equation

$$\begin{aligned} 0 &= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{A}(\mathbf{x},t) \\ &= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{\mathbf{A}}(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(\frac{1}{c^2}\partial_{tt}\tilde{\mathbf{A}}(\mathbf{k},t) + \mathbf{k}^2\mathbf{A}(\mathbf{k},t)\right)e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned} \tag{5.140}$$

Now operate with $\int d^3\mathbf{x}e^{-i\mathbf{p}\cdot\mathbf{x}}$

$$\begin{aligned} 0 &= \int d^3\mathbf{x}e^{-i\mathbf{p}\cdot\mathbf{x}}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(\frac{1}{c^2}\partial_{tt}\tilde{\mathbf{A}}(\mathbf{k},t) + \mathbf{k}^2\mathbf{A}(\mathbf{k},t)\right)e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int d^3\mathbf{k}\delta^3(\mathbf{p}-\mathbf{k})\left(\frac{1}{c^2}\partial_{tt}\tilde{\mathbf{A}}(\mathbf{k},t) + \mathbf{k}^2\mathbf{A}(\mathbf{k},t)\right) \end{aligned} \tag{5.141}$$

Since this is true for all $\mathbf{p}$ we have

$$\partial_{tt}\tilde{\mathbf{A}}(\mathbf{p},t) = -c^2\mathbf{p}^2\tilde{\mathbf{A}}(\mathbf{p},t) \tag{5.142}$$

For every value of momentum we have a harmonic oscillator!

$$\ddot{x} = -\omega^2 x \tag{5.143}$$

Fourier modes of EM potential in vacuum obey

$$\partial_{tt}\tilde{\mathbf{A}}(\mathbf{k},t) = -c^2\mathbf{k}^2\tilde{\mathbf{A}}(\mathbf{k},t) \tag{5.144}$$

Because we are operating in the Coulomb gauge we must also have zero divergence. Let us see how that translates to our Fourier representation implies

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{x}, t) \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \boldsymbol{\nabla} \cdot \left(e^{i\mathbf{k}\cdot\mathbf{x}} \cdot \tilde{\mathbf{A}}(\mathbf{k}, t) \right) \end{aligned} \tag{5.145}$$

The chain rule for the divergence in this case takes the form

$$\boldsymbol{\nabla} \cdot (\phi \mathbf{B}) = (\boldsymbol{\nabla}\phi) \cdot \mathbf{B} + \phi \boldsymbol{\nabla} \cdot \mathbf{B}. \tag{5.146}$$

But since our vector function $\tilde{\mathbf{A}}$ is not a function of spatial coordinates we have

$$0 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} (i\mathbf{k} \cdot \tilde{\mathbf{A}}(\mathbf{k}, t)). \tag{5.147}$$

This has two immediate consequences. The first is that our momentum space potential is perpendicular to the wave number vector at all points in momentum space, and the second gives us a conjugate relation (substitute $\mathbf{k} \rightarrow -\mathbf{k}'$ after taking conjugates for that one)

$$\begin{aligned} \mathbf{k} \cdot \tilde{\mathbf{A}}(\mathbf{k}, t) &= 0 \\ \tilde{\mathbf{A}}(-\mathbf{k}, t) &= \tilde{\mathbf{A}}^*(\mathbf{k}, t). \end{aligned} \tag{5.148}$$

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\frac{1}{2} \tilde{\mathbf{A}}(\mathbf{k}, t) + \frac{1}{2} \tilde{\mathbf{A}}^*(-\mathbf{k}, t) \right) \tag{5.149}$$

Since out system is essentially a harmonic oscillator at each point in momentum space

$$\begin{aligned} \partial_{tt} \tilde{\mathbf{A}}(\mathbf{k}, t) &= -\omega_k^2 \tilde{\mathbf{A}}(\mathbf{k}, t) \\ \omega_k^2 &= c^2 \mathbf{k}^2 \end{aligned} \tag{5.150}$$

our general solution is of the form

$$\begin{aligned} \tilde{\mathbf{A}}(\mathbf{k}, t) &= e^{i\omega_k t} \mathbf{a}_+(\mathbf{k}) + e^{-i\omega_k t} \mathbf{a}_-(\mathbf{k}) \\ \tilde{\mathbf{A}}^*(\mathbf{k}, t) &= e^{-i\omega_k t} \mathbf{a}_+^*(\mathbf{k}) + e^{i\omega_k t} \mathbf{a}_-^*(\mathbf{k}) \end{aligned} \tag{5.151}$$

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{2} \left(e^{i\omega_k t} (\mathbf{a}_+(\mathbf{k}) + \mathbf{a}_-^*(-\mathbf{k})) + e^{-i\omega_k t} (\mathbf{a}_-(\mathbf{k}) + \mathbf{a}_+^*(-\mathbf{k})) \right) \tag{5.152}$$

Define

$$\boldsymbol{\beta}(\mathbf{k}) \equiv \frac{1}{2}(\mathbf{a}_-(\mathbf{k}) + \mathbf{a}_+^*(-\mathbf{k})) \tag{5.153}$$

so that

$$\boldsymbol{\beta}(-\mathbf{k}) = \frac{1}{2}(\mathbf{a}_+^*(\mathbf{k}) + \mathbf{a}_-(-\mathbf{k})) \tag{5.154}$$

Our solution now takes the form

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(e^{i(\mathbf{k}\cdot\mathbf{x}+\omega_k t)} \boldsymbol{\beta}^*(-\mathbf{k}) + e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_k t)} \boldsymbol{\beta}(\mathbf{k}) \right) \tag{5.155}$$

Claim: This is now manifestly real. To see this, consider the first term with $\mathbf{k} = -\mathbf{k}'$, noting that $\int_{-\infty}^{\infty} dk = \int_{\infty}^{\infty} -dk' = \int_{-\infty}^{\infty} dk'$ with $dk = -dk'$

$$\int \frac{d^3\mathbf{k}'}{(2\pi)^3} e^{i(-\mathbf{k}'\cdot\mathbf{x}+\omega_k t)} \boldsymbol{\beta}^*(\mathbf{k}') \tag{5.156}$$

Dropping primes this is the conjugate of the second term.

Claim: We have $\mathbf{k} \cdot \boldsymbol{\beta}(\mathbf{k}) = 0$.

Since we have $\mathbf{k} \cdot \tilde{\mathbf{A}}(\mathbf{k}, t) = 0$, eq. (5.151) implies that we have $\mathbf{k} \cdot \mathbf{a}_\pm(\mathbf{k}) = 0$. With each of these vector integration constants being perpendicular to $\mathbf{k}$ at that point in momentum space, so must be the linear combination of these constants $\boldsymbol{\beta}(\mathbf{k})$.

Reading Covering chapter 6 material from the text [11], and lecture notes RelEMpp114-127.pdf.

5.13 REVIEW. SOLUTION TO THE WAVE EQUATION

Recall that in the Coulomb gauge

$$\begin{aligned} A^0 &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{A} &= 0 \end{aligned} \tag{5.157}$$

our equation to solve is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{A} = 0. \tag{5.158}$$

We found that the general solution was

$$\mathbf{A}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(e^{i(\mathbf{k}\cdot\mathbf{x}+\omega_k t)}\boldsymbol{\beta}^*(-\mathbf{k}) + e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_k t)}\boldsymbol{\beta}(\mathbf{k})\right) \tag{5.159}$$

where

$$\mathbf{k}\cdot\boldsymbol{\beta}(\mathbf{k}) = 0 \tag{5.160}$$

It is clear that this is a solution since

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)e^{i(\mathbf{k}\cdot\mathbf{x}\pm\omega_k t)} = 0 \tag{5.161}$$

5.14 MOVING TO PHYSICALLY RELEVANT RESULTS

Since the most general solution is a sum over **k**, it is enough to consider only a single **k**, or equivalently, take

$$\begin{aligned}\boldsymbol{\beta}(\mathbf{k}) &= \boldsymbol{\beta}(2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})\\ \boldsymbol{\beta}^*(-\mathbf{k}) &= \boldsymbol{\beta}^*(2\pi)^3\delta^3(-\mathbf{k}-\mathbf{p})\end{aligned} \tag{5.162}$$

but we have the freedom to pick a real and constant $\boldsymbol{\beta}$. Now our solution is

$$\mathbf{A}(\mathbf{x}, t) = \boldsymbol{\beta}\left(e^{-i(\mathbf{p}\cdot\mathbf{x}+\omega_k t)} + e^{i(\mathbf{p}\cdot\mathbf{x}-\omega_k t)}\right) = \boldsymbol{\beta}\cos(\omega t - \mathbf{p}\cdot\mathbf{x}) \tag{5.163}$$

where

$$\boldsymbol{\beta}\cdot\mathbf{p} = 0. \tag{5.164}$$

Note that the more general case, utilizing complex $\boldsymbol{\beta}$, leads to eliptically polarized fields. This is handled very elegantly (and compactly) in §48 of the text.

Let us choose

$$\mathbf{p} = (p, 0, 0) \tag{5.165}$$

Since

$$\mathbf{p} \cdot \boldsymbol{\beta} = p_x \beta_x \tag{5.166}$$

we must have

$$\boldsymbol{\beta}_x = 0 \tag{5.167}$$

so

$$\boldsymbol{\beta} = (0, \beta_y, \beta_z) \tag{5.168}$$

Claim: The Coulomb gauge $0 = \boldsymbol{\nabla} \cdot \mathbf{A} = (\boldsymbol{\beta} \cdot \mathbf{p}) \sin(\omega t - \mathbf{p} \cdot \mathbf{x})$ implies that there are two linearly independent choices of $\boldsymbol{\beta}$ and $\mathbf{p}$.

FIXME: missing exactly how this is?

PICTURE:

$\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$, $\mathbf{p}$ all mutually perpendicular.

$$\begin{aligned}\mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial ct} \\ &= -\frac{\boldsymbol{\beta}}{c}\frac{\partial}{\partial t}\cos(\omega t - \mathbf{p} \cdot \mathbf{x}) \\ &= \frac{1}{c}\boldsymbol{\beta}\omega \sin(\omega t - \mathbf{p} \cdot \mathbf{x})\end{aligned} \tag{5.169}$$

(recall: $\omega(\mathbf{p}) = c|\mathbf{p}|$)

$$\boxed{\mathbf{E} = \boldsymbol{\beta}|\mathbf{p}| \sin(\omega t - \mathbf{p} \cdot \mathbf{x})} \tag{5.170}$$

$$\begin{aligned}\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\ &= \boldsymbol{\nabla} \times (\boldsymbol{\beta} \cos(\omega t - \mathbf{p} \cdot \mathbf{x}) \\ &= (\boldsymbol{\nabla} \cos(\omega t - \mathbf{p} \cdot \mathbf{x})) \times \boldsymbol{\beta} \\ &= \sin(\omega t - \mathbf{p} \cdot \mathbf{x}) \mathbf{p} \times \boldsymbol{\beta}\end{aligned} \tag{5.171}$$

$$\boxed{\mathbf{B} = (\mathbf{p} \times \boldsymbol{\beta}) \sin(\omega t - \mathbf{p} \cdot \mathbf{x})} \tag{5.172}$$

Observe also that $\mathbf{E}$ and $\mathbf{B}$ are not independent. We have

$$\hat{\mathbf{p}} \times \mathbf{E} = (\hat{\mathbf{p}} \times \boldsymbol{\beta}) |\mathbf{p}| \sin(\omega t - \mathbf{p} \cdot \mathbf{x}) = \mathbf{B} \tag{5.173}$$

Example: $\mathbf{p} \parallel \mathbf{e}_1$, $\mathbf{B} \parallel \mathbf{e}_2$ or $\mathbf{e}_3$
(since we have two linearly independent choices)

Example: take $\boldsymbol{\beta} \parallel \mathbf{e}_2$

$$\begin{aligned} \mathbf{E} &= \boldsymbol{\beta} p \sin(cpt - px) \\ \mathbf{B} &= (\mathbf{p} \times \boldsymbol{\beta}) \sin(cpt - px) \end{aligned} \tag{5.174}$$

At $t = 0$

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\beta} p \sin(px) \\ B_z &= -|\boldsymbol{\beta}| \mathbf{e}_3 cp \sin(px) \end{aligned} \tag{5.175}$$

PICTURE: two oscillating mutually perpendicular sinusoids.
So physically, we see that $\mathbf{p}$ is the direction of propagation. We have always

$$\mathbf{p} \perp \mathbf{E} \tag{5.176}$$

and we have two possible polarizations.
Convention is usually to take the direction of oscillation of $\mathbf{E}$ the polarization of the wave.
This is the starting point for the field of optics, because the polarization of the incident wave, is strongly tied to how much of the wave will reflect off of a surface with a given index of refraction n.

5.15 EM WAVES CARRYING ENERGY AND MOMENTUM

Maxwell field in vacuum is the sum of plane monochromatic waves, two per wave vector.

PICTURE:

$$\begin{aligned} \mathbf{E} &\parallel \mathbf{e}_3 \\ \mathbf{B} &\parallel \mathbf{e}_1 \\ \mathbf{k} &\parallel \mathbf{e}_2 \end{aligned} \tag{5.177}$$

PICTURE:

$$\begin{aligned} \mathbf{B} &\parallel -\mathbf{e}_3 \\ \mathbf{E} &\parallel \mathbf{e}_1 \\ \mathbf{k} &\parallel \mathbf{e}_2 \end{aligned} \tag{5.178}$$

(two linearly independent polarizations)
Our wave frequency is

$$\omega_{\mathbf{k}} = c|\mathbf{k}| \tag{5.179}$$

The wavelength, the value such that $x \rightarrow x + \frac{2\pi}{k}$
FIXME:DIY: see:

$$\sin(kct - kx) \tag{5.180}$$

$$\lambda_{\mathbf{k}} = \frac{2\pi}{k} \tag{5.181}$$

period

$$T = \frac{2\pi}{kc} = \frac{\lambda_{\mathbf{k}}}{c} \tag{5.182}$$

5.16 ENERGY AND MOMENTUM OF EM WAVES

Classical mechanics motivation To motivate our approach, let us recall one route from our equations of motion in classical mechanics, to the energy conservation relation. Our EOM in one dimension is

$$m\frac{d}{dt}\dot{x} = -\mathcal{U}'(x). \tag{5.183}$$

We can multiply both sides by what we take the time derivative of

$$m\dot{x}\frac{d\dot{x}}{dt} = -\dot{x}\mathcal{U}'(x), \tag{5.184}$$

and then manipulate it a bit so that we have time derivatives on both sides

$$\frac{d}{dt}\frac{m\dot{x}^2}{2} = -\frac{d\mathcal{U}(x)}{dt}. \tag{5.185}$$

Taking differences, we have

$$\frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + \mathcal{U}(x)\right) = 0, \tag{5.186}$$

which allows us to find a conservation relationship that we label energy conservation ($\mathcal{E} = K + \mathcal{U}$).

Doing the same thing for Maxwell's equations Poppitz claims we have very little tricks in physics, and we really just do the same thing for our EM case. Our equations are a bit messier to start with, and for the vacuum, our non-divergence equations are

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{j} \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0\end{aligned} \tag{5.187}$$

We can dot these with $\mathbf{E}$ and $\mathbf{B}$ respectively, repeating the trick of "multiplying" by what we take the time derivative of

$$\begin{aligned}\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \frac{1}{c}\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{E} \cdot \mathbf{j} \\ \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) + \frac{1}{c}\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} &= 0,\end{aligned} \tag{5.188}$$

and then take differences

$$\frac{1}{c}\left(\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}\right) + \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = -\frac{4\pi}{c}\mathbf{E} \cdot \mathbf{j}. \tag{5.189}$$

Claim:

$$-\mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) + \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla} \cdot (\mathbf{B} \times \mathbf{E}). \tag{5.190}$$

This is almost trivial with an expansion of the RHS in tensor notation

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{B} \times \mathbf{E}) &= \partial_\alpha e^{\alpha\beta\sigma} B^\beta E^\sigma \\ &= e^{\alpha\beta\sigma} (\partial_\alpha B^\beta) E^\sigma + e^{\alpha\beta\sigma} B^\beta (\partial_\alpha E^\sigma) \\ &= \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}). \qquad \square \end{aligned} \tag{5.191}$$

Regrouping we have

$$\frac{1}{2c} \frac{\partial}{\partial t} \left(\mathbf{B}^2 + \mathbf{E}^2 \right) + \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) = -\frac{4\pi}{c} \mathbf{E} \cdot \mathbf{j}. \tag{5.192}$$

A final rescaling makes the units natural

$$\frac{\partial}{\partial t} \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + \boldsymbol{\nabla} \cdot \left(\frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right) = -\mathbf{E} \cdot \mathbf{j}. \tag{5.193}$$

We define the cross product term as the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}. \tag{5.194}$$

Suppose we integrate over a spatial volume. This gives us

$$\frac{\partial}{\partial t} \int_V d^3\mathbf{x} \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + \int_V d^3\mathbf{x} \boldsymbol{\nabla} \cdot \mathbf{S} = -\int_V d^3\mathbf{x} \mathbf{E} \cdot \mathbf{j}. \tag{5.195}$$

Our Poynting integral can be converted to a surface integral utilizing Stokes theorem

$$\int_V d^3\mathbf{x} \boldsymbol{\nabla} \cdot \mathbf{S} = \int_{\partial V} d^2\sigma \mathbf{n} \cdot \mathbf{S} = \int_{\partial V} d^2\boldsymbol{\sigma} \cdot \mathbf{S} \tag{5.196}$$

We make the interpretations

$$\begin{aligned} \int_V d^3\mathbf{x} \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} &= \text{energy} \\ \int_V d^3\mathbf{x} \boldsymbol{\nabla} \cdot \mathbf{S} &= \text{momentum change through surface per unit time} \\ -\int_V d^3\mathbf{x} \mathbf{E} \cdot \mathbf{j} &= \text{work done} \end{aligned} \tag{5.197}$$

Justifying the sign, and clarifying work done by what, above. Recall that the energy term of the Lorentz force equation was

$$\frac{d\mathcal{E}_{\text{kinetic}}}{dt} = e\mathbf{E} \cdot \mathbf{v} \tag{5.198}$$

and

$$\mathbf{j} = e\rho\mathbf{v} \tag{5.199}$$

so

$$\int_V d^3\mathbf{x}\mathbf{E} \cdot \mathbf{j} \tag{5.200}$$

represents the rate of change of kinetic energy of the charged particles as they move through through a field. If this is positive, then the charge distribution has gained energy. The negation of this quantity would represent energy transfer to the field from the charge distribution, the work done on the field by the charge distribution.

Aside: As a four vector relationship In tutorial today (after this lecture, but before typing up these lecture notes in full), we used $\mathcal{U}$ for the energy density term above

$$\mathcal{U} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}. \tag{5.201}$$

This allows us to group the quantities in our conservation relationship above nicely

$$\frac{\partial \mathcal{U}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}. \tag{5.202}$$

It appears natural to write eq. (5.202) in the form of a four divergence. Suppose we define

$$P^i = (\mathcal{U}/c, \mathbf{S}/c^2) \tag{5.203}$$

then we have

$$\partial_i P^i = -\mathbf{E} \cdot \mathbf{j}/c^2. \tag{5.204}$$

Since the LHS has the appearance of a four scalar, this seems to imply that $\mathbf{E} \cdot \mathbf{j}$ is a Lorentz invariant. It is curious that we have only the four scalar that comes from the energy term of the Lorentz force on the RHS of the conservation relationship. Peeking ahead at the text, this appears to be why a rank two energy tensor T^{ij} is introduced. For a relativistically natural quantity, we ought to have a conservation relationship also associated with each of the momentum change components of the four vector Lorentz force equation too.

Reading Covering chapter 6 material §31, and starting chapter 8 material from the text [11], and lecture notes RelEMpp128-135.pdf.

5.17 REVIEW. ENERGY DENSITY AND POYNTING VECTOR

Last time we showed that Maxwell's equations imply

$$\frac{\partial}{\partial t}\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} = -\mathbf{j}c\dot{\mathbf{E}} - \boldsymbol{\nabla} \cdot \mathbf{S} \tag{5.205}$$

In the lecture, Professor Poppitz said he was free here to use a full time derivative. When asked why, it was because he was considering **E** and **B** here to be functions of time only, since they were measured at a fixed point in space. This is really the same thing as using a time partial, so in these notes I will just be explicit and stick to using partials.

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \tag{5.206}$$

$$\frac{\partial}{\partial t}\int_V \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} = -\int_V \mathbf{j} \cdot \mathbf{E} - \int_{\partial V} d^2\boldsymbol{\sigma} \cdot \mathbf{S} \tag{5.207}$$

Any change in the energy must either due to currents, or energy escaping through the surface.

$$\begin{aligned} \mathcal{E} &= \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} = \text{Energy density of the EM field} \\ \mathbf{S} &= \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} = \text{Energy flux of the EM fields} \end{aligned} \tag{5.208}$$

The energy flux of the EM field: this is the energy flowing through $d^2\mathbf{A}$ in unit time ($\mathbf{S} \cdot d^2\mathbf{A}$).

5.18 HOW ABOUT ELECTROMAGNETIC WAVES?

In a plane wave moving in direction **k**.

PICTURE: $\mathbf{E} \parallel \hat{\mathbf{z}}$, $\mathbf{B} \parallel \hat{\mathbf{x}}$, $\mathbf{k} \parallel \hat{\mathbf{y}}$.

So, $\mathbf{S} \parallel \mathbf{k}$ since $\mathbf{E} \times \mathbf{B} \sim \mathbf{k}$.

$|\mathbf{S}|$ for a plane wave is the amount of energy through unit area perpendicular to **k** in unit time.

Recall that we calculated

$$\begin{aligned} \mathbf{B} &= (\mathbf{k} \times \boldsymbol{\beta}) \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) \\ \mathbf{E} &= \boldsymbol{\beta}|\mathbf{k}| \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) \end{aligned} \tag{5.209}$$

Since we had $\mathbf{k}\cdot\boldsymbol{\beta} = 0$, we have $|\mathbf{E}| = |\mathbf{B}|$, and our Poynting vector follows nicely

$$\begin{aligned}\mathbf{S} &= \frac{\mathbf{k}}{|\mathbf{k}|}\frac{c}{4\pi}\mathbf{E}^2 \\ &= \frac{\mathbf{k}}{|\mathbf{k}|}c\frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi} \\ &= \frac{\mathbf{k}}{|\mathbf{k}|}e\mathcal{E}\end{aligned} \tag{5.210}$$

$$[\mathbf{S}] = \frac{\text{energy}}{\text{time}\times\text{area}} = \frac{\text{momentum}\times\text{speed}}{\text{time}\times\text{area}} \tag{5.211}$$

$$\begin{aligned}\left[\frac{\mathbf{S}}{c^2}\right] &= \frac{\text{momentum}}{\text{time}\times\text{area}\times\text{speed}} \\ &= \frac{\text{momentum}}{\text{area}\times\text{distance}} \\ &= \frac{\text{momentum}}{\text{volume}}\end{aligned} \tag{5.212}$$

So we wee that $\mathbf{S}/c^2$ is indeed rightly called "the momentum density" of the EM field.
We will later find that $\mathcal{E}$ and $\mathbf{S}$ are components of a rank-2 four tensor

$$T^{ij} = \begin{bmatrix}\mathcal{E} & \frac{S^1}{c^2} & \frac{S^2}{c^2} & \frac{S^3}{c^2} \\ \frac{S^1}{c^2} & & & \\ \frac{S^1}{c^2} & & \left[\sigma^{\alpha\beta}\right] & \\ \frac{S^1}{c^2} & & & \end{bmatrix} \tag{5.213}$$

where $\sigma^{\alpha\beta}$ is the stress tensor. We will get to all this in more detail later.
For EM wave we have

$$\mathbf{S} = \hat{\mathbf{k}}c\mathcal{E} \tag{5.214}$$

(this is the energy flux)

$$\frac{\mathbf{S}}{c^2} = \hat{\mathbf{k}}\frac{\mathcal{E}}{c} \tag{5.215}$$

(the momentum density of the wave).

$$c\left|\frac{\mathbf{S}}{c^2}\right| = \mathcal{E} \tag{5.216}$$

(recall $\mathcal{E} = c\mathbf{p}$ for massless particles.

EM waves carry energy and momentum so when absorbed or reflected these are transferred to bodies.

Kepler speculated that this was the fact because he had observed that the tails of the comets were being pushed by the sunlight, since the tails faced away from the sun.

Maxwell also suggested that light would extort a force (presumably he wrote down the "Maxwell stress tensor" T^{ij} that is named after him).

This was actually measured later in 1901, by Peter Lebedev (Russia).

PICTURE: pole with flags in vacuum jar. Black (absorber) on one side, and Silver (reflector) on the other. Between the two of these, momentum conservation will introduce rotation (in the direction of the silver).

This is actually a tricky experiment and requires the vacuum, since the black surface warms up, and heats up the nearby gas molecules, which causes a rotation in the opposite direction due to just these thermal effects.

Another example (a factor) that prevents star collapse under gravitation is the radiation pressure of the light.

5.19 PROBLEMS

Exercise 5.1 Energy, momentum, etc., of EM waves

a. Energy and momentum density
   Calculate the energy density, energy flux, and momentum density of a plane monochromatic linearly polarized electromagnetic wave.
b. Calculate the values of these quantities averaged over a period.
c. Imagine that a plane monochromatic linearly polarized wave incident on a surface (let the angle between the wave vector and the normal to the surface be θ) is completely reflected. Find the pressure that the EM wave exerts on the surface.
d. To plug in some numbers, note that the intensity of sunlight hitting the Earth is about $1300 W/m^2$ (the intensity is the average power per unit area transported by the wave). If sunlight strikes a perfect absorber, what is the pressure exerted? What if it strikes a perfect reflector? What fraction of the atmospheric pressure does this amount to?

Answer for Exercise 5.1

Part a. Because it does not add too much complexity, I am going to calculate these using the more general elliptically polarized wave solutions. Our vector potential (in the Coulomb gauge $\phi = 0$, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$) has the form

$$\mathbf{A} = \operatorname{Re} \boldsymbol{\beta} e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}. \tag{5.217}$$

The elliptical polarization case only differs from the linear by allowing $\boldsymbol{\beta}$ to be complex, rather than purely real or purely imaginary. Observe that the Coulomb gauge condition $\boldsymbol{\nabla} \cdot \mathbf{A}$ implies

$$\boldsymbol{\beta} \cdot \mathbf{k} = 0, \tag{5.218}$$

a fact that will kill of terms in a number of places in the following manipulations.

Also observe that for this to be a solution to the wave equation operator

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta, \tag{5.219}$$

the frequency and wave vector must be related by the condition

$$\frac{\omega}{c} = |\mathbf{k}| = k. \tag{5.220}$$

For the time and spatial phase let us write

$$\theta = \omega t - \mathbf{k} \cdot \mathbf{x}. \tag{5.221}$$

In the Coulomb gauge, our electric and magnetic fields are

$$\begin{aligned}\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \operatorname{Re} \frac{-i\omega}{c} \boldsymbol{\beta} e^{i\theta} \\ \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} = \operatorname{Re} i\boldsymbol{\beta} \times \mathbf{k} e^{i\theta}\end{aligned} \tag{5.222}$$

Similar to §48 of the text [11], let us split $\boldsymbol{\beta}$ into a phase and perpendicular vector components so that

$$\boldsymbol{\beta} = \mathbf{b} e^{-i\alpha} \tag{5.223}$$

where $\mathbf{b}$ has a real square

$$\mathbf{b}^2 = |\boldsymbol{\beta}|^2. \tag{5.224}$$

This allows a split into two perpendicular real vectors

$$\mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2, \tag{5.225}$$

where $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$ since $\mathbf{b}^2 = \mathbf{b}_1^2 - \mathbf{b}_2^2 + 2\mathbf{b}_1 \cdot \mathbf{b}_2$ is real.
Our electric and magnetic fields are now reduced to

$$\begin{aligned}\mathbf{E} &= \mathrm{Re}\left(\frac{-i\omega}{c}\mathbf{b}e^{i(\theta-\alpha)}\right)\\ \mathbf{B} &= \mathrm{Re}\left(i\mathbf{b}\times\mathbf{k}e^{i(\theta-\alpha)}\right)\end{aligned} \tag{5.226}$$

or explicitly in terms of $\mathbf{b}_1$ and $\mathbf{b}_2$

$$\begin{aligned}\mathbf{E} &= \frac{\omega}{c}(\mathbf{b}_1\sin(\theta-\alpha) + \mathbf{b}_2\cos(\theta-\alpha))\\ \mathbf{B} &= (\mathbf{k}\times\mathbf{b}_1)\sin(\theta-\alpha) + (\mathbf{k}\times\mathbf{b}_2)\cos(\theta-\alpha)\end{aligned} \tag{5.227}$$

The special case of interest for this problem, since it only strictly asked for linear polarization, is where $\alpha = 0$ and one of $\mathbf{b}_1$ or $\mathbf{b}_2$ is zero (i.e. $\boldsymbol{\beta}$ is strictly real or strictly imaginary). The case with $\boldsymbol{\beta}$ strictly real, as done in class, is

$$\begin{aligned}\mathbf{E} &= \frac{\omega}{c}\mathbf{b}_1\sin(\theta-\alpha)\\ \mathbf{B} &= (\mathbf{k}\times\mathbf{b}_1)\sin(\theta-\alpha)\end{aligned} \tag{5.228}$$

Now lets calculate the energy density and Poynting vectors. We will need a few intermediate results.

$$\begin{aligned}(\mathrm{Re}\,\mathbf{d}e^{i\phi})^2 &= \frac{1}{4}(\mathbf{d}e^{i\phi} + \mathbf{d}^*e^{-i\phi})^2\\ &= \frac{1}{4}(\mathbf{d}^2e^{2i\phi} + (\mathbf{d}^*)^2e^{-2i\phi} + 2|\mathbf{d}|^2)\\ &= \frac{1}{2}\left(|\mathbf{d}|^2 + \mathrm{Re}(\mathbf{d}e^{i\phi})^2\right),\end{aligned} \tag{5.229}$$

and

$$(\mathrm{Re}\,\mathbf{d}e^{i\phi}) \times (\mathrm{Re}\,\mathbf{e}e^{i\phi}) = \frac{1}{4}(\mathbf{d}e^{i\phi} + \mathbf{d}^* e^{-i\phi}) \times (\mathbf{e}e^{i\phi} + \mathbf{e}^* e^{-i\phi}) \\ = \frac{1}{2}\,\mathrm{Re}\left(\mathbf{d} \times \mathbf{e}^* + (\mathbf{d} \times \mathbf{e})e^{2i\phi}\right). \tag{5.230}$$

Let us use arrowed vectors for the phasor parts

$$\vec{E} = \frac{-i\omega}{c}\mathbf{b}e^{i(\theta-\alpha)} \\ \vec{B} = i\mathbf{b} \times \mathbf{k}e^{i(\theta-\alpha)}, \tag{5.231}$$

where we can recover our vector quantities by taking real parts $\mathbf{E} = \mathrm{Re}\,\vec{E}$, $\mathbf{B} = \mathrm{Re}\,\vec{B}$. Our energy density in terms of these phasors is then

$$\mathcal{E} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = \frac{1}{16\pi}\left(\left|\vec{E}\right|^2 + \left|\vec{B}\right|^2 + \mathrm{Re}(\vec{E}^2 + \vec{B}^2)\right). \tag{5.232}$$

This is

$$\mathcal{E} = \frac{1}{16\pi}\left(\frac{\omega^2}{c^2}|\mathbf{b}|^2 + |\mathbf{b} \times \mathbf{k}|^2 - \mathrm{Re}\left(\frac{\omega^2}{c^2}\mathbf{b}^2 + (\mathbf{b} \times \mathbf{k})^2\right)e^{2i(\theta-\alpha)}\right). \tag{5.233}$$

Note that $\omega^2/c^2 = \mathbf{k}^2$, and $|\mathbf{b} \times \mathbf{k}| = |\mathbf{b}|^2\mathbf{k}^2$ (since $\mathbf{b} \cdot \mathbf{k} = 0$). Also $(\mathbf{b} \times \mathbf{k})^2 = \mathbf{b}^2\mathbf{k}^2$, so we have

$$\boxed{\mathcal{E} = \frac{\mathbf{k}^2}{8\pi}\left(|\mathbf{b}|^2 - \mathrm{Re}\,\mathbf{b}^2 e^{2i(\theta-\alpha)}\right).} \tag{5.234}$$

Now, for the Poynting vector. We have

$$S = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} = \frac{c}{8\pi}\,\mathrm{Re}\left(\vec{E} \times \vec{B}^* + \vec{E} \times \vec{B}\right). \tag{5.235}$$

This is

$$S = \frac{c}{8\pi}\,\mathrm{Re}\left(-k\mathbf{b} \times (\mathbf{b}^* \times \mathbf{k}) + k\mathbf{b} \times (\mathbf{b} \times \mathbf{k})e^{2i(\theta-\alpha)}\right) \tag{5.236}$$

Reducing the terms we get $\mathbf{b} \times (\mathbf{b}^* \times \mathbf{k}) = -\mathbf{k}|\mathbf{b}|^2$, and $\mathbf{b} \times (\mathbf{b} \times \mathbf{k}) = -\mathbf{k}\mathbf{b}^2$, leaving

$$\boxed{S = \frac{c\hat{\mathbf{k}}\mathbf{k}^2}{8\pi}\left(|\mathbf{b}|^2 - \mathrm{Re}\,\mathbf{b}^2 e^{2i(\theta-\alpha)}\right) = c\hat{\mathbf{k}}\mathcal{E}} \tag{5.237}$$

Now, the text in §47 defines the energy flux as the Poynting vector, and the momentum density as $\mathbf{S}/c^2$, so we just divide eq. (5.237) by c^2 for the momentum density and we are done. For the linearly polarized case (all that was actually asked for, but less cool to calculate), where $\mathbf{b}$ is real, we have

$$
\begin{aligned}
\text{Energy density} &= \mathcal{E} = \frac{\mathbf{k}^2\mathbf{b}^2}{8\pi}(1 - \cos(2(\omega t - \mathbf{k}\cdot\mathbf{x}))) \\
\text{Energy flux} &= \mathbf{S} = c\hat{\mathbf{k}}\mathcal{E} \\
\text{Momentum density} &= \frac{1}{c^2}\mathbf{S} = \frac{\hat{\mathbf{k}}}{c}\mathcal{E}.
\end{aligned}
\tag{5.238}
$$

Part b. We want to average over one period, the time T such that $\omega T = 2\pi$, so the average is

$$
\langle f \rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} f dt.
\tag{5.239}
$$

It is clear that this will just kill off the sinusoidal terms, leaving

$$
\begin{aligned}
\text{Average Energy density} &= \langle\mathcal{E}\rangle = \frac{\mathbf{k}^2|\mathbf{b}|^2}{8\pi} \\
\text{Average Energy flux} &= \langle\mathbf{S}\rangle = c\hat{\mathbf{k}}\mathcal{E} \\
\text{Average Momentum density} &= \frac{1}{c^2}\langle\mathbf{S}\rangle = \frac{\hat{\mathbf{k}}}{c}\mathcal{E}.
\end{aligned}
\tag{5.240}
$$

Part c. The magnitude of the momentum of light is related to its energy by

$$
\mathbf{p} = \frac{\mathcal{E}}{c}
\tag{5.241}
$$

and can thus loosely identify the magnitude of the force as

$$
\begin{aligned}
\frac{d\mathbf{p}}{dt} &= \frac{1}{c}\frac{\partial}{\partial t}\int \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} d^3\mathbf{x} \\
&= \int d^2\boldsymbol{\sigma}\cdot\frac{\mathbf{S}}{c}.
\end{aligned}
\tag{5.242}
$$

With pressure as the force per area, we could identify

$$
\frac{\mathbf{S}}{c}
\tag{5.243}
$$

as the instantaneous (directed) pressure on a surface. What is that for linearly polarized light? We have from above for the linear polarized case (where $|\mathbf{b}|^2 = \mathbf{b}^2$)

$$\mathbf{S} = \frac{c\hat{\mathbf{k}}\mathbf{k}^2\mathbf{b}^2}{8\pi}(1 - \cos(2(\omega t - \mathbf{k}\cdot\mathbf{x}))) \tag{5.244}$$

If we look at the magnitude of the average pressure from the radiation, we have

$$\left|\frac{\langle\mathbf{S}\rangle}{c}\right| = \frac{\mathbf{k}^2\mathbf{b}^2}{8\pi}. \tag{5.245}$$

Part d. With atmospheric pressure at $101.3kPa$, and the pressure from the light at $1300W/3x10^8 m/s$, we have roughly $4x10^-5Pa$ of pressure from the sunlight being only $\sim 10^-10$ of the total atmospheric pressure. Wow. Very tiny!

Would it make any difference if the surface is a perfect absorber or a reflector? Consider a ball hitting a wall. If it manages to embed itself in the wall, the wall will have to move a bit to conserve momentum. However, if the ball bounces off twice the momentum has been transferred to the wall. The numbers above would be for perfect absorbtion, so double them for a perfect reflector.

Exercise 5.2 Spherical EM waves

Suppose you are given:

$$\begin{aligned}&\vec{E}(r,\theta,\phi,t)\\&= A\frac{\sin\theta}{r}\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\hat{\boldsymbol{\phi}}\end{aligned} \tag{5.246}$$

where $\omega = k/c$ and $\hat{\boldsymbol{\phi}}$ is the unit vector in the ϕ-direction. This is a simple example of a spherical wave.

a. Show that $\vec{E}$ obeys all four Maxwell equations in vacuum and find the associated magnetic field.
b. Calculate the Poynting vector. Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I} \equiv \left\langle\vec{S}\right\rangle$. Where does it point to? How does it depend on r?
c. Integrate the intensity vector flux through a spherical surface centered at the origin to find the total power radiated.

Answer for Exercise 5.2

Part a.

Maxwell equation verification and magnetic field Our vacuum Maxwell equations to verify are

$$\begin{aligned}\nabla \cdot \vec{E} &= 0 \\ \nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} &= 0 \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} &= 0.\end{aligned} \tag{5.247}$$

We will also need the spherical polar forms of the divergence and curl operators, as found in §1.4 of [4]

$$\begin{aligned}\nabla \cdot \vec{v} &= \frac{1}{r^2}\partial_r(r^2 v_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\partial_\phi v_\phi \\ \nabla \times \vec{v} &= \frac{1}{r\sin\theta}\left(\partial_\theta(\sin\theta v_\phi) - \partial_\phi v_\theta\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\phi v_r - \partial_r(r v_\phi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\partial_r(r v_\theta) - \partial_\theta v_r\right)\hat{\boldsymbol{\phi}}\end{aligned} \tag{5.248}$$

We can start by verifying the divergence equation for the electric field. Observe that our electric field has only an E_ϕ component, so our divergence is

$$\nabla \cdot \vec{E} = \frac{1}{r\sin\theta}\partial_\phi\left(A\frac{\sin\theta}{r}\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\right) = 0. \tag{5.249}$$

We have a zero divergence since the component E_ϕ has no ϕ dependence (whereas $\vec{E}$ itself does since the unit vector $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}(\phi)$).

All of the rest of Maxwell's equations require $\vec{B}$ so we will have to first calculate that before progressing further.

A aside on approaches attempted to find $\vec{B}$ I tried two approaches without success to calculate $\vec{B}$. First I hoped that I could just integrate $-\vec{E}$ to obtain $\vec{A}$ and then take the curl. Doing so gave me a result that had $\nabla \times \vec{B} \neq 0$. I hunted for an algebraic error that would account for this, but could not find one.

The second approach that I tried, also without success, was to simply take the cross product $\hat{\mathbf{r}} \times \vec{E}$. This worked in the monochromatic plane wave case where we had

$$\begin{aligned}\vec{B} &= (\vec{k} \times \vec{\beta})\sin(\omega t - \vec{k}\cdot\vec{x}) \\ \vec{E} &= \vec{\beta}\left|\vec{k}\right|\sin(\omega t - \vec{k}\cdot\vec{x})\end{aligned} \tag{5.250}$$

since one can easily show that $\vec{B} = \vec{k} \times \vec{E}$. Again, I ended up with a result for $\vec{B}$ that did not have a zero divergence.

Finding $\vec{B}$ with a more systematic approach Following [6] §16.2, let us try a phasor approach, assuming that all the solutions, whatever they are, have all the time dependence in a $e^{-i\omega t}$ term.

Let us write our fields as

$$\begin{aligned}\vec{E} &= \mathrm{Re}(\mathbf{E}e^{-i\omega t}) \\ \vec{B} &= \mathrm{Re}(\mathbf{B}e^{-i\omega t}).\end{aligned} \tag{5.251}$$

Substitution back into Maxwell's equations thus requires equality in the real parts of

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= -i\frac{\omega}{c}\mathbf{E} \\ \nabla \times \mathbf{E} &= i\frac{\omega}{c}\mathbf{B}\end{aligned} \tag{5.252}$$

With $k = \omega/c$ we can now directly compute the magnetic field phasor

$$\mathbf{B} = -\frac{i}{k}\nabla \times \mathbf{E}. \tag{5.253}$$

The electric field of this problem can be put into phasor form by noting

$$\vec{E} = A\frac{\sin\theta}{r}\,\mathrm{Re}\left(e^{i(kr-\omega t)} - \frac{i}{kr}e^{i(kr-\omega t)}\right)\hat{\boldsymbol{\phi}}, \tag{5.254}$$

which allows for reading off the phasor part directly

$$\mathbf{E} = A\frac{\sin\theta}{r}\left(1 - \frac{i}{kr}\right)e^{ikr}\hat{\boldsymbol{\phi}}. \tag{5.255}$$

Now we can compute the magnetic field phasor $\mathbf{B}$. Since we have only a ϕ component in our field, the curl will have just $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ components. This is reasonable since we expect it to be perpendicular to $\mathbf{E}$.

$$\nabla \times (v_\phi \hat{\boldsymbol{\phi}}) = \frac{1}{r\sin\theta}\partial_\theta(\sin\theta v_\phi)\hat{\mathbf{r}} - \frac{1}{r}\partial_r(r v_\phi)\hat{\boldsymbol{\theta}}. \tag{5.256}$$

Chugging through all the algebra we have

$$\begin{aligned} ik\mathbf{B} &= \nabla \times \mathbf{E} \\ &= \frac{2A\cos\theta}{r^2}\left(1 - \frac{i}{kr}\right)e^{ikr}\hat{\mathbf{r}} - \frac{A\sin\theta}{r}\frac{\partial}{\partial r}\left(\left(1 - \frac{i}{kr}\right)e^{ikr}\right)\hat{\boldsymbol{\theta}} \\ &= \frac{2A\cos\theta}{r^2}\left(1 - \frac{i}{kr}\right)e^{ikr}\hat{\mathbf{r}} - \frac{A\sin\theta}{r}\left(ik + \frac{1}{r} + \frac{i}{kr^2}\right)e^{ikr}\hat{\boldsymbol{\theta}}, \end{aligned} \tag{5.257}$$

so our magnetic phasor is

$$\mathbf{B} = \frac{2A\cos\theta}{kr^2}\left(-i - \frac{1}{kr}\right)e^{ikr}\hat{\mathbf{r}} - \frac{A\sin\theta}{r}\left(1 - \frac{i}{kr} + \frac{1}{k^2r^2}\right)e^{ikr}\hat{\boldsymbol{\theta}} \tag{5.258}$$

Multiplying by $e^{-i\omega t}$ and taking real parts gives us the messy magnetic field expression

$$\begin{aligned} \vec{B} &= \frac{A}{r}\frac{2\cos\theta}{kr}\left(\sin(kr - \omega t) - \frac{1}{kr}\cos(kr - \omega t)\right)\hat{\mathbf{r}} \\ &\quad - \frac{A}{r}\frac{\sin\theta}{kr}\left(\sin(kr - \omega t) + \frac{k^2r^2 + 1}{kr}\cos(kr - \omega t)\right)\hat{\boldsymbol{\theta}}. \end{aligned} \tag{5.259}$$

Since this was constructed directly from $\nabla \times \vec{E} + \frac{1}{c}\partial\vec{B}/\partial t = 0$, this implicitly verifies one more of Maxwell's equations, leaving only $\nabla \cdot \vec{B}$, and $\nabla \times \vec{B} - \frac{1}{c}\partial\vec{E}/\partial t = 0$. Neither of these looks particularly fun to verify, however, we can take a small shortcut and use the phasors to verify without the explicit time dependence.

From eq. (5.258) we have for the divergence

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \frac{2A\cos\theta}{kr^2}\frac{\partial}{\partial r}\left(\left(-i - \frac{1}{kr}\right)e^{ikr}\right) - \frac{A2\cos\theta}{r^2}\left(1 - \frac{i}{kr} + \frac{1}{k^2r^2}\right)e^{ikr} \\ &= \frac{2A\cos\theta}{r^2}e^{ikr}\left(\frac{1}{k}\left(\frac{1}{kr^2} + ik\left(-i - \frac{1}{kr}\right)\right) - \left(1 - \frac{i}{kr} + \frac{1}{k^2r^2}\right)\right) \\ &= 0. \qquad \square \end{aligned} \tag{5.260}$$

Let us also verify the last of Maxwell's equations in phasor form. The time dependence is knocked out, and we want to see that taking the curl of the magnetic phasor returns us (scaled) the electric phasor. That is

$$\nabla \times \mathbf{B} = -i\frac{\omega}{c}\mathbf{E} \tag{5.261}$$

With only r and θ components in the magnetic phasor we have

$$\nabla \times (v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}}) = -\frac{1}{r \sin\theta} \partial_\phi v_\theta \hat{\mathbf{r}} + \frac{1}{r} \frac{1}{\sin\theta} \partial_\phi v_r \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\partial_r (r v_\theta) - \partial_\theta v_r \right) \hat{\boldsymbol{\phi}} \tag{5.262}$$

Immediately, we see that with no explicit ϕ dependence in the coordinates, we have no $\hat{\mathbf{r}}$ nor $\hat{\boldsymbol{\theta}}$ terms in the curl, which is good. Our curl is now just

$$\begin{aligned}
\nabla \times \mathbf{B} &= \frac{1}{r} \left(A \sin\theta \partial_r \left(1 - \frac{i}{kr} + \frac{1}{k^2 r^2} \right) e^{ikr} + \frac{2A \sin\theta}{kr^2} \left(-i - \frac{1}{kr} \right) e^{ikr} \right) \hat{\boldsymbol{\phi}} \\
&= A \sin\theta \frac{1}{r} \left(\partial_r \left(1 - \frac{i}{kr} + \frac{1}{k^2 r^2} \right) e^{ikr} + \frac{2}{kr^2} \left(-i - \frac{1}{kr} \right) e^{ikr} \right) \hat{\boldsymbol{\phi}} \\
&= A \sin\theta e^{ikr} \frac{1}{r} \\
&\qquad \left((ik) \left(1 - \frac{i}{kr} + \frac{1}{k^2 r^2} \right) + \left(\frac{i}{kr^2} - \frac{2}{k^2 r^3} \right) + \frac{2}{kr^2} \left(-i - \frac{1}{kr} \right) \right) \hat{\boldsymbol{\phi}} \\
&= A \sin\theta e^{ikr} \frac{1}{r} \left(ik + \frac{1}{r} - \frac{4}{k^2 r^3} \right) \hat{\boldsymbol{\phi}}
\end{aligned} \tag{5.263}$$

What we expect is $\nabla \times \mathbf{B} = -ik\mathbf{E}$ which is

$$-ik\mathbf{E} = A \sin\theta e^{ikr} \frac{1}{r} \left(-ik - \frac{1}{r} \right) \hat{\boldsymbol{\phi}} \tag{5.264}$$

FIXME: Somewhere I must have made a sign error, because these are not matching! Have an extra $1/r^3$ term and the wrong sign on the $1/r$ term.

Part b. Our Poynting vector is

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}, \tag{5.265}$$

which we could calculate from eq. (5.246), and eq. (5.259). However, that looks like it is going to be a mess to multiply out. Let us use instead the trick from §48 of the course text [11], and work with the complex quantities directly, noting that we have

$$\begin{aligned}
(\operatorname{Re} \mathbf{E} e^{i\alpha}) \times (\operatorname{Re} \mathbf{B} e^{i\alpha}) &= \frac{1}{4} (\mathbf{E} e^{i\alpha} + \mathbf{E}^* e^{-i\alpha}) \times (\mathbf{B} e^{i\alpha} + \mathbf{B}^* e^{-i\alpha}) \\
&= \frac{1}{2} \operatorname{Re} \left(\mathbf{E} \times \mathbf{B}^* + (\mathbf{E} \times \mathbf{B}) e^{2i\alpha} \right).
\end{aligned} \tag{5.266}$$

Now we can do the Poynting calculation using the simpler relations eq. (5.255), eq. (5.258). Let us also write

$$\begin{aligned} \mathbf{E} &= Ae^{ikr}E_\phi\hat{\boldsymbol{\phi}} \\ \mathbf{B} &= Ae^{ikr}(B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}}) \end{aligned} \tag{5.267}$$

where

$$\begin{aligned} E_\phi &= \frac{\sin\theta}{r}\left(1 - \frac{i}{kr}\right) \\ B_r &= -\frac{2\cos\theta}{kr^2}\left(i + \frac{1}{kr}\right) \\ B_\theta &= -\frac{\sin\theta}{r}\left(1 - \frac{i}{kr} + \frac{1}{k^2r^2}\right) \end{aligned} \tag{5.268}$$

So our Poynting vector is

$$\vec{S} = \frac{A^2c}{2\pi}\,\mathrm{Re}\left(E_\phi\hat{\boldsymbol{\phi}} \times (B_r^*\hat{\mathbf{r}} + B_\theta^*\hat{\boldsymbol{\theta}}) + E_\phi\hat{\boldsymbol{\phi}} \times (B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}})e^{2i(kr-\omega t)}\right) \tag{5.269}$$

Note that our unit vector basis $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ was rotated from $\{\hat{\mathbf{z}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}\}$, so we have

$$\begin{aligned} \hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} &= \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} &= \hat{\mathbf{r}} \\ \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{\phi}}, \end{aligned} \tag{5.270}$$

and plug this into our Poynting expression

$$\vec{S} = \frac{A^2c}{2\pi}\,\mathrm{Re}\left(E_\phi B_r^*\hat{\boldsymbol{\theta}} - E_\phi B_\theta^*\hat{\mathbf{r}} + (E_\phi B_r\hat{\boldsymbol{\theta}} - E_\phi B_\theta\hat{\mathbf{r}})e^{2i(kr-\omega t)}\right) \tag{5.271}$$

Now we have to multiply out our terms. We have

$$\begin{aligned} E_\phi B_r^* &= -\frac{\sin\theta}{r}\frac{2\cos\theta}{kr^2}\left(1 - \frac{i}{kr}\right)\left(-i + \frac{1}{kr}\right) \\ &= -\frac{\sin(2\theta)}{kr^3}\left(-i - \frac{i}{k^2r^2}\right), \end{aligned} \tag{5.272}$$

Since this has no real part, there is no average contribution to $\vec{S}$ in the $\hat{\boldsymbol{\theta}}$ direction. What do we have for the time dependent part

$$\begin{aligned} E_\phi B_r &= -\frac{\sin\theta}{r}\frac{2\cos\theta}{kr^2}\left(1 - \frac{i}{kr}\right)\left(i + \frac{1}{kr}\right) \\ &= -\frac{\sin(2\theta)}{kr^3}\left(i + \frac{2}{kr} - \frac{i}{k^2r^2}\right) \end{aligned} \tag{5.273}$$

This is non zero, so we have a time dependent $\hat{\boldsymbol{\theta}}$ contribution that averages out. Moving on

$$\begin{aligned} -E_\phi B_\theta^* &= \frac{\sin^2\theta}{r^2}\left(1 - \frac{i}{kr}\right)\left(1 + \frac{i}{kr} + \frac{1}{k^2r^2}\right) \\ &= \frac{\sin^2\theta}{r^2}\left(1 + \frac{2}{k^2r^2} - \frac{i}{k^3r^3}\right). \end{aligned} \tag{5.274}$$

This is non-zero, so the steady state Poynting vector is in the outwards radial direction. The last piece is

$$\begin{aligned} -E_\phi B_\theta &= \frac{\sin^2\theta}{r^2}\left(1 - \frac{i}{kr}\right)\left(1 - \frac{i}{kr} + \frac{1}{k^2r^2}\right) \\ &= \frac{\sin^2\theta}{r^2}\left(1 - \frac{2i}{kr} - \frac{i}{k^3r^3}\right). \end{aligned} \tag{5.275}$$

Assembling all the results we have

$$\begin{aligned} \vec{S} &= \frac{A^2c}{2\pi}\frac{\sin^2\theta}{r^2}\left(1 + \frac{2}{k^2r^2}\right)\hat{\mathbf{r}} \\ &+ \frac{A^2c}{2\pi} \\ &\operatorname{Re}\left(\left(-\frac{\sin(2\theta)}{kr^3}\left(i + \frac{2}{kr} - \frac{i}{k^2r^2}\right)\hat{\boldsymbol{\theta}} + \frac{\sin^2\theta}{r^2}\left(1 - \frac{2i}{kr} - \frac{i}{k^3r^3}\right)\hat{\mathbf{r}}\right)e^{2i(kr-\omega t)}\right) \end{aligned} \tag{5.276}$$

We can read off the intensity directly

$$\vec{I} = \left\langle\vec{S}\right\rangle = \frac{A^2c\sin^2\theta}{2\pi r^2}\left(1 + \frac{2}{k^2r^2}\right)\hat{\mathbf{r}} \tag{5.277}$$

Part c. Through a surface of radius r, integration of the intensity vector eq. (5.277) is

$$\begin{aligned}\int r^2 \sin\theta d\theta d\phi \vec{I} &= \int r^2 \sin\theta d\theta d\phi \frac{A^2 c \sin^2\theta}{2\pi r^2}\left(1 + \frac{2}{k^2 r^2}\right)\hat{\mathbf{r}} \\ &= A^2 c \left(1 + \frac{2}{k^2 r^2}\right)\hat{\mathbf{r}} \int_0^\pi \sin^3\theta d\theta \\ &= A^2 c \left(1 + \frac{2}{k^2 r^2}\right)\hat{\mathbf{r}} \frac{1}{12}(\cos(3\theta) - 9\cos\theta)\Big|_0^\pi .\end{aligned} \tag{5.278}$$

Our average power through the surface is therefore

$$\int d^2\boldsymbol{\sigma}\vec{I} = \frac{4A^2 c}{3}\left(1 + \frac{2}{k^2 r^2}\right)\hat{\mathbf{r}}. \tag{5.279}$$

Notes on grading of my solution This was the graded portion.

FIXME1: I lost a mark in the spot I expected, where I failed to verify one of the Maxwell equations. I will still need to figure out what got messed up there.

What occured to me later, also mentioned in the grading of the solution was that Maxwell's equations in the space-time domain could have been used to solve for $\partial \mathbf{B}/\partial t$ instead of all the momentum space logic (which simplified some things, but probably complicated others).

FIXME2: I lost a mark on eq. (5.277) with a big X beside it. I will have to read the graded solution to see why.

FIXME3: Lost a mark for the final average power result eq. (5.279). Again, I will have to go back and figure out why.

6

LIENARD-WIECHERT POTENTIALS

6.1 SOLVING MAXWELL'S EQUATION

Our equations are

$$\begin{aligned} \epsilon^{ijkl}\partial_j F_{kl} &= 0 \\ \partial_i F^{ik} &= \frac{4\pi}{c} j^k, \end{aligned} \tag{6.1}$$

where we assume that $j^k(\mathbf{x}, t)$ is a given. Our task is to find F^{ik}, the $(\mathbf{E}, \mathbf{B})$ fields.

Proceed by finding A^i. First, as usual when $F_{ij} = \partial_i A_j - \partial_j A_i$. The Bianchi identity is satisfied so we focus on the current equation.

In terms of potentials

$$\partial_i(\partial^i A^k - \partial^k A^i) = \frac{4\pi}{c} j^k \tag{6.2}$$

or

$$\partial_i \partial^i A^k - \partial^k(\partial_i A^i) = \frac{4\pi}{c} j^k \tag{6.3}$$

We want to work in the Lorentz gauge $\partial_i A^i = 0$. This is justified by the simplicity of the remaining problem

$$\partial_i \partial^i A^k = \frac{4\pi}{c} j^k \tag{6.4}$$

Write

$$\partial_i \partial^i = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta = \Box \tag{6.5}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{6.6}$$

This $\Box$ is the d'Alembert operator ("d'Alembertian").
Our equation is

$$\Box A^k = \frac{4\pi}{c} j^k \tag{6.7}$$

(in the Lorentz gauge)
If we learn how to solve (**), then we have learned all.
Method: Green's function's
In electrostatics where $j^0 = 0$, $A^0 \neq 0$ only, we have

$$\Delta A^0 = -4\pi\rho \tag{6.8}$$

Solution

$$\Delta_{\mathbf{x}} G(\mathbf{x} - \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}') \tag{6.9}$$

PICTURE:

$$\rho(\mathbf{x}') d^3\mathbf{x}' \tag{6.10}$$

(a small box)
acting through distance $|\mathbf{x} - \mathbf{x}'|$, acting at point $\mathbf{x}$. With $G(\mathbf{x}, \mathbf{x}') = -1/4\pi|\mathbf{x} - \mathbf{x}'|$, we have

$$\begin{aligned} \int d^3\mathbf{x}' \Delta_{\mathbf{x}} G(\mathbf{x} - \mathbf{x}')\rho(\mathbf{x}') &= \int d^3\mathbf{x}' \delta^3(\mathbf{x} - \mathbf{x}')\rho(\mathbf{x}') \\ &= \rho(\mathbf{x}) \end{aligned} \tag{6.11}$$

Also since G is deemed a linear operator, we have $\Delta_{\mathbf{x}} G = G\Delta_{\mathbf{x}}$, we find

$$\begin{aligned} \rho(\mathbf{x}) &= \int d^3\mathbf{x}' \Delta_{\mathbf{x}} G(\mathbf{x} - \mathbf{x}') 4\pi\rho(\mathbf{x}') \\ &= \int d^3\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|}\rho(\mathbf{x}'). \end{aligned} \tag{6.12}$$

We end up finding that

$$\phi(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \tag{6.13}$$

thus solving the problem. We wish next to do this for the Maxwell equation eq. (6.7).

The Green's function method is effective, but I can not help but consider it somewhat of a cheat, since one has to through higher powers know what the Green's function is. In the electrostatics case, at least we can work from the potential function and take its Laplacian to find that this is equivalent (thus implictly solving for the Green's function at the same time). It will be interesting to see how we do this for the forced d'Alembertian equation.

Reading Covering chapter 8 material from the text [11], and lecture notes RelEMpp136-146.pdf.

6.2 SOLVING THE FORCED WAVE EQUATION

See the notes for a complex variables and Fourier transform method of deriving the Green's function. In class, we will just pull it out of a magic hat. We wish to solve

$$\Box A^k = \partial_i \partial^i A^k = \frac{4\pi}{c} j^k \tag{6.14}$$

(with a $\partial_i A^i = 0$ gauge choice).

Our Green's method utilizes

$$\Box_{(\mathbf{x},t)} G(\mathbf{x} - \mathbf{x}', t - t') = \delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t') \tag{6.15}$$

If we know such a function, our solution is simple to obtain

$$A^k(\mathbf{x}, t) = \int d^3\mathbf{x}' dt' \frac{4\pi}{c} j^k(\mathbf{x}', t') G(\mathbf{x} - \mathbf{x}', t - t') \tag{6.16}$$

Proof:

$$\begin{aligned}
\Box_{(\mathbf{x},t)} A^k(\mathbf{x}, t) &= \int d^3\mathbf{x}' dt' \frac{4\pi}{c} j^k(\mathbf{x}', t') \Box_{(\mathbf{x},t)} G(\mathbf{x} - \mathbf{x}', t - t') \\
&= \int d^3\mathbf{x}' dt' \frac{4\pi}{c} j^k(\mathbf{x}', t') \delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t') \\
&= \frac{4\pi}{c} j^k(\mathbf{x}, t)
\end{aligned} \tag{6.17}$$

Claim:

$$G(\mathbf{x}, t) = \frac{\delta(t - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|} \tag{6.18}$$

This is the retarded Green's function of the operator $\Box$, where

$$\Box G(\mathbf{x}, t) = \delta^3(\mathbf{x})\delta(t) \tag{6.19}$$

6.3 ELABORATING ON THE WAVE EQUATION GREEN'S FUNCTION

The Green's function eq. (K.27) is a distribution that is non-zero only on the future lightcone. Observe that for $t < 0$ we have

$$\begin{aligned}\delta\left(t - \frac{|\mathbf{x}|}{c}\right) &= \delta\left(-|t| - \frac{|\mathbf{x}|}{c}\right) \\ &= 0.\end{aligned} \tag{6.20}$$

We say that G is supported only on the future light cone. At $\mathbf{x} = 0$, only the contributions for $t > 0$ matter. Note that in the "old days", Green's functions used to be called influence functions, a name that works particularly well in this case. We have other Green's functions for the d'Alembertian. The one above is called the retarded Green's functions and we also have an advanced Green's function. Writing + for advanced and − for retarded these are

$$G_{\pm} = \frac{\delta\left(t \pm \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|} \tag{6.21}$$

There are also causal and non-causal variations that will not be of interest for this course.

This arms us now to solve any problem in the Lorentz gauge

$$A^k(\mathbf{x}, t) = \frac{1}{c}\int d^3\mathbf{x}' dt' \frac{\delta\left(t - t' - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{4\pi|\mathbf{x} - \mathbf{x}'|} j^k(\mathbf{x}', t') + \text{An arbitrary collection of EM waves.} \tag{6.22}$$

The additional EM waves are the possible contributions from the homogeneous equation.

Since $\delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c)$ is non-zero only when $t' = t - |\mathbf{x} - \mathbf{x}'|/c)$, the non-homogeneous parts of eq. (6.22) reduce to

$$A^k(\mathbf{x}, t) = \frac{1}{c}\int d^3\mathbf{x}' \frac{j^k(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c)}{4\pi|\mathbf{x} - \mathbf{x}'|}. \tag{6.23}$$

Our potentials at time t and spatial position $\mathbf{x}$ are completely specified in terms of the sums of the currents acting at the retarded time $t - |\mathbf{x} - \mathbf{x}'|/c$. The field can only depend on the charge and current distribution in the past. Specifically, it can only depend on the charge and current distribution on the past light cone of the spacetime point at which we measure the field.

Example 6.1: The Green's function, a charged particle moving on a worldline

$$(ct, \mathbf{x}_c(t)) \tag{6.24}$$

(c for classical)
For this particle

$$\begin{aligned} \rho(\mathbf{x}, t) &= e\delta^3(\mathbf{x} - \mathbf{x}_c(t)) \\ \mathbf{j}(\mathbf{x}, t) &= e\dot{\mathbf{x}}_c(t)\delta^3(\mathbf{x} - \mathbf{x}_c(t)) \end{aligned} \tag{6.25}$$

$$\begin{aligned} \begin{bmatrix} A^0(\mathbf{x}, t) \\ \mathbf{A}(\mathbf{x}, t) \end{bmatrix} &= \frac{1}{c} \int d^3\mathbf{x}' dt' \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c}{|\mathbf{x} - \mathbf{x}'|} \begin{bmatrix} ce \\ e\dot{\mathbf{x}}_c(t) \end{bmatrix} \delta^3(\mathbf{x} - \mathbf{x}_c(t)) \\ &= \int_{-\infty}^{\infty} \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}_c(t')|/c}{|\mathbf{x}_c(t') - \mathbf{x}|} \begin{bmatrix} e \\ e\frac{\dot{\mathbf{x}}_c(t)}{c} \end{bmatrix} \end{aligned} \tag{6.26}$$

PICTURE: light cones, and curved worldline. Pick an arbitrary point $(\mathbf{x}_0, t_0)$, and draw the past light cone, looking at where this intersects with the trajectory

For the arbitrary point $(\mathbf{x}_0, t_0)$ we see that this point and the retarded time $(\mathbf{x}_c(t_r), t_r)$ obey the relation

$$c(t_0 - t_r) = |\mathbf{x}_0 - \mathbf{x}_c(t_r)| \tag{6.27}$$

This retarded time is unique. There is only one such intersection.
Our job is to calculate

$$\int_{-\infty}^{\infty} \delta(f(x))g(x) = \frac{g(x_*)}{|f'(x_*)|} \tag{6.28}$$

where $f(x_*) = 0$.

$$f(t') = t - t' - \left|\mathbf{x} - \mathbf{x}_c(t')\right|/c \tag{6.29}$$

$$
\begin{aligned}
\frac{\partial f}{\partial t'} &= -1 - \frac{1}{c}\frac{\partial}{\partial t'}\sqrt{(\mathbf{x} - \mathbf{x}_c(t')) \cdot (\mathbf{x} - \mathbf{x}_c(t'))} \\
&= -1 + \frac{1}{c}\frac{(\mathbf{x} - \mathbf{x}_c(t')) \cdot \mathbf{v}_c(t')}{|\mathbf{x} - \mathbf{x}_c(t')|}
\end{aligned}
\tag{6.30}
$$

This is with

$$
\mathbf{v}_c = \frac{\partial \mathbf{x}_c}{\partial t'}.
\tag{6.31}
$$

Putting things back together, the potentials due to a moving charge are

$$
\begin{aligned}
\begin{bmatrix} A^0(\mathbf{x}, t) \\ \mathbf{A}(\mathbf{x}, t) \end{bmatrix} &= e\frac{1}{|\mathbf{x}_c(t_r) - \mathbf{x}|}\begin{bmatrix} 1 \\ \frac{\mathbf{v}_c}{c} \end{bmatrix}\frac{1}{\left|-1 + \frac{1}{c}\frac{(\mathbf{x}-\mathbf{x}_c(t_r))\cdot\mathbf{v}_c(t_r)}{|\mathbf{x}-\mathbf{x}_c(t_r)|}\right|} \\
&= e\begin{bmatrix} 1 \\ \frac{\mathbf{v}_c}{c} \end{bmatrix}\frac{1}{||\mathbf{x} - \mathbf{x}_c(t_r)| - (\mathbf{x} - \mathbf{x}_c(t_r)) \cdot \mathbf{v}_c(t_r)/c|}
\end{aligned}
\tag{6.32}
$$

Provided $|\mathbf{x} - \mathbf{x}_c| > (\mathbf{x} - \mathbf{x}_c(t_r)) \cdot \mathbf{v}_c(t_r)/c$, we have the Lienard-Wiechert potentials.

$$
\begin{bmatrix} A^0(\mathbf{x}, t) \\ \mathbf{A}(\mathbf{x}, t) \end{bmatrix} = e\begin{bmatrix} 1 \\ \frac{\mathbf{v}_c}{c} \end{bmatrix}\frac{1}{|\mathbf{x} - \mathbf{x}_c| - (\mathbf{x} - \mathbf{x}_c(t_r)) \cdot \mathbf{v}_c(t_r)/c}
\tag{6.33}
$$

FIXME: What provides the previous inequality required to get to this final point?

Reading Covering chapter 8 material from the text [11], and lecture notes RelEMpp136-146.pdf.

6.4 FIELDS FROM THE LIENARD-WIECHERT POTENTIALS

(We finished off with the scalar and vector potentials in class, but I have put those notes with the previous lecture).

To find **E** and **B** need

$\frac{\partial t_r}{\partial t}$, and $\boldsymbol{\nabla} t_r(\mathbf{x}, t)$

where

$$t - t_r = |\mathbf{x} - \mathbf{x}_c(t_r)| \tag{6.34}$$

implicit definition of $t_r(\mathbf{x}, t)$
In HW5 you will show

$$\frac{\partial t_r}{\partial t} = \frac{|\mathbf{x} - \mathbf{x}_c(t_r)|}{|\mathbf{x} - \mathbf{x}_c(t_r)| - \frac{\mathbf{v}_c}{c} \cdot (\mathbf{x} - \mathbf{x}_c(t_r))} \tag{6.35}$$

$$\boldsymbol{\nabla} t_r = \frac{1}{c} \frac{\mathbf{x} - \mathbf{x}_c(t_r)}{|\mathbf{x} - \mathbf{x}_c(t_r)| - \frac{\mathbf{v}_c}{c} \cdot (\mathbf{x} - \mathbf{x}_c(t_r))} \tag{6.36}$$

and then use this to show that the electric and magnetic fields due to a moving charge are

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \frac{eR}{(\mathbf{R} \cdot \mathbf{u})^3} \Big((c^2 - \mathbf{v}_c^2)\mathbf{u} + \mathbf{R} \times (\mathbf{u} \times \mathbf{a}_c) \Big) \\ &= \frac{\mathbf{R}}{R} \times \mathbf{E} \\ \mathbf{u} &= c\frac{\mathbf{R}}{R} - \mathbf{v}_c, \end{aligned} \tag{6.37}$$

where everything is evaluated at the retarded time $t_r = t - |\mathbf{x} - \mathbf{x}_c(t_r)|/c$.

This looks quite a bit different than what we find in §63 (63.8) in the text, but a little bit of expansion shows they are the same.

6.5 CHECK. PARTICLE AT REST

With

$$\begin{aligned} \mathbf{x}_c &= \mathbf{x}_0 \\ X_c^k &= (ct, \mathbf{x}_0) \\ |\mathbf{x} - \mathbf{x}_c(t_r)| &= c(t - t_r) \end{aligned} \tag{6.38}$$

As illustrated in fig. 6.1 the retarded position is

$$\mathbf{x}_c(t_r) = \mathbf{x}_0, \tag{6.39}$$

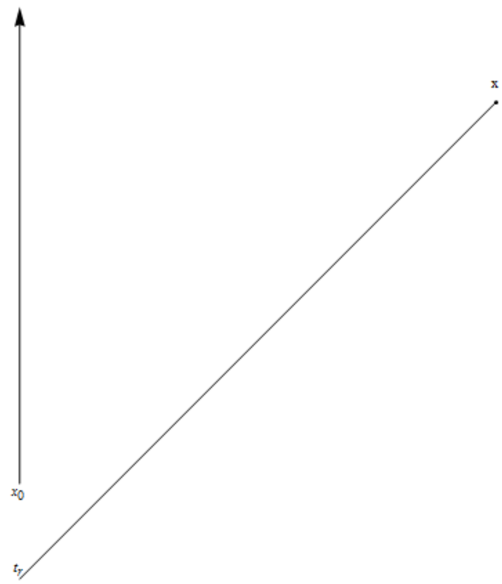

Figure 6.1: Retarded time for particle at rest

for

$$\mathbf{u} = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} c, \tag{6.40}$$

and

$$\mathbf{E} = e \frac{\cancel{|\mathbf{x} - \mathbf{x}_0|}}{(c|\mathbf{x} - \mathbf{x}_0|)^3} c^3 \frac{\mathbf{x} - \mathbf{x}_0}{\cancel{|\mathbf{x} - \mathbf{x}_0|}}, \tag{6.41}$$

which is Coulomb's law

$$\mathbf{E} = e \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^3} \tag{6.42}$$

6.6 CHECK. PARTICLE MOVING WITH CONSTANT VELOCITY

This was also computed in full in homework 5. The end result was

$$\mathbf{E} = e \frac{\mathbf{x} - \mathbf{v}t}{|\mathbf{x} - \mathbf{v}t|^3} \frac{1 - \boldsymbol{\beta}^2}{\left(1 - \frac{(\mathbf{x} \times \boldsymbol{\beta})^2}{|\mathbf{x} - \mathbf{v}t|^2}\right)^{3/2}} \tag{6.43}$$

Writing

$$\begin{aligned}\frac{\mathbf{x}\times\boldsymbol{\beta}}{|\mathbf{x}-\mathbf{v}t|} &= \frac{1}{c}\frac{(\mathbf{x}-\mathbf{v}t)\times\mathbf{v}}{|\mathbf{x}-\mathbf{v}t|} \\ &= \frac{|\mathbf{v}|}{c}\frac{(\mathbf{x}-\mathbf{v}t)\times\mathbf{v}}{|\mathbf{x}-\mathbf{v}t||\mathbf{v}|}\end{aligned} \tag{6.44}$$

We can introduce an angular dependence between the charge's translated position and its velocity

$$\sin^2\theta = \left|\frac{\mathbf{v}\times(\mathbf{x}-\mathbf{v}t)}{|\mathbf{v}||\mathbf{x}-\mathbf{v}t|}\right|^2, \tag{6.45}$$

and write the field as

$$\mathbf{E} = \underbrace{\boxed{e\frac{\mathbf{x}-\mathbf{v}t}{|\mathbf{x}-\mathbf{v}t|^3}}}_{(*)}\frac{1-\boldsymbol{\beta}^2}{\left(1-\frac{\mathbf{v}^2}{c^2}\sin^2\theta\right)^{3/2}} \tag{6.46}$$

Observe that $(*)$ = Coulomb's law measured from the instantaneous position of the charge.

The electric field $\mathbf{E}$ has a time dependence, strongest when perpendicular to the instantaneous position when $\theta = \pi/2$, since the denominator is smallest ($\mathbf{E}$ largest) when $\mathbf{v}/c$ is not small. This is strongly θ dependent.

Compare

$$\begin{aligned}\frac{|\mathbf{E}(\theta=\pi/2)| - |\mathbf{E}(\theta=\pi/2+\Delta\theta)|}{|\mathbf{E}(\theta=\pi/2)|} &\approx \frac{\frac{1}{(1-\mathbf{v}^2/c^2)^{3/2}} - \frac{1}{(1-\mathbf{v}^2/c^2(1-(\Delta\theta)^2))^{3/2}}}{\frac{1}{(1-\mathbf{v}^2/c^2)^{3/2}}} \\ &= 1 - \left(\frac{1-\mathbf{v}^2/c^2}{1-\mathbf{v}^2/c^2+\mathbf{v}^2/c^2(\Delta\theta)^2}\right)^{3/2} \\ &= 1 - \left(\frac{1}{1+\mathbf{v}^2/c^2\frac{(\Delta\theta)^2}{1-\mathbf{v}^2/c^2}}\right)^{3/2}\end{aligned} \tag{6.47}$$

Here we used

$$\sin(\theta+\pi/2) = \frac{e^{i(\theta+\pi/2)} - e^{-i(\theta+\pi/2)}}{2i} = \cos\theta \tag{6.48}$$

and

$$\cos^2 \Delta\theta \approx \left(1 - \frac{(\Delta\theta)^2}{2}\right)^2 \approx 1 - (\Delta\theta)^2 \tag{6.49}$$

FIXME: he writes:

$$\Delta\theta \le \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \tag{6.50}$$

I do not see where that comes from.
PICTURE: Various **E**'s up, and **v** perpendicular to that, strongest when charge is moving fast.

6.7 BACK TO EXTRACTING PHYSICS FROM THE LIENARD-WIECHERT FIELD EQUATIONS

Imagine that we have a localized particle motion with

$$|\mathbf{x}_c(t_r)| < l \tag{6.51}$$

The velocity vector

$$\mathbf{u} = c\frac{\mathbf{x} - \mathbf{x}_c(t_r)}{|\mathbf{x} - \mathbf{x}_c|} \tag{6.52}$$

does not grow as distance from the source, so from eq. (6.37), we have for $|\mathbf{x}| \gg l$

$$\mathbf{B}, \mathbf{E} \sim \frac{1}{|\mathbf{x}|^2}(\cdots) + \frac{1}{\mathbf{x}}(\text{acceleration term}) \tag{6.53}$$

The acceleration term will dominate at large distances from the source. Our Poynting magnitude is

$$|\mathbf{S}| \sim |\mathbf{E} \times \mathbf{B}| \sim \frac{1}{\mathbf{x}^2}(\text{acceleration})^2. \tag{6.54}$$

We can ask about

$$\oint d^2\boldsymbol{\sigma} \cdot \mathbf{S} \sim R^2 \frac{1}{R^2}(\text{acceleration})^2 \sim (\text{acceleration})^2 \tag{6.55}$$

In the limit, for the radiation of EM waves

$$\lim_{R\to\infty} \oint d^2\boldsymbol{\sigma} \cdot \mathbf{S} \neq 0 \tag{6.56}$$

The energy flux through a sphere of radius R is called the radiated power.

Reading Covering chapter 8 material from the text [11], and lecture notes RelEMpp147-165.pdf.

6.8 MULTIPOLE EXPANSION OF THE FIELDS

$$A^i(\mathbf{x}, t) = \frac{1}{c} \int d^3\mathbf{x}' \, j^i\left(\mathbf{x}', t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right) \frac{1}{|\mathbf{x} - \mathbf{x}'|} \tag{6.57}$$

This integral is over the region of space where the sources j^i are non-vanishing, but this region is limited. The value $|\mathbf{x}'| \le l$, so we can expand the denominator in multipole expansion

$$\begin{aligned} \frac{1}{|\mathbf{x} - \mathbf{x}'|} &= \frac{1}{\sqrt{(\mathbf{x} - \mathbf{x}')^2}} \\ &= \frac{1}{\sqrt{\mathbf{x}^2 + \mathbf{x}'^2 - 2\mathbf{x} \cdot \mathbf{x}'}} \\ &= \frac{1}{|\mathbf{x}|} \frac{1}{\sqrt{1 + \frac{\mathbf{x}'^2}{\mathbf{x}^2} - 2\frac{\hat{\mathbf{r}}}{|\mathbf{x}|} \cdot \mathbf{x}'}} \\ &\approx \frac{1}{|\mathbf{x}|} \frac{1}{\sqrt{1 - 2\frac{\hat{\mathbf{r}}}{|\mathbf{x}|} \cdot \mathbf{x}'}} \\ &\approx \frac{1}{|\mathbf{x}|} \left(1 + \frac{\hat{\mathbf{r}}}{|\mathbf{x}|} \cdot \mathbf{x}'\right). \end{aligned} \tag{6.58}$$

Neglecting all but the first order term in the expansion we have

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot \mathbf{x}'. \tag{6.59}$$

Similarly, for the retarded time we have

$$\begin{aligned} t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} &\approx t - \frac{|\mathbf{x}|}{c}\left(1 - \frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^2}\right) \\ &= t - \frac{|\mathbf{x}|}{c} + \frac{\mathbf{x} \cdot \mathbf{x}'}{c|\mathbf{x}|} \end{aligned} \tag{6.60}$$

We can now do a first order Taylor expansion of the current j^i about the retarded time

$$j^i\left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} + \frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|} + \cdots\right) \approx j^i\left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c}\right) + \frac{\partial j^i}{\partial t}\left(\mathbf{x}, t - \frac{|\mathbf{x}|}{c}\right)\frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|}. \tag{6.61}$$

To elucidate the physics, imagine that time dependence of the source is periodic with angular frequency ω_0. For example:

$$j^i = A(\mathbf{x})e^{-i\omega t}. \tag{6.62}$$

Here we have

$$\frac{\partial j^i}{\partial t} = -i\omega_0 j^i. \tag{6.63}$$

So, for the magnitude of the second term we have

$$\left|\frac{\partial j^i}{\partial t}\frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|}\right| = \omega_0\left|j^i\frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|}\right|. \tag{6.64}$$

Requiring second term much less than the first term means

$$\left|\omega_0\frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|}\right| \ll 1. \tag{6.65}$$

But recall

$$\left|\frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|}\right| \le l, \tag{6.66}$$

so for our Taylor expansion to be valid we have the following constraints on the angular velocity and the position vectors for our charge and measurement position

$$\left|\omega_0\frac{\mathbf{x}\cdot\mathbf{x}'}{c|\mathbf{x}|}\right| \le \frac{\omega_0 l}{c} \ll 1. \tag{6.67}$$

This is a physical requirement size of the wavelength of the emitter (if the wavelength does not meet this requirement, this expansion does not work). The connection to the wavelength can be observed by noting that we have

$$\begin{aligned} \frac{\omega_0}{c} &= k \\ 2\pi k &= \frac{1}{\lambda} \\ \implies \frac{\omega_0}{c} &\sim \frac{1}{\lambda} \end{aligned} \tag{6.68}$$

6.9 PUTTING THE PIECES TOGETHER. POTENTIALS AT A DISTANCE

Moral: We will utilize two expansions (we need two small parameters)

1. $|\mathbf{x}| \gg l$
2. $\lambda \gg l$

Plugging into our current

$$A^i(\mathbf{x}, t) \approx \frac{1}{c} \int d^3\mathbf{x}' \left(j^i \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) + \frac{\partial j^i}{\partial t} \left(\mathbf{x}, t - \frac{|\mathbf{x}|}{c} \right) \frac{\mathbf{x} \cdot \mathbf{x}'}{c|\mathbf{x}|} \right) \left(\frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot \mathbf{x}' \right) \tag{6.69}$$

$$\begin{aligned} A^0(\mathbf{x}, t) \approx & \frac{1}{|\mathbf{x}|} \int d^3\mathbf{x}' \rho \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) \\ & + \frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot \int d^3\mathbf{x}' \mathbf{x}' \rho \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) + \frac{\mathbf{x}}{c|\mathbf{x}|^2} \cdot \int d^3\mathbf{x}' \mathbf{x}' \frac{\partial \rho}{\partial t} \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right). \end{aligned} \tag{6.70}$$

The first term is the total charge evaluated at the retarded time. In the second term (and in the third, where its derivative is taken) we have

$$\int d^3\mathbf{x}' \mathbf{x}' \rho \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) = \mathbf{d}(t_r), \tag{6.71}$$

which is the dipole moment evaluated at the retarded time $t_r = t - |\mathbf{x}|/c$. In the last term we can pull out the time derivative (because we are integrating over $\mathbf{x}'$)

$$\begin{aligned} \frac{1}{|\mathbf{x}|^2} \mathbf{x} \cdot \int d^3\mathbf{x}' \mathbf{x}' \frac{\partial}{\partial t} \rho \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) &= \frac{1}{|\mathbf{x}|^2} \mathbf{x} \cdot \frac{\partial}{\partial t} \int d^3\mathbf{x}' \mathbf{x}' \rho \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) \\ &= \frac{1}{|\mathbf{x}|^2} \mathbf{x} \cdot \frac{\partial}{\partial t} \mathbf{d} \left(t - \frac{|\mathbf{x}|}{c} \right) \end{aligned} \tag{6.72}$$

For the spatial components of the current lets just keep the first term

$$\begin{aligned} A^\alpha(\mathbf{x}, t) &\approx \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' j^\alpha \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) \\ &= \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' (\boldsymbol{\nabla}_{\mathbf{x}'} x^\alpha) \cdot \mathbf{j} \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) \\ &= \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' \left(\boldsymbol{\nabla} \cdot \left(x'^\alpha \mathbf{j} \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) \right) - x'^\alpha \boldsymbol{\nabla}_{\mathbf{x}'} \cdot \mathbf{j} \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) \right) \\ &= \frac{1}{c|\mathbf{x}|} \oint_{S^2_\infty} d^2\boldsymbol{\sigma} \cdot x'^\alpha \mathbf{j} \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) + \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' x'^\alpha \frac{\partial}{\partial t} \rho \left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c} \right) \end{aligned} \tag{6.73}$$

There is two tricks used here. One was writing the unit vector $\mathbf{e}_\alpha = \boldsymbol{\nabla} x^\alpha$. The other was use of the continuity equation $\partial\rho/\partial t + \boldsymbol{\nabla}\cdot\mathbf{j} = 0$. This first trick was mentioned as one of the few tricks of physics that will often be repeated since there are not many good ones.

With the first term vanishing on the boundary (since j^i is localized), and pulling the time derivatives out of the integral, we can summarize the dipole potentials as

$$\boxed{\begin{aligned} A^0(\mathbf{x}, t) &= \frac{Q\left(t - \frac{|\mathbf{x}|}{c}\right)}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{d}\left(t - \frac{|\mathbf{x}|}{c}\right)}{|\mathbf{x}|^3} + \frac{\mathbf{x}\cdot\dot{\mathbf{d}}\left(t - \frac{|\mathbf{x}|}{c}\right)}{c|\mathbf{x}|^2} \\ \mathbf{A}(\mathbf{x}, t) &= \frac{1}{c|\mathbf{x}|}\dot{\mathbf{d}}\left(t - \frac{|\mathbf{x}|}{c}\right). \end{aligned}} \tag{6.74}$$

Example 6.2: Electric dipole radiation

PICTURE: two closely separated oppositely charges, wiggling along the line connecting them (on the z-axis). $-q$ at rest, while $+q$ oscillates.

$$z_+(t) = z_0 + a\sin\omega t. \tag{6.75}$$

Since we have put the $-q$ charge at the origin, it has no contribution to the dipole moment, and we have

$$\mathbf{d}(t) = \mathbf{e}_3 q(z_0 + a\sin\omega t). \tag{6.76}$$

Thus

$$\begin{aligned} A^0(\mathbf{x}, t) &= \frac{1}{|\mathbf{x}|^3}\mathbf{x}\cdot\mathbf{d}\left(t - \frac{|\mathbf{x}|}{c}\right) + \frac{1}{c|\mathbf{x}|^2}\mathbf{x}\cdot\dot{\mathbf{d}}\left(t - \frac{|\mathbf{x}|}{c}\right) \\ \mathbf{A}(\mathbf{x}, t) &= \frac{\dot{\mathbf{d}}\left(t - \frac{|\mathbf{x}|}{c}\right)}{c|\mathbf{x}|} \end{aligned} \tag{6.77}$$

so with $t_r = t - |\mathbf{x}|/c$, and $z = \mathbf{x} \cdot \mathbf{e}_3$ in the dipole dot product, we have

$$\begin{aligned} A^0(\mathbf{x}, t) &= \frac{zq}{|\mathbf{x}|^3}(z_0 + a\sin(\omega t_r)) + \frac{zq}{c|\mathbf{x}|^2} a\omega\cos(\omega t_r) \\ \mathbf{A}(\mathbf{x}, t) &= \frac{1}{c|\mathbf{x}|}\mathbf{e}_3 q a\omega\cos(\omega t_r) \end{aligned} \tag{6.78}$$

These hold provided $|\mathbf{x}| \gg (z_0, a)$ and $\omega l/c \ll 1$. Recall that $\omega\lambda = c/2\pi$, which has dimensions of velocity.

FIXME: think through and justify $\omega l = v$.

Observe that $\omega l \sim v$ so this is a requirement that our charged positive particle is moving with $|\mathbf{v}|/c \ll 1$.

Now we will take derivatives. The first term of the scalar potential will be ignored since the $1/|\mathbf{x}|^2$ is non-radiative.

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla} A^0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ &= -\frac{za\omega q}{|\mathbf{x}|^2 c}(-\omega\sin(\omega t_r))\left(-\frac{1}{c}\boldsymbol{\nabla}|\mathbf{x}|\right) - \frac{1}{c^2|\mathbf{x}|}\mathbf{e}_3 q a\omega^2(-\sin(\omega t_r)). \end{aligned} \tag{6.79}$$

We have used $\boldsymbol{\nabla} t_r = -\boldsymbol{\nabla}|\mathbf{x}|/c$, and $\boldsymbol{\nabla}|\mathbf{x}| = \hat{\mathbf{r}}$, and $\partial_t t_r = 1$.

$$\mathbf{E} = \frac{qa\omega^2}{c^2|\mathbf{x}|}\sin(\omega t_r)\left(\mathbf{e}_3 - \frac{z}{|\mathbf{x}|}\hat{\mathbf{r}}\right) \tag{6.80}$$

So,

$$|\mathbf{S}| \sim \omega^4 \tag{6.81}$$

The power is proportional to ω^4. Higher frequency radiation has more power : this is why the sky is blue! It all comes from the fact that the electric field is proportional to the squared acceleration ($\sim \omega^2$).

Reading Covering chapter 8 material from the text [11], and lecture notes RelEMpp147-165.pdf.

6.10 WHERE WE LEFT OFF

For a localized charge distribution, we would arrived at expressions for the scalar and vector potentials far from the point where the charges and currents were localized. This was then used to consider the specific case of a dipole system where one of the charges had a sinusoidal oscillation. The charge positions for the negative and positive charges respectively were

$$\begin{aligned} z_- &= 0 \\ z_+ &= \mathbf{e}_3(z_0 + a\sin(\omega t)), \end{aligned} \tag{6.82}$$

so that our dipole moment $\mathbf{d} = \int \rho(\mathbf{x}')\mathbf{x}'$ is

$$\mathbf{d} = \mathbf{e}_3 q(z_0 + a\sin(\omega t)). \tag{6.83}$$

The scalar potential, to first order in a number of Taylor expansions at our point far from the source, evaluated at the retarded time $t_r = t - |\mathbf{x}|/c$, was found to be

$$A^0(\mathbf{x}, t) = \frac{zq}{|\mathbf{x}|^3}(z_0 + a\sin(\omega t_r)) + \frac{zq}{c|\mathbf{x}|^2} a\omega\cos(\omega t_r), \tag{6.84}$$

and our vector potential, also with the same approximations, was

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{c|\mathbf{x}|}\mathbf{e}_3 q a\omega\cos(\omega t_r). \tag{6.85}$$

We found that the electric field (neglecting any non-radiation terms that died off as inverse square in the distance) was

$$\mathbf{E} = \frac{a\omega^2 q}{c^2|\mathbf{x}|}\sin(\omega(t - |\mathbf{x}|/c))\left(\mathbf{e}_3 - \hat{\mathbf{r}}\frac{z}{|\mathbf{x}|}\right). \tag{6.86}$$

6.11 DIRECT COMPUTATION OF THE MAGNETIC RADIATION FIELD

Taking the curl of the vector potential eq. (6.86) for the magnetic field, we will neglect the contribution from the $1/|\mathbf{x}|$ since that will be inverse square, and die off too quickly far from the source

$$\begin{aligned}\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\ &= \boldsymbol{\nabla} \times \frac{1}{c|\mathbf{x}|}\mathbf{e}_3 q a \omega \cos(\omega(t - |\mathbf{x}|/c)) \\ &\approx -\frac{qa\omega}{c|\mathbf{x}|}\mathbf{e}_3 \times \boldsymbol{\nabla} \cos(\omega(t - |\mathbf{x}|/c)) \\ &= -\frac{qa\omega}{c|\mathbf{x}|}\left(-\frac{\omega}{c}\right)(\mathbf{e}_3 \times \boldsymbol{\nabla}|\mathbf{x}|) \sin(\omega(t - |\mathbf{x}|/c)),\end{aligned} \tag{6.87}$$

which is

$$\mathbf{B} = \frac{qa\omega^2}{c^2|\mathbf{x}|}(\mathbf{e}_3 \times \hat{\mathbf{r}}) \sin(\omega(t - |\mathbf{x}|/c)). \tag{6.88}$$

Comparing to eq. (6.86), we see that this equals $\hat{\mathbf{r}} \times \mathbf{E}$ as expected.

6.12 AN ASIDE: A TIDIER FORM FOR THE ELECTRIC DIPOLE FIELD

We can rewrite the electric field eq. (6.86) in terms of the retarded time dipole

$$\mathbf{E} = \frac{1}{c^2|\mathbf{x}|}\left(-\ddot{\mathbf{d}}(t_r) + \hat{\mathbf{r}}(\ddot{\mathbf{d}}(t_r) \cdot \hat{\mathbf{r}})\right), \tag{6.89}$$

where

$$\ddot{\mathbf{d}}(t) = -qa\omega^2 \sin(\omega t)\mathbf{e}_3 \tag{6.90}$$

Then using the vector identity

$$(\mathbf{A} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\mathbf{A} + (\hat{\mathbf{r}} \cdot \mathbf{A})\hat{\mathbf{r}}, \tag{6.91}$$

we have for the fields

$$\boxed{\begin{aligned}\mathbf{E} &= \frac{1}{c^2|\mathbf{x}|}(\ddot{\mathbf{d}}(t_r) \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} \\ \mathbf{B} &= \hat{\mathbf{r}} \times \mathbf{E}.\end{aligned}} \tag{6.92}$$

6.13 CALCULATING THE ENERGY FLUX

Our Poynting vector, the energy flux, is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{c}{4\pi}\left(\frac{qa\omega^2}{c^2|\mathbf{x}|}\right)^2 \sin^2(\omega(t-|\mathbf{x}|/c))\left(\mathbf{e}_3 - \hat{\mathbf{r}}\frac{z}{|\mathbf{x}|}\right)\times(\hat{\mathbf{r}}\times\mathbf{e}_3). \tag{6.93}$$

Expanding just the cross terms we have

$$\begin{aligned}\left(\mathbf{e}_3 - \hat{\mathbf{r}}\frac{z}{|\mathbf{x}|}\right)\times(\hat{\mathbf{r}}\times\mathbf{e}_3) &= -(\hat{\mathbf{r}}\times\mathbf{e}_3)\times\mathbf{e}_3 - \frac{z}{|\mathbf{x}|}(\mathbf{e}_3\times\hat{\mathbf{r}})\times\hat{\mathbf{r}} \\ &= -(-\hat{\mathbf{r}} + \mathbf{e}_3(\mathbf{e}_3\cdot\hat{\mathbf{r}})) - \frac{z}{|\mathbf{x}|}(-\mathbf{e}_3 + \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{e}_3)) \\ &= \hat{\mathbf{r}} - \cancel{\mathbf{e}_3(\mathbf{e}_3\cdot\hat{\mathbf{r}})} + \frac{z}{|\mathbf{x}|}(\cancel{\mathbf{e}_3} - \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{e}_3)) \\ &= \hat{\mathbf{r}}(1-(\hat{\mathbf{r}}\cdot\mathbf{e}_3)^2).\end{aligned} \tag{6.94}$$

Note that we have utilized $\hat{\mathbf{r}}\cdot\mathbf{e}_3 = z/|\mathbf{x}|$ to do the cancellations above, and for the final grouping. Since $\hat{\mathbf{r}}\cdot\mathbf{e}_3 = \cos\theta$, the direction cosine of the unit radial vector with the z-axis, we have for the direction of the Poynting vector

$$\begin{aligned}\hat{\mathbf{r}}(1-(\hat{\mathbf{r}}\cdot\mathbf{e}_3)^2) &= \hat{\mathbf{r}}(1-\cos^2\theta) \\ &= \hat{\mathbf{r}}\sin^2\theta.\end{aligned} \tag{6.95}$$

Our Poynting vector is found to be directed radially outwards, and is

$$\mathbf{S} = \frac{c}{4\pi}\left(\frac{qa\omega^2}{c^2|\mathbf{x}|}\right)^2 \sin^2(\omega(t-|\mathbf{x}|/c))\sin^2\theta\hat{\mathbf{r}}. \tag{6.96}$$

The intensity is constant along the curves

$$|\sin\theta| \sim r \tag{6.97}$$

PICTURE: dipole lobes diagram with $\mathbf{d}$ up along the z axis, and $\hat{\mathbf{r}}$ pointing in an arbitrary direction.

FIXME: understand how this lobes picture comes from our result above.

PICTURE: field diagram along spherical north-south great circles, and the electric field $\mathbf{E}$ along what looks like it is the $\hat{\boldsymbol{\theta}}$ direction, and $\mathbf{B}$ along what appear to be the $\hat{\boldsymbol{\phi}}$ direction, and $\mathbf{S}$ pointing radially out.

Utilizing the spherical unit vectors to express the field directions In class we see the picture showing these spherical unit vector directions. We can see this algebraically as well. Recall that we have for our unit vectors

$$\begin{aligned}\hat{\mathbf{r}} &= \mathbf{e}_1 \sin\theta\cos\phi + \mathbf{e}_2 \sin\theta\sin\phi + \mathbf{e}_3\cos\theta \\ \hat{\boldsymbol{\phi}} &= \sin\theta(\mathbf{e}_2\cos\phi - \mathbf{e}_1\sin\phi) \\ \hat{\boldsymbol{\theta}} &= \cos\theta(\mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi) - \mathbf{e}_3\sin\theta,\end{aligned} \tag{6.98}$$

with the volume element orientation governed by cyclic permutations of

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}. \tag{6.99}$$

We can now express the direction of the magnetic field in terms of the spherical unit vectors

$$\begin{aligned}\mathbf{e}_3 \times \hat{\mathbf{r}} &= \mathbf{e}_3 \times (\mathbf{e}_1 \sin\theta\cos\phi + \mathbf{e}_2 \sin\theta\sin\phi + \mathbf{e}_3\cos\theta) \\ &= \mathbf{e}_3 \times (\mathbf{e}_1 \sin\theta\cos\phi + \mathbf{e}_2 \sin\theta\sin\phi) \\ &= \mathbf{e}_2 \sin\theta\cos\phi - \mathbf{e}_1\sin\theta\sin\phi \\ &= \sin\theta(\mathbf{e}_2\cos\phi - \mathbf{e}_1\sin\phi) \\ &= \sin\theta\hat{\boldsymbol{\phi}}.\end{aligned} \tag{6.100}$$

The direction of the electric field was in the direction of $(\ddot{\mathbf{d}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}$ where $\mathbf{d}$ was directed along the z-axis. This is then

$$\begin{aligned}(\mathbf{e}_3 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} &= -\sin\theta\hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} \\ &= -\sin\theta\hat{\boldsymbol{\theta}}\end{aligned} \tag{6.101}$$

$$\boxed{\begin{aligned}\mathbf{E} &= \frac{qa\omega^2}{c^2|\mathbf{x}|}\sin(\omega t_r)\sin\theta\hat{\boldsymbol{\theta}} \\ \mathbf{B} &= -\frac{qa\omega^2}{c^2|\mathbf{x}|}\sin(\omega t_r)\sin\theta\hat{\boldsymbol{\phi}} \\ \mathbf{S} &= \left(\frac{qa\omega^2}{c^2|\mathbf{x}|}\right)^2 \sin^2(\omega t_r)\sin^2\theta\hat{\mathbf{r}}\end{aligned}} \tag{6.102}$$

6.14 CALCULATING THE POWER

Integrating $\mathbf{S}$ over a spherical surface, we can calculate the power

FIXME: remind myself why Power is an appropriate label for this integral.

This is

$$\begin{aligned} P(r,t) &= \oint d^2\boldsymbol{\sigma}\cdot\mathbf{S} \\ &= \int \cancel{r^2}\sin\theta d\theta d\phi \frac{c}{4\pi}\left(\frac{qa\omega^2}{c^2\cancel{|\mathbf{x}|}}\right)^2 \sin^2(\omega(t-|\mathbf{x}|/c))\sin^2\theta \\ &= \frac{q^2a^2\omega^4}{2c^3}\sin^2(\omega(t-r/c))\overbrace{\boxed{\int \sin^3\theta d\theta}}^{=4/3} \end{aligned} \tag{6.103}$$

$$P(r,t) = \frac{2}{3}\frac{q^2a^2\omega^4}{c^3}\sin^2(\omega(t-r/c)) = \frac{q^2a^2\omega^4}{3c^3}(1-\cos(2\omega(t-r/c)) \tag{6.104}$$

Averaging over a period kills off the cosine term

$$\langle P(r,t)\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt P(t) = \frac{q^2a^2\omega^4}{3c^3} = \frac{2}{3c^3}\left\langle \ddot{d}(t_r)\right\rangle \tag{6.105}$$

and we once again see that higher frequencies radiate more power (i.e. why the sky is blue).

6.15 TYPES OF RADIATION

We have seen now radiation from localized current distributions, and called that electric dipole radiation. There are many other sources of electrodynamic radiation, of which here are a couple.

- Magnetic dipole radiation.

  This will be covered more in more depth in the tutorial. Picture of a positive circulating current $I = I_o \sin\omega t$ given, and a magnetic dipole moment $\boldsymbol{\mu} = \pi b^2 I \mathbf{e}_3$.

  This sort of current loop is a source of magnetic dipole radiation.

- Cyclotron radiation.

  This is the label for acceleration induced radiation (at high velocities) by particles moving in a uniform magnetic field.

PICTURE: circular orbit with speed $v = \omega r$. The particle trajectories are

$$\begin{aligned} x &= r \cos \omega t \\ y &= r \sin \omega t \end{aligned} \tag{6.106}$$

This problem can be treated as two electric dipoles out of phase by 90 degrees.

PICTURE: 4 lobe dipole picture, with two perpendicular dipole moment arrows. Resulting superposition sort of smeared together.

6.16 PROBLEMS

Exercise 6.1 Sinusoidal current density on an infinite flat conducting sheet

An infinitely thin flat conducting surface lying in the $x - z$ plane carries a surface current density:

$$\boldsymbol{\kappa} = \mathbf{e}_3 \theta(t) \kappa_0 \sin \omega t \tag{6.107}$$

Here $\mathbf{e}_3$ is a unit vector in the z direction, κ_0 is the peak value of the current density, and $\theta(t)$ is the theta function: $\theta(t < 0) - 0, \theta(t > 0) = 1$.

a. Write down the equations determining the electromagnetic potentials. Specify which gauge you choose to work in.
b. Find the electromagnetic potentials outside the plane.
c. Find the electric and magnetic fields outside the plane.
d. Give a physical interpretation of the results of the previous section. Do they agree with your qualitative expectations?
e. Find the direction and magnitude of the energy flux outside the plane.
f. Consider a point at some distance from the plane. Sketch the intensity of the electromagnetic field near this point as a function of time. Explain physically.
g. Consider now a point near the plane. Are the electric and magnetic fields you found continuous across the conducting plane? Explain.

Answer for Exercise 6.1

Part a. Determining the electromagnetic potentials. Augmenting the surface current density with a delta function we can form the current density for the system

$$\mathbf{J} = \delta(y) \boldsymbol{\kappa} = \mathbf{e}_3 \theta(t) \delta(y) \kappa_0 \sin \omega t. \tag{6.108}$$

With only a current distribution specified use of the Coulomb gauge allows for setting the scalar potential on the surface equal to zero, so that we have

$$\begin{aligned}
\Box \mathbf{A} &= \frac{4\pi \mathbf{J}}{c} \\
\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{B}
\end{aligned} \tag{6.109}$$

Utilizing our Green's function

$$G(\mathbf{x}, t) = \frac{\delta(t - |\mathbf{x}|/c)}{4\pi|\mathbf{x}|} = \delta^3(\mathbf{x})\delta(t), \tag{6.110}$$

we can invert our vector potential equation, solving for **A**

$$\begin{aligned}
\mathbf{A}(\mathbf{x}, t) &= \int d^3\mathbf{x}' dt' \Box_{\mathbf{x}',t'} G(\mathbf{x} - \mathbf{x}', t - t')\mathbf{A}(\mathbf{x}', t') \\
&= \int d^3\mathbf{x}' dt' G(\mathbf{x} - \mathbf{x}', t - t') \frac{4\pi \mathbf{J}(\mathbf{x}', t')}{c} \\
&= \int d^3\mathbf{x}' dt' \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|/c)}{4\pi|\mathbf{x} - \mathbf{x}'|} \frac{4\pi \mathbf{J}(\mathbf{x}', t')}{c} \\
&= \int d^3\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c}{c|\mathbf{x} - \mathbf{x}'|} \\
&= \frac{1}{c}\int dx' dy' dz' \mathbf{e}_3 \theta(t - \left|\mathbf{x} - \mathbf{x}'\right|/c)\delta(y)\kappa_0 \\
&\qquad \sin(\omega(t - \left|\mathbf{x} - \mathbf{x}'\right|/c)) \frac{1}{|\mathbf{x} - \mathbf{x}'|} \\
&= \frac{\mathbf{e}_3\kappa_0}{c}\int dx' dz' \theta(t - \left|\mathbf{x} - (x', 0, z')\right|/c) \\
&\qquad \sin(\omega(t - \left|\mathbf{x} - (x', 0, z')\right|/c)) \frac{1}{|\mathbf{x} - (x', 0, z')|} \\
&= \frac{\mathbf{e}_3\kappa_0}{c}\int dx' dz' \theta\left(t - \frac{1}{c}\sqrt{(x - x')^2 + y^2 + (z - z')^2}\right) \\
&\qquad \frac{\sin\left(\omega\left(t - \frac{1}{c}\sqrt{(x - x')^2 + y^2 + (z - z')^2}\right)\right)}{\sqrt{(x - x')^2 + y^2 + (z - z')^2}}
\end{aligned} \tag{6.111}$$

Now a switch to polar coordinates makes sense. Let us use

$$\begin{aligned}
x' - x &= r\cos\alpha \\
z' - z &= r\sin\alpha
\end{aligned} \tag{6.112}$$

This gives us

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \frac{\mathbf{e}_3\kappa_0}{c}\int_{r=0}^{\infty}\int_{\alpha=0}^{2\pi} r dr d\alpha \theta\left(t - \frac{1}{c}\sqrt{r^2+y^2}\right)\frac{\sin\left(\omega\left(t-\frac{1}{c}\sqrt{r^2+y^2}\right)\right)}{\sqrt{r^2+y^2}} \\ &= \frac{2\pi\mathbf{e}_3\kappa_0}{c}\int_{r=0}^{\infty} r dr \theta\left(t - \frac{1}{c}\sqrt{r^2+y^2}\right)\frac{\sin\left(\omega\left(t-\frac{1}{c}\sqrt{r^2+y^2}\right)\right)}{\sqrt{r^2+y^2}}\end{aligned} \tag{6.113}$$

Since the theta function imposes a

$$t - \frac{1}{c}\sqrt{r^2+y^2} > 0 \tag{6.114}$$

constraint, equivalent to

$$c^2t^2 > r^2 + y^2, \tag{6.115}$$

we can reduce the upper range of the integral

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \frac{2\pi\mathbf{e}_3\kappa_0}{c}\int_{r=0}^{\sqrt{c^2t^2-y^2}} r dr \frac{\sin\left(\omega\left(t-\frac{1}{c}\sqrt{r^2+y^2}\right)\right)}{\sqrt{r^2+y^2}}\theta\left(t - \frac{1}{c}\sqrt{r^2+y^2}\right) \\ &= \frac{2\pi\mathbf{e}_3\kappa_0}{c}\int_{r=0}^{\sqrt{c^2t^2-y^2}} \frac{\omega r}{c}\frac{\omega dr}{c} \\ &\quad \frac{\sin\left(\omega\left(t-\frac{1}{c}\sqrt{r^2+y^2}\right)\right)}{\frac{\omega}{c}\sqrt{r^2+y^2}}\frac{c}{\omega}\theta\left(t - \frac{1}{\omega}\sqrt{\frac{\omega^2r^2}{c^2}+k^2y^2}\right) \\ &= \frac{2\pi\mathbf{e}_3\kappa_0}{\omega}\int_{u=0}^{\sqrt{\omega^2t^2-k^2y^2}} u du \\ &\quad \frac{\sin\left(\omega t - \sqrt{u^2+k^2y^2}\right)}{\sqrt{u^2+k^2y^2}}\theta\left(t - \frac{1}{\omega}\sqrt{u^2+k^2y^2}\right)\end{aligned} \tag{6.116}$$

Here $k = \omega/c$, and $u = kr$. One more change of variables

$$\begin{aligned}v^2 &= u^2 + k^2y^2 \\ v dv &= u du,\end{aligned} \tag{6.117}$$

gives us

$$
\begin{aligned}
udu \frac{\sin\left(\omega t - \sqrt{u^2+k^2y^2}\right)}{\sqrt{u^2+k^2y^2}} &= vdv \frac{\sin\left(\omega t - |v|\right)}{|v|} \\
&= dv \frac{d}{dv} \cos(\omega t - |v|)
\end{aligned}
\tag{6.118}
$$

Omitting the integration limits temporarily we want to evaluate

$$
\begin{aligned}
&\int dv \theta(t - |v|/\omega) \frac{d}{dv} \cos(\omega t - |v|) \\
&= \int dv \frac{d}{dv} \left(\cos(\omega t - |v|)\theta(t - |v|/\omega)\right) - \int dv \cos(\omega t - |v|) \frac{d}{dv} \theta(t - |v|/\omega) \\
&= \cos(\omega t - |v|)\theta(t - |v|/\omega) - \int dv \cos(\omega t - |v|)\delta(t - |v|/\omega) \left(-\frac{\operatorname{sgn} v}{\omega}\right)
\end{aligned}
\tag{6.119}
$$

This last integral only takes a value at $v = |v| = \sqrt{u^2+k^2y^2} = \omega t$, and recalling that $\delta(ax) = \delta(x)/|a|$ we have

$$
- \int dv \cos(\omega t - |v|)\delta(t - |v|/\omega) \left(-\frac{\operatorname{sgn} v}{\omega}\right) = \cos(0) = 1.
\tag{6.120}
$$

However because we are integrating over a definite range, this entire term therefore vanishes. We are left with

$$
\begin{aligned}
\mathbf{A}(\mathbf{x}, t) &= \frac{2\pi \mathbf{e}_3 \kappa_0}{\omega} \cos(\omega t - \sqrt{u^2+k^2y^2}) \theta \left(t - \frac{\sqrt{u^2+k^2y^2}}{\omega} \right) \Bigg|_{u=0}^{\sqrt{\omega^2 t^2 - k^2 y^2}} \\
&= \frac{2\pi \mathbf{e}_3 \kappa_0}{\omega} \left(\cos(\omega t - \omega|t|)\theta(t - |t|) - \cos(\omega t - k|y|)\theta(t - |y|/c) \right)
\end{aligned}
\tag{6.121}
$$

For $t \geq 0$, $\theta(t - |t|) = \theta(0) = 1/2$, but is zero for $t < 0$, so we have

$$
\mathbf{A}(\mathbf{x}, t) = \frac{2\pi\kappa_0}{\omega} \mathbf{e}_3 \left(\frac{1}{2} - \cos(\omega(t - |y|/c))\theta(t - |y|/c) \right).
\tag{6.122}
$$

However, since we take either spatial or time derivatives of the vector potential to get the fields, the constant term will not effect the result, so it is equivalent to write just

$$
\mathbf{A}(\mathbf{x}, t) = -\frac{2\pi\kappa_0}{\omega} \mathbf{e}_3 \cos(\omega(t - |y|/c))\theta(t - |y|/c).
\tag{6.123}
$$

Part c. Find the electric and magnetic fields outside the plane Our electric field can be calculated by inspection. For $t > |y|/c$ we have

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\frac{2\pi\kappa_0\omega}{c^2}\mathbf{e}_3 \sin(\omega(t - |y|/c)). \tag{6.124}$$

For the magnetic field we have, also for $t > |y|/c$ we have

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\
&= -\frac{2\pi\kappa_0}{c}\mathbf{e}_3 \times \boldsymbol{\nabla}(1 - \cos(\omega(t - |y|/c)))) \\
&= \frac{2\pi\kappa_0}{c}(-\sin(\omega(t - |y|/c))\mathbf{e}_3 \times \boldsymbol{\nabla}\omega(t - |y|/c) \\
&= \frac{2\pi\kappa_0\omega}{c^2}\sin(\omega(t - |y|/c))\mathbf{e}_3 \times \boldsymbol{\nabla}|y| \\
&= \frac{2\pi\kappa_0\omega}{c^2}\sin(\omega(t - |y|/c))\mathbf{e}_3 \times \mathbf{e}_2,
\end{aligned} \tag{6.125}$$

which gives us

$$\begin{aligned}
\mathbf{E} &= -\frac{2\pi\kappa_0\omega}{c^2}\mathbf{e}_3 \sin(\omega(t - |y|/c))\theta(t - |y|/c) \\
\mathbf{B} &= -\frac{2\pi\kappa_0\omega}{c^2}\mathbf{e}_1 \sin(\omega(t - |y|/c))\theta(t - |y|/c)
\end{aligned} \tag{6.126}$$

Part d. Give a physical interpretation of the results of the previous section It was expected that the lack of boundary on the conducting sheet would make the potential away from the plane only depend on the y components of the spatial distance, and this is precisely what we find performing the grunt work of the integration.

Given that we had a sinusoidal forcing function for our wave equation, it seems logical that we also find our non-homogeneous solution to the wave equation has sinusoidal dependence. We find that the sinusoidal current results in sinusoidal potentials and fields very much like one has in the electric circuits problem that we solve with phasors in engineering applications.

We find that the electric and magnetic fields are oriented parallel to the plane containing the surface current density, with the electric field in the direction of the current, and the magnetic field perpendicular to that, but having energy propagate outwards from the plane.

We see that the step function for the current results in a transient response, which is intuitively pleasing. The application of the current does not result in an instantanious field in all space, but instead there is time required for the field to propagate to the point of measurement. The time required for the field to propagate is the time for light to reach that point $t = |y|/c$.

Part e. Find the direction and magnitude of the energy flux outside the plane Our energy flux, the Poynting vector, is

$$\mathbf{S} = \frac{c}{4\pi} \left(\frac{2\pi\kappa_0\omega}{c^2} \right)^2 \sin^2(\omega(t - |y|/c)\mathbf{e}_3 \times \mathbf{e}_1. \tag{6.127}$$

This is

$$\mathbf{S} = \frac{\pi\kappa_0^2\omega^2}{c^3} \sin^2(\omega(t - |y|/c)\mathbf{e}_2 = \frac{\pi\kappa_0^2\omega^2}{2c^3}(1 - \cos(2\omega(t - |y|/c)))\mathbf{e}_2. \tag{6.128}$$

This energy flux is directed outwards along the y axis, with magnitude oscillating around an average value of

$$|\langle S \rangle| = \frac{\pi\kappa_0^2\omega^2}{2c^3}. \tag{6.129}$$

Part f. Sketch the intensity of the electromagnetic field far from the plane I am assuming here that this question does not refer to the flux intensity $\langle \mathbf{S} \rangle$, since that is constant, and boring to sketch.

The time varying portion of either the electric or magnetic field is proportional to

$$\sin(\omega t - \omega|y|/c) \tag{6.130}$$

We have a sinusoid as a function of time, of period $T = 2\pi/\omega$ where the phase is adjusted at each position by the factor $\omega|y|/c$. Every increase of $\Delta y = 2\pi c/\omega$ shifts the waveform back.

A sketch is attached.

Part g. Continuity across the plane? It is sufficient to consider either the electric or magnetic field for the continuity question since the continuity is dictated by the sinusoidal term for both fields.

The point in time only changes the phase, so let us consider the electric field at $t = 0$, and an infinitesimal distance $y = \pm\epsilon c/\omega$. At either point we have

$$\mathbf{E}(0, \pm\epsilon c/\omega, 0, 0) = \frac{2\pi\kappa_0\omega}{c^2}\mathbf{e}_3\epsilon \tag{6.131}$$

In the limit as $\epsilon \to 0$ the field strength matches on either side of the plane (and happens to equal zero for this $t = 0$ case).

We have a discontinuity in the spatial derivative of either field near the plate, but not for the fields themselves. A plot illustrates this nicely

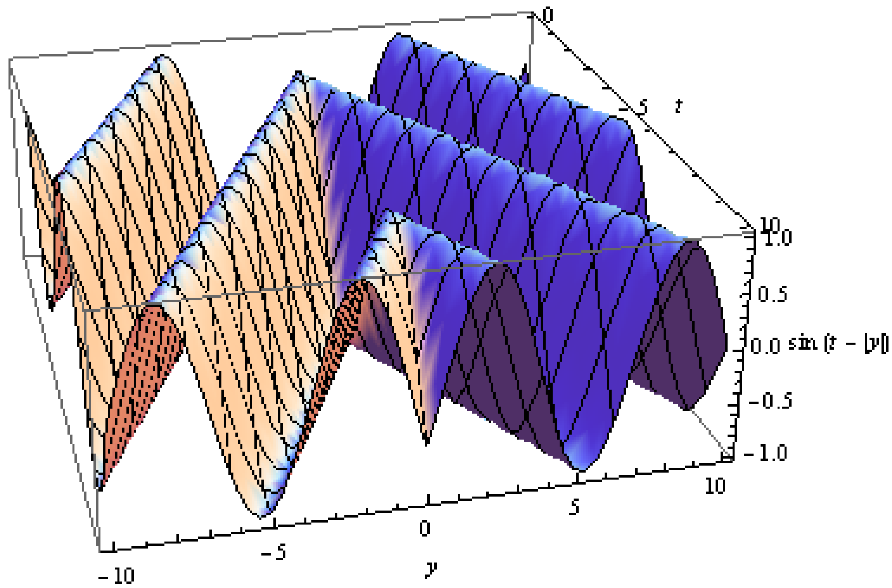

Figure 6.2: $\sin(t - |y|)$

FIXME: this plot was from before I had reintroduced the θ function I had dropped. It is not right, and does not display the transient response that I expected but did not see in the calculation I had submitted originally.

Grading notes This was the graded question (I lost 1.5 marks). I got my units wrong when I integrated to find $\mathbf{A}$, resulting in an additional ω/c in every result from that point on. I should have done a dimensional analysis check. I also dropped the θ function thinking that incorporating that into the integral bounds was enough. Without this we do not have the $t > |y|/c$ propagation rate for the fields, and they counterinutively (and erroneously) appear at all points in space. I have fixed up the units and reworked the bits utilizing the θ functions now, and believe it to be correct. I had had trouble with the interpretation part of the question initially since my result did not make sense to me.

Exercise 6.2 Charged particle in a circle

From the 2008 PHY453 exam, given a particle of charge q moving in a circle of radius a at constant angular frequency ω.

a. Find the Lienard-Wiechert potentials for points on the z-axis.

b. Find the electric and magnetic fields at the center.

Answer for Exercise 6.2

When I tried this I did it for points not just on the z-axis. It turns out that we also got this question on the exam (but stated slightly differently). Since I will not get to see my exam solution again, let us work through this at a leisurely rate, and see if things look right. The problem as stated in this old practice exam is easier since it does not say to calculate the fields from the four potentials, so there was nothing preventing one from just grinding away and plugging stuff into the Lienard-Wiechert equations for the fields (as I did when I tried it for practice).

Part a. The potentials. Let us set up our coordinate system in cylindrical coordinates. For the charged particle and the point that we measure the field, with $i = \mathbf{e}_1\mathbf{e}_2$

$$\begin{aligned} \mathbf{x}(t) &= a\mathbf{e}_1 e^{i\omega t} \\ \mathbf{r} &= z\mathbf{e}_3 + \rho\mathbf{e}_1 e^{i\phi} \end{aligned} \tag{6.132}$$

Here I am using the geometric product of vectors (if that is unfamiliar then just substitute

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \rightarrow \{\sigma_1, \sigma_2, \sigma_3\} \tag{6.133}$$

We can do that since the Pauli matrices also have the same semantics (with a small difference since the geometric square of a unit vector is defined as the unit scalar, whereas the Pauli matrix square is the identity matrix). The semantics we require of this vector product are just $\mathbf{e}_\alpha^2 = 1$ and $\mathbf{e}_\alpha\mathbf{e}_\beta = -\mathbf{e}_\beta\mathbf{e}_\alpha$ for any $\alpha \neq \beta$.

I will also be loose with notation and use $\mathrm{Re}(X) = \langle X \rangle$ to select the scalar part of a multivector (or with the Pauli matrices, the portion proportional to the identity matrix).

Our task is to compute the Lienard-Wiechert potentials. Those are

$$\begin{aligned} A^0 &= \frac{q}{R^*} \\ \mathbf{A} &= A^0 \frac{\mathbf{v}}{c}, \end{aligned} \tag{6.134}$$

where

$$\begin{aligned} \mathbf{R} &= \mathbf{r} - \mathbf{x}(t_r) \\ R = |\mathbf{R}| &= c(t - t_r) \\ R^* &= R - \frac{\mathbf{v}}{c} \cdot \mathbf{R} \\ \mathbf{v} &= \frac{d\mathbf{x}}{dt_r}. \end{aligned} \tag{6.135}$$

We will need (eventually)

$$\begin{aligned}\mathbf{v} &= a\omega \mathbf{e}_2 e^{i\omega t_r} = a\omega(-\sin\omega t_r, \cos\omega t_r, 0) \\ \dot{\mathbf{v}} &= -a\omega^2 \mathbf{e}_1 e^{i\omega t_r} = -a\omega^2(\cos\omega t_r, \sin\omega t_r, 0)\end{aligned} \tag{6.136}$$

and also need our retarded distance vector

$$\mathbf{R} = z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a e^{i\omega t_r}), \tag{6.137}$$

From this we have

$$\begin{aligned}R^2 &= z^2 + \left|\mathbf{e}_1(\rho e^{i\phi} - a e^{i\omega t_r})\right|^2 \\ &= z^2 + \rho^2 + a^2 - 2\rho a(\mathbf{e}_1 \rho e^{i\phi}) \cdot (\mathbf{e}_1 e^{i\omega t_r}) \\ &= z^2 + \rho^2 + a^2 - 2\rho a \operatorname{Re}(e^{i(\phi - \omega t_r)}) \\ &= z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \omega t_r)\end{aligned} \tag{6.138}$$

So

$$R = \sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \omega t_r)}. \tag{6.139}$$

Next we need

$$\begin{aligned}\mathbf{R} \cdot \mathbf{v}/c &= (z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a e^{i\omega t_r})) \cdot \left(a\frac{\omega}{c}\mathbf{e}_2 e^{i\omega t_r}\right) \\ &= a\frac{\omega}{c} \operatorname{Re}(i(\rho e^{-i\phi} - a e^{-i\omega t_r}) e^{i\omega t_r}) \\ &= a\frac{\omega}{c}\rho \operatorname{Re}(i e^{-i\phi + i\omega t_r}) \\ &= a\frac{\omega}{c}\rho \sin(\phi - \omega t_r)\end{aligned} \tag{6.140}$$

So we have

$$R^* = \sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \omega t_r)} - a\frac{\omega}{c}\rho \sin(\phi - \omega t_r) \tag{6.141}$$

Writing $k = \omega/c$, and having a peek back at eq. (6.134), our potentials are now solved for

$$\boxed{\begin{aligned}A^0 &= \frac{q}{\sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - k c t_r)}} \\ \mathbf{A} &= A^0 a k(-\sin k c t_r, \cos k c t_r, 0).\end{aligned}} \tag{6.142}$$

The caveat is that t_r is only specified implicitly, according to

$$ct_r = ct - \sqrt{z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - kct_r)}. \tag{6.143}$$

There does not appear to be much hope of solving for t_r explicitly in closed form.

Part b.

General fields for this system With

$$\mathbf{R}^* = \mathbf{R} - \frac{\mathbf{v}}{c} R, \tag{6.144}$$

the fields are

$$\begin{aligned}\mathbf{E} &= q(1 - \mathbf{v}^2/c^2)\frac{\mathbf{R}^*}{R^{*3}} + \frac{q}{R^{*3}} \mathbf{R} \times (\mathbf{R}^* \times \dot{\mathbf{v}}/c^2) \\ \mathbf{B} &= \frac{\mathbf{R}}{R} \times \mathbf{E}.\end{aligned} \tag{6.145}$$

In there we have

$$1 - \mathbf{v}^2/c^2 = 1 - a^2 \frac{\omega^2}{c^2} = 1 - a^2 k^2 \tag{6.146}$$

and

$$\begin{aligned}\mathbf{R}^* &= z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a e^{ikct_r}) - ak\mathbf{e}_2 e^{ikct_r} R \\ &= z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a(1 - kRi)e^{ikct_r})\end{aligned} \tag{6.147}$$

Writing this out in coordinates is not particularly illuminating, but can be done for completeness without too much trouble

$$\mathbf{R}^* = (\rho \cos\phi - a \cos t_r + akR \sin t_r, \rho \sin\phi - a \sin t_r - akR \cos t_r, z) \tag{6.148}$$

In one sense the problem could be considered solved, since we have all the pieces of the puzzle. The outstanding question is whether or not the resulting mess can be simplified at all. Let us see if the cross product reduces at all. Using

$$\mathbf{R} \times (\mathbf{R}^* \times \dot{\mathbf{v}}/c^2) = \mathbf{R}^*(\mathbf{R} \cdot \dot{\mathbf{v}}/c^2) - \frac{\dot{\mathbf{v}}}{c^2}(\mathbf{R} \cdot \mathbf{R}^*) \tag{6.149}$$

Perhaps one or more of these dot products can be simplified? One of them does reduce nicely

$$\begin{aligned}
\mathbf{R}^* \cdot \mathbf{R} &= (\mathbf{R} - R\mathbf{v}/c) \cdot \mathbf{R} \\
&= R^2 - (\mathbf{R} \cdot \mathbf{v}/c)R \\
&= R^2 - Rak\rho \sin(\phi - kct_r) \\
&= R(R - ak\rho \sin(\phi - kct_r))
\end{aligned} \tag{6.150}$$

$$\begin{aligned}
\mathbf{R} \cdot \dot{\mathbf{v}}/c^2 &= \left(z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a e^{i\omega t_r})\right) \cdot (-ak^2 \mathbf{e}_1 e^{i\omega t_r}) \\
&= -ak^2 \left\langle \mathbf{e}_1(\rho e^{i\phi} - a e^{i\omega t_r})\mathbf{e}_1 e^{i\omega t_r}) \right\rangle \\
&= -ak^2 \left\langle (\rho e^{i\phi} - a e^{i\omega t_r}) e^{-i\omega t_r}) \right\rangle \\
&= -ak^2 \left\langle \rho e^{i\phi - i\omega t_r} - a \right\rangle \\
&= -ak^2 (\rho \cos(\phi - kct_r) - a)
\end{aligned} \tag{6.151}$$

Putting this cross product back together we have

$$\begin{aligned}
\mathbf{R} \times (\mathbf{R}^* \times \dot{\mathbf{v}}/c^2) &= ak^2(a - \rho\cos(\phi - kct_r))\mathbf{R}^* + ak^2 \mathbf{e}_1 e^{ikct_r} R(R - ak\rho \sin(\phi - kct_r)) \\
&= ak^2(a - \rho\cos(\phi - kct_r))\left(z\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a(1 - kRi)e^{ikct_r})\right) \\
&\qquad + ak^2 R \mathbf{e}_1 e^{ikct_r}(R - ak\rho \sin(\phi - kct_r))
\end{aligned} \tag{6.152}$$

Writing

$$\phi_r = \phi - kct_r, \tag{6.153}$$

this can be grouped into similar terms

$$\begin{aligned}
\mathbf{R} \times (\mathbf{R}^* \times \dot{\mathbf{v}}/c^2) &= ak^2(a - \rho\cos\phi_r) z\mathbf{e}_3 \\
&\quad + ak^2 \mathbf{e}_1 (a - \rho\cos\phi_r)\rho e^{i\phi} \\
&\quad + ak^2 \mathbf{e}_1 \left(-a(a - \rho\cos\phi_r)(1 - kRi) + R(R - ak\rho\sin\phi_r)\right) e^{ikct_r}
\end{aligned} \tag{6.154}$$

The electric field pieces can now be collected. Not expanding out the R^* from eq. (6.141), this is

$$\begin{aligned}
\mathbf{E} &= \frac{q}{(R^*)^3} z\mathbf{e}_3 \left(1 - a\rho k^2 \cos\phi_r\right) \\
&\quad + \frac{q}{(R^*)^3} \rho \mathbf{e}_1 \left(1 - a\rho k^2 \cos\phi_r\right) e^{i\phi} \\
&\quad + \frac{q}{(R^*)^3} a\mathbf{e}_1 \Big(-\left(1 + ak^2(a - \rho\cos\phi_r)\right)(1 - kRi)(1 - a^2k^2) + k^2 R(R - ak\rho\sin\phi_r)\Big) e^{ikct_r}
\end{aligned} \tag{6.155}$$

Along the z-axis where $\rho = 0$ what do we have?

$$R = \sqrt{z^2 + a^2} \tag{6.156a}$$

$$A^0 = \frac{q}{R} \tag{6.156b}$$

$$\mathbf{A} = A^0 ak\mathbf{e}_2 e^{ikct_r} \tag{6.156c}$$

$$ct_r = ct - \sqrt{z^2 + a^2} \tag{6.156d}$$

$$\begin{aligned}\mathbf{E} &= \frac{q}{R^3} z\mathbf{e}_3 \\ &+ \frac{q}{R^3} a\mathbf{e}_1 \left(-(1 - a^4k^4)(1 - kRi) + k^2R^2 \right) e^{ikct_r}\end{aligned} \tag{6.156e}$$

$$\mathbf{B} = \frac{z\mathbf{e}_3 - a\mathbf{e}_1 e^{ikct_r}}{R} \times \mathbf{E} \tag{6.156f}$$

The magnetic term here looks like it can be reduced a bit.

An approximation near the center Unlike the old exam I did, where it did not specify that the potentials had to be used to calculate the fields, and the problem was reduced to one of algebraic manipulation, our exam explicitly asked for the potentials to be used to calculate the fields.

There was also the restriction to compute them near the center. Setting $\rho = 0$ so that we are looking only near the z-axis, we have

$$\begin{aligned}A^0 &= \frac{q}{\sqrt{z^2 + a^2}} \\ \mathbf{A} &= \frac{qak\mathbf{e}_2 e^{ikct_r}}{\sqrt{z^2 + a^2}} = \frac{qak(-\sin kct_r, \cos kct_r, 0)}{\sqrt{z^2 + a^2}} \\ t_r &= t - R/c = t - \sqrt{z^2 + a^2}/c\end{aligned} \tag{6.157}$$

Now we are set to calculate the electric and magnetic fields directly from these. Observe that we have a spatial dependence in due to the t_r quantities and that will have an effect when we operate with the gradient.

In the exam I had asked Simon (our TA) if this question was asking for the fields at the origin (ie: in the plane of the charge's motion in the center) or along the z-axis. He said in the plane. That would simplify things, but perhaps too much since A^0 becomes constant (in my exam attempt I somehow fudged this to get what I wanted for the $v = 0$ case, but that must have been wrong, and was the result of rushed work).

Let us now proceed with the field calculation from these potentials

$$\begin{aligned}\mathbf{E} &= -\boldsymbol{\nabla} A^0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A}.\end{aligned} \tag{6.158}$$

For the electric field we need

$$\begin{aligned}\boldsymbol{\nabla} A^0 &= q\mathbf{e}_3 \partial_z (z^2 + a^2)^{-1/2} \\ &= -q\mathbf{e}_3 \frac{z}{(\sqrt{z^2 + a^2})^3},\end{aligned} \tag{6.159}$$

and

$$\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{qak^2 \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 e^{ikct_r}}{\sqrt{z^2 + a^2}}. \tag{6.160}$$

Putting these together, our electric field near the z-axis is

$$\mathbf{E} = q\mathbf{e}_3 \frac{z}{(\sqrt{z^2 + a^2})^3} + \frac{qak^2 \mathbf{e}_1 e^{ikct_r}}{\sqrt{z^2 + a^2}}. \tag{6.161}$$

(another mistake I made on the exam, since I somehow fooled myself into forcing what I knew had to be in the gradient term, despite having essentially a constant scalar potential (having taken $z = 0$)).

What do we get for the magnetic field. In that case we have

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{A}(z) &= \mathbf{e}_\alpha \times \partial_\alpha \mathbf{A} \\
&= \mathbf{e}_3 \times \partial_z \frac{qak\mathbf{e}_2 e^{ikct_r}}{\sqrt{z^2+a^2}} \\
&= \mathbf{e}_3 \times (\mathbf{e}_2 e^{ikct_r}) qak \frac{\partial}{\partial z} \frac{1}{\sqrt{z^2+a^2}} + qak \frac{1}{\sqrt{z^2+a^2}} \mathbf{e}_3 \times (\mathbf{e}_2 \partial_z e^{ikct_r}) \\
&= -\mathbf{e}_3 \times (\mathbf{e}_2 e^{ikct_r}) qak \frac{z}{(\sqrt{z^2+a^2})^3} \\
&\quad + qak \frac{1}{\sqrt{z^2+a^2}} \mathbf{e}_3 \times \left(\mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 kce^{ikct_r} \partial_z (t - \sqrt{z^a+a^2}/c) \right) \\
&= -\mathbf{e}_3 \times (\mathbf{e}_2 e^{ikct_r}) qak \frac{z}{(\sqrt{z^2+a^2})^3} - qak^2 \frac{z}{z^2+a^2} \mathbf{e}_3 \times \left(\mathbf{e}_1 k e^{ikct_r} \right) \\
&= -\frac{qakz\mathbf{e}_3}{z^2+a^2} \times \left(\frac{\mathbf{e}_2 e^{ikct_r}}{\sqrt{z^2+a^2}} + k\mathbf{e}_1 e^{ikct_r} \right)
\end{aligned} \tag{6.162}$$

For the direction vectors in the cross products above we have

$$\begin{aligned}
\mathbf{e}_3 \times (\mathbf{e}_2 e^{i\mu}) &= \mathbf{e}_3 \times (\mathbf{e}_2 \cos\mu - \mathbf{e}_1 \sin\mu) \\
&= -\mathbf{e}_1 \cos\mu - \mathbf{e}_2 \sin\mu \\
&= -\mathbf{e}_1 e^{i\mu}
\end{aligned} \tag{6.163}$$

and

$$\begin{aligned}
\mathbf{e}_3 \times (\mathbf{e}_1 e^{i\mu}) &= \mathbf{e}_3 \times (\mathbf{e}_1 \cos\mu + \mathbf{e}_2 \sin\mu) \\
&= \mathbf{e}_2 \cos\mu - \mathbf{e}_1 \sin\mu \\
&= \mathbf{e}_2 e^{i\mu}
\end{aligned} \tag{6.164}$$

Putting everything, and summarizing results for the fields, we have

$$\begin{aligned}
\mathbf{E} &= q\mathbf{e}_3 \frac{z}{(\sqrt{z^2+a^2})^3} + \frac{qak^2 \mathbf{e}_1 e^{i\omega t_r}}{\sqrt{z^2+a^2}} \\
\mathbf{B} &= \frac{qakz}{z^2+a^2} \left(\frac{\mathbf{e}_1}{\sqrt{z^2+a^2}} - k\mathbf{e}_2 \right) e^{i\omega t_r}
\end{aligned} \tag{6.165}$$

The electric field expression above compares well to eq. (6.156e). We have the Coulomb term and the radiation term. It is harder to compare the magnetic field to the exact result eq. (6.156f) since I did not expand that out.

FIXME: A question to consider. If all this worked should we not also get

$$\mathbf{B} \stackrel{?}{=} \frac{z\mathbf{e}_3 - \mathbf{e}_1 a e^{i\omega t_r}}{\sqrt{z^2 + a^2}} \times \mathbf{E}. \tag{6.166}$$

However, if I do this check I get

$$\mathbf{B} = \frac{qaz}{z^2 + a^2} \left(\frac{1}{z^2 + a^2} + k^2 \right) \mathbf{e}_2 e^{i\omega t_r}. \tag{6.167}$$

Without geometric algebra I tried the problem of calculating the Lienard-Wiechert potentials for circular motion once again in [7] but with the added generalization that allowed the particle to have radial or z-axis motion. Really that was no longer a circular motion problem, but really just a calculation where I was playing with the use of cylindrical coordinates to describe the motion.

It occurred to me that this can be done without any use of Geometric Algebra (or Pauli matrices), which is probably how I should have attempted it on the exam. Let us use a hybrid coordinate vector and complex number representation to describe the particle position

$$\mathbf{x}_c = \begin{bmatrix} a e^{i\theta} \\ h \end{bmatrix}, \tag{6.168}$$

with the field measurement position of

$$\mathbf{r} = \begin{bmatrix} \rho e^{i\phi} \\ z \end{bmatrix}. \tag{6.169}$$

The particle velocity is

$$\mathbf{v}_c = \begin{bmatrix} (\dot{a} + ia\dot{\theta}) e^{i\theta} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & ie^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{a} \\ a\dot{\theta} \\ \dot{h} \end{bmatrix} \tag{6.170}$$

We also want the vectorial difference between the field measurement position and the particle position

$$\mathbf{R} = \mathbf{r} - \mathbf{x}_c = \begin{bmatrix} e^{i\phi} & -e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z - h \end{bmatrix}. \tag{6.171}$$

The dot product between $\mathbf{R}$ and $\mathbf{v}_c$ is then

$$\begin{aligned}
\mathbf{v}_c \cdot \mathbf{R} &= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \mathrm{Re} \left(\begin{bmatrix} e^{-i\theta} & 0 \\ -ie^{-i\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\phi} & -e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \mathrm{Re} \left(\begin{bmatrix} e^{i(\phi-\theta)} & -1 & 0 \\ -ie^{i(\phi-\theta)} & i & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \begin{bmatrix} \cos(\phi-\theta) & -1 & 0 \\ \sin(\phi-\theta) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix}.
\end{aligned} \tag{6.172}$$

Expansion of the final matrix products is then

$$\mathbf{v}_c \cdot \mathbf{R} = \dot{h}(z-h) - a\dot{a} + \rho\dot{a}\cos(\phi-\theta) + \rho a^2\dot{\theta}\sin(\phi-\theta) \tag{6.173}$$

The other quantity that we want is $\mathbf{R}^2$, which is

$$\begin{aligned}
\mathbf{R}^2 &= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \mathrm{Re} \left(\begin{bmatrix} e^{-i\phi} & 0 \\ -e^{-i\theta} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\phi} & -e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} 1 & -\cos(\phi-\theta) & 0 \\ -\cos(\phi-\theta) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix}
\end{aligned} \tag{6.174}$$

The retarded time at which the field is measured is therefore defined implicitly by

$$R = \sqrt{(\rho^2 + (a(t_r))^2 + (z-h(t_r))^2 - 2a(t_r)\rho\cos(\phi-\theta(t_r))} = c(t-t_r). \tag{6.175}$$

Together eq. (6.170), eq. (6.173), and eq. (6.175) define the four potentials

$$\begin{aligned}
A^0 &= \frac{q}{R - \mathbf{R}\cdot\mathbf{v}_c/c} \\
\mathbf{A} &= \frac{\mathbf{v}_c}{c} A^0,
\end{aligned} \tag{6.176}$$

where all quantities are evaluated at the retarded time t_r given by eq. (6.175).
In the homework (and in the text [11] §63) we found for **E** and **B**

$$
\begin{aligned}
\mathbf{E} &= e(1-\boldsymbol{\beta}_c^2)\frac{\hat{\mathbf{R}}-\boldsymbol{\beta}_c}{R^2(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3} + e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3}\hat{\mathbf{R}}\times((\hat{\mathbf{R}}-\boldsymbol{\beta}_c)\times\mathbf{a}_c/c^2) \\
\mathbf{B} &= \hat{\mathbf{R}}\times\mathbf{E}.
\end{aligned}
\tag{6.177}
$$

Expanding out the cross products this yields

$$
\begin{aligned}
\mathbf{E} &= e(1-\boldsymbol{\beta}_c^2)\frac{\hat{\mathbf{R}}-\boldsymbol{\beta}_c}{R^2(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3} + e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3}(\hat{\mathbf{R}}-\boldsymbol{\beta}_c)\left(\hat{\mathbf{R}}\cdot\frac{\mathbf{a}_c}{c^2}\right) - e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^2}\frac{\mathbf{a}_c}{c^2} \\
\mathbf{B} &= e(1-\boldsymbol{\beta}_c^2)\frac{\boldsymbol{\beta}_c\times\hat{\mathbf{R}}}{R^2(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3} + e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3}(\boldsymbol{\beta}_c\times\hat{\mathbf{R}})\left(\hat{\mathbf{R}}\cdot\frac{\mathbf{a}_c}{c^2}\right) + e\frac{1}{R(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^2}\frac{\mathbf{a}_c}{c^2}\times\hat{\mathbf{R}}
\end{aligned}
\tag{6.178}
$$

While longer, it is nice to call out the symmetry between **E** and **B** explicitly. As a side note, how do these combine in the Geometric Algebra formalism where we have $F = \mathbf{E} + I\mathbf{B}$? That gives us

$$
F = e\frac{1}{(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}_c)^3}\left(\left(\frac{1-\boldsymbol{\beta}_c^2}{R^2} + \frac{\hat{\mathbf{R}}\cdot\mathbf{a}_c}{cR}\right)\left(\hat{\mathbf{R}}-\boldsymbol{\beta}_c+\hat{\mathbf{R}}\wedge(\hat{\mathbf{R}}-\boldsymbol{\beta}_c)\right) + \frac{1}{R}\left(\frac{\mathbf{a}_c}{c^2} + \frac{\mathbf{a}_c}{c^2}\wedge\hat{\mathbf{R}}\right)\right)
\tag{6.179}
$$

I had guess a multivector of the form $\mathbf{a} + \mathbf{a}\wedge\hat{\mathbf{b}}$, can be tidied up a bit more, but this will not be persued here. Instead let us write out the fields corresponding to the potentials of eq. (6.176) explicitly. We need to calculate $\mathbf{a}_c$, $\mathbf{v}_c\times\mathbf{R}$, $\mathbf{a}_c\times\mathbf{R}$, and $\mathbf{a}_c\cdot\mathbf{R}$. For the acceleration we get

$$
\mathbf{a}_c = \begin{bmatrix} \left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)e^{i\theta} \\ \ddot{h} \end{bmatrix}
\tag{6.180}
$$

Dotted with **R** we have

$$
\begin{aligned}
\mathbf{a}_c\cdot\mathbf{R} &= \begin{bmatrix} \left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)e^{i\theta} \\ \ddot{h} \end{bmatrix} \cdot \begin{bmatrix} \rho e^{i\phi} - a e^{i\theta} \\ h \end{bmatrix} \\
&= h\ddot{h} + \mathrm{Re}\left(\left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)\left(\rho e^{i(\theta-\phi)} - a\right)\right),
\end{aligned}
\tag{6.181}
$$

which gives us

$$\mathbf{a}_c \cdot \mathbf{R} = h\ddot{h} + (\ddot{a} - a\dot{\theta}^2)(\rho\cos(\phi-\theta) - a) + (a\ddot{\theta} + 2\dot{a}\dot{\theta})\rho\sin(\phi-\theta). \tag{6.182}$$

Now, how do we handle the cross products in this complex number, scalar hybrid format? With some playing around such a cross product can be put into the following tidy form

$$\begin{bmatrix} z_1 \\ h_1 \end{bmatrix} \times \begin{bmatrix} z_2 \\ h_2 \end{bmatrix} = \begin{bmatrix} i(h_1 z_2 - h_2 z_1) \\ \operatorname{Im}(z_1^* z_2) \end{bmatrix}. \tag{6.183}$$

This is a sensible result. Crossing with $\mathbf{e}_3$ will rotate in the $x-y$ plane, which accounts for the factors of i in the complex portion of the cross product. The imaginary part has only contributions from the portions of the vectors z_1 and z_2 that are perpendicular to each other, so while the real part of $z_1^* z_2$ measures the colinearity, the imaginary part is a measure of the amount perpendicular.

Using this for our velocity cross product we have

$$\begin{aligned}
\mathbf{v}_c \times \mathbf{R} &= \begin{bmatrix} (\dot{a} + ia\dot{\theta})e^{i\theta} \\ \dot{h} \end{bmatrix} \times \begin{bmatrix} \rho e^{i\phi} - ae^{i\theta} \\ h \end{bmatrix} \\
&= \begin{bmatrix} i\left(\dot{h}(\rho e^{i\phi} - ae^{i\theta}) - h(\dot{a} + ia\dot{\theta})e^{i\theta}\right) \\ \operatorname{Im}\left((\dot{a} - ia\dot{\theta})(\rho e^{i(\phi-\theta)} - a)\right) \end{bmatrix}
\end{aligned} \tag{6.184}$$

which is

$$\mathbf{v}_c \times \mathbf{R} = \begin{bmatrix} i(\dot{h}\rho e^{i\phi} - (h\dot{a} + iha\dot{\theta} + a\dot{h})e^{i\theta}) \\ \dot{a}\rho\sin(\phi-\theta) - a\dot{\theta}\rho\cos(\phi-\theta) + a^2\dot{\theta} \end{bmatrix}. \tag{6.185}$$

The last thing required to write out the fields is

$$\begin{aligned}
\mathbf{a}_c \times \mathbf{R} &= \begin{bmatrix} \left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right) e^{i\theta} \\ \ddot{h} \end{bmatrix} \times \begin{bmatrix} \rho e^{i\phi} - ae^{i\theta} \\ z - h \end{bmatrix} \\
&= \begin{bmatrix} i\ddot{h}(\rho e^{i\phi} - ae^{i\theta}) - i(z-h)\left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right) e^{i\theta} \\ \operatorname{Im}\left(\left(\ddot{a} - a\dot{\theta}^2 - i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)(\rho e^{i(\phi-\theta)} - a)\right) \end{bmatrix}
\end{aligned} \tag{6.186}$$

So the acceleration cross product is

$$\mathbf{a}_c \times \mathbf{R} = \begin{bmatrix} i\ddot{h}\rho e^{i\phi} - i\left(\ddot{h}a + (z-h)\left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)\right)e^{i\theta} \\ \left(\ddot{a} - a\dot{\theta}^2\right)\rho\sin(\phi-\theta) - (a\ddot{\theta} + 2\dot{a}\dot{\theta})(\rho\cos(\phi-\theta) - a) \end{bmatrix} \tag{6.187}$$

Putting all the results together creates something that is too long to easily write, but can at least be summarized

$$\begin{aligned}
\mathbf{E} &= \frac{e}{(R - \mathbf{R}\cdot\boldsymbol{\beta}_c)^3}\left(\left(1 - \boldsymbol{\beta}_c^2 + \mathbf{R}\cdot\frac{\mathbf{a}_c}{c^2}\right)(\mathbf{R} - \boldsymbol{\beta}_c R) - R(R - \mathbf{R}\cdot\boldsymbol{\beta}_c)\frac{\mathbf{a}_c}{c^2}\right) \\
\mathbf{B} &= \frac{e}{(R - \mathbf{R}\cdot\boldsymbol{\beta}_c)^3}\left(\left(1 - \boldsymbol{\beta}_c^2 + \mathbf{R}\cdot\frac{\mathbf{a}_c}{c^2}\right)(\boldsymbol{\beta}_c \times \mathbf{R}) - (R - \mathbf{R}\cdot\boldsymbol{\beta}_c)\frac{\mathbf{a}_c}{c^2}\times\mathbf{R}\right) \\
1 - \boldsymbol{\beta}_c^2 &= 1 - (\dot{a}^2 + a^2\dot{\theta}^2 + \dot{h}^2)/c^2 \\
R &= \sqrt{(\rho^2 + (a(t_r))^2 + (z - h(t_r))^2 - 2a(t_r)\rho\cos(\phi - \theta(t_r))} = c(t - t_r) \\
\mathbf{R} - \boldsymbol{\beta}_c R &= \begin{bmatrix} \rho e^{i\phi} - (a + (\dot{a} + ia\dot{\theta})R/c)e^{i\theta} \\ z - h - \dot{h}R/c \end{bmatrix} \\
\boldsymbol{\beta}_c \cdot \mathbf{R} &= \frac{1}{c}\left(\dot{h}(z-h) - a\dot{a} + \rho\dot{a}\cos(\phi-\theta) + \rho a^2\dot{\theta}\sin(\phi-\theta)\right) \\
\boldsymbol{\beta}_c \times \mathbf{R} &= \frac{1}{c}\begin{bmatrix} i(\dot{h}\rho e^{i\phi} - (h\dot{a} + iha\dot{\theta} + a\dot{h})e^{i\theta}) \\ \dot{a}\rho\sin(\phi-\theta) - a\dot{\theta}\rho\cos(\phi-\theta) + a^2\dot{\theta} \end{bmatrix} \\
\frac{\mathbf{a}_c}{c^2} &= \frac{1}{c^2}\begin{bmatrix} \left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)e^{i\theta} \\ \ddot{h} \end{bmatrix} \\
\frac{\mathbf{a}_c}{c^2}\cdot\mathbf{R} &= \frac{1}{c^2}\left(h\ddot{h} + (\ddot{a} - a\dot{\theta}^2)(\rho\cos(\phi-\theta) - a) + (a\ddot{\theta} + 2\dot{a}\dot{\theta})\rho\sin(\phi-\theta)\right) \\
\frac{\mathbf{a}_c}{c^2}\times\mathbf{R} &= \frac{1}{c^2}\begin{bmatrix} i\ddot{h}\rho e^{i\phi} - i\left(\ddot{h}a + (z-h)\left(\ddot{a} - a\dot{\theta}^2 + i(a\ddot{\theta} + 2\dot{a}\dot{\theta})\right)\right)e^{i\theta} \\ \left(\ddot{a} - a\dot{\theta}^2\right)\rho\sin(\phi-\theta) - (a\ddot{\theta} + 2\dot{a}\dot{\theta})(\rho\cos(\phi-\theta) - a) \end{bmatrix}.
\end{aligned} \tag{6.188}$$

This is a whole lot more than the exam question asked for, since it is actually the most general solution to the electric and magnetic fields associated with an arbitrary charged particle (when that motion is described in cylindrical coordinates). The exam question had $\theta = kct$ and $\dot{a} = 0, h = 0$, which kills a number of the terms

$$
\begin{aligned}
1 - \boldsymbol{\beta}_c^2 + \frac{\mathbf{a}_c}{c^2} \cdot \mathbf{R} &= 1 - ak^2 \rho \cos(\phi - kct_r) \\
R &= \sqrt{(\rho^2 + a^2 + z^2 - 2a\rho\cos(\phi - kct_r)} = c(t - t_r) \\
\mathbf{R} - \boldsymbol{\beta}_c R &= \begin{bmatrix} \rho e^{i\phi} - a(1 + ikR)e^{ikct_r} \\ z \end{bmatrix} \\
\boldsymbol{\beta}_c \cdot \mathbf{R} &= \rho a^2 k \sin(\phi - kct_r) \\
\boldsymbol{\beta}_c \times \mathbf{R} &= \begin{bmatrix} 0 \\ ak(a - \rho\cos(\phi - kct_r)) \end{bmatrix} \\
\frac{\mathbf{a}_c}{c^2} &= \begin{bmatrix} -ak^2 e^{ikct_r} \\ 0 \end{bmatrix} \\
\frac{\mathbf{a}_c}{c^2} \times \mathbf{R} &= \begin{bmatrix} izak^2 e^{ikct_r} \\ -ak^2 \rho \sin(\phi - kct_r) \end{bmatrix}.
\end{aligned}
\tag{6.189}
$$

This is still messy, but is a satisfactory solution to the problem.

The exam question also asked only about the $\rho = 0$, so ϕ also becomes irrelevant. In that case we have along the z-axis the fields are given by

$$
\begin{aligned}
\mathbf{E}(z) &= \frac{e}{R^3} \begin{bmatrix} -a(1 + ikR - k^2 R^2) e^{ik(ct - R)} \\ z \end{bmatrix} \\
\mathbf{B}(z) &= \frac{e}{R^3} \begin{bmatrix} -Rizak^2 e^{ik(ct - R)} \\ a^2 k \end{bmatrix} \\
R &= \sqrt{a^2 + z^2}
\end{aligned}
\tag{6.190}
$$

Similar to when things were calculated from the potentials directly, I get a different result from $\hat{\mathbf{R}} \times \mathbf{E}$

$$
\hat{\mathbf{R}} \times \mathbf{E}(z) = \frac{e}{R^3} \begin{bmatrix} akz(1 + ikR)e^{ik(ct - R)} \\ -a^2 k \end{bmatrix}
\tag{6.191}
$$

compared to the value of $\mathbf{B}$ that was directly calculated above. With the sign swapped in the z-axis term of $\mathbf{B}(z)$ here I had guess I have got an algebraic error hiding somewhere?

7 ENERGY MOMENTUM TENSOR

7.1 ENERGY MOMENTUM CONSERVATION

We have defined

$$
\begin{aligned}
\mathcal{E} &= \frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi} && \text{Energy density} \\
\frac{\mathbf{S}}{c^2} &= \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B} && \text{Momentum density}
\end{aligned}
\tag{7.1}
$$

(where $\mathbf{S}$ was defined as the energy flow).

Dimensional analysis arguments and analogy with classical mechanics were used to motivate these definitions, as opposed to starting with the field action to find these as a consequence of a symmetry. We also saw that we had a conservation relationship that had the appearance of a four divergence of a four vector. With

$$P^i = (\mathcal{U}/c, \mathbf{S}/c^2), \tag{7.2}$$

that was

$$\partial_i P^i = -\mathbf{E}\cdot\mathbf{j}/c^2 \tag{7.3}$$

The left had side has the appearance of a Lorentz scalar, since it contracts two four vectors, but the right hand side is the continuum equivalent to the energy term of the Lorentz force law and cannot be a Lorentz scalar. The conclusion has to be that P^i is not a four vector, and it is natural to assume that these are components of a rank 2 four tensor instead (since we have got just one component of a rank 1 four tensor on the RHS). We want to know find out how the EM energy and momentum densities transform.

Classical mechanics reminder Recall that in particle mechanics when we had a Lagrangian that had no explicit time dependence

$$\mathcal{L}(q, \dot{q}, \not{t}), \tag{7.4}$$

that energy resulted from time translation invariance. We found this by taking the full derivative of the Lagrangian, and employing the EOM for the system to find a conserved quantity

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(q,\dot{q}) &= \frac{\partial \mathcal{L}}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} \\ &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} \\ &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q}\right)\end{aligned} \tag{7.5}$$

Taking differences we have

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L}\right) = 0, \tag{7.6}$$

and we labeled this conserved quantity the energy

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \dot{q}}\dot{q} - \mathcal{L} \tag{7.7}$$

Our approach from the EM field action Our EM field action was

$$S = -\frac{1}{16\pi c}\int d^4 x F_{ij}F^{ij}. \tag{7.8}$$

The squared field tensor $F_{ij}F^{ij}$ only depends on the fields $A^i(\mathbf{x},t)$ or its derivatives $\partial_j A^i(\mathbf{x},t)$, and not on the coordinates $\mathbf{x}, t$ themselves. This is very similar to the particle action with no explicit time dependence

$$S = \int dt \left(\frac{m\dot{q}^2}{2} + V(q)\right). \tag{7.9}$$

For the particle case we obtained our conservation relationship by taking time derivatives of the Lagrangian. These are very similar with the action having no explicit dependence on space or time, only on the field, so what will we get if we take the coordinate partials of the EM Lagrangian density?

We will chew on this tomorrow and calculate

$$\frac{\partial}{\partial x^k}\left(F_{ij}F^{ij}\right) \tag{7.10}$$

in full gory details. We will find that instead of finding a single conserved quantity $C^A(\mathbf{x},t)$, we instead find a quantity that only changes through escape from the boundary of a surface.

Reading Covering §32, §33 of chapter 4 in the text [11], and lecture notes RelEMpp166-180.pdf.

7.2 TOTAL DERIVATIVE OF THE LAGRANGIAN DENSITY

Rather cleverly, our Professor avoided the spacetime translation arguments of the text. Inspired by an approach possible in classical mechanics to find that we have a conserved quantity derivable from a force law, he proceeds directly to taking the derivative of the Lagrangian density (see previous lecture notes for details building up to this).

I will proceed in exactly the same fashion.

$$\begin{aligned}
\partial_k \left(-\frac{1}{16\pi c} F_{ij} F^{ij} \right) &= -\frac{1}{8\pi c} (\partial_k F_{ij}) F^{ij} \\
&= -\frac{1}{8\pi c} (\partial_k F_{ij}) F^{ij} \\
&= -\frac{1}{8\pi c} (\partial_k (\partial_i A_j - \partial_j A_i)) F^{ij} \\
&= -\frac{1}{4\pi c} (\partial_k \partial_i A_j) F^{ij} \\
&= -\frac{1}{4\pi c} (\partial_i \partial_k A_j) F^{ij} \\
&= -\frac{1}{4\pi c} (\partial_m \partial_k A_j) F^{mj} \\
&= -\frac{1}{4\pi c} \left(\partial_m ((\partial_k A_j) F^{mj}) - (\partial_m F^{mj}) \partial_k A_j \right) \\
&= -\frac{1}{4\pi c} \left(\partial_m ((\partial_k A_j) F^{mj}) - (\partial_m F^{ma}) \partial_k A_a \right) \\
&= -\frac{1}{4\pi c} \left(\partial_m ((\partial_k A_j) F^{mj}) - \left(\frac{4\pi}{c} j^a \right) \partial_k A_a \right) \\
&= -\frac{1}{4\pi c} \partial_m ((\partial_k A_j) F^{mj}) + \left(\frac{1}{c^2} j^a \right) \partial_k A_a
\end{aligned} \tag{7.11}$$

Multiplying through by c and renaming our derivative index using a delta function we have

$$\partial_k \left(-\frac{1}{16\pi} F_{ij} F^{ij} \right) = \partial_m \delta^m{}_k \left(-\frac{1}{16\pi} F_{ij} F^{ij} \right) = -\frac{1}{4\pi} \partial_m ((\partial_k A_j) F^{mj}) + \left(\frac{1}{c} j^a \right) \partial_k A_a \tag{7.12}$$

We can now group the ∂_m terms for

$$\partial_m \left(-\frac{1}{4\pi} (\partial_k A_j) F^{mj} + \delta^m{}_k \frac{1}{16\pi} F_{ij} F^{ij} \right) = -\left(\frac{1}{c} j^a \right) \partial_k A_a \tag{7.13}$$

Knowing the end goal, a quantity that is expressed in terms of F^{ij} let us raise the k indices, and any of the A_i's that are along side of those

$$\partial_m \left(-\frac{1}{4\pi} (\partial^k A^n) F^{mj} g_{nj} + g^{mk} \frac{1}{16\pi} F_{ij} F^{ij} \right) = -\left(\frac{1}{c} j_a \right) \partial^k A^a. \tag{7.14}$$

Next, we want to get rid of the explicit vector potential dependence

$$\begin{aligned}
&\partial_m \left(-\frac{1}{4\pi} (\partial^k A^n) F^{mj} g_{nj} \right) \\
&= \partial_m \left(-\frac{1}{4\pi} (F^{kn} + \partial^n A^k) F^{mj} g_{nj} \right) \\
&= \partial_m \left(-\frac{1}{4\pi} F^{kn} F^{mj} g_{nj} - \frac{1}{4\pi} (\partial_m (\partial^n A^k)) F^{mj} g_{nj} - \frac{1}{4\pi} (\partial^n A^k)(\partial_m F^{mj}) g_{nj} \right) \\
&= \partial_m \left(-\frac{1}{4\pi} F^{kn} F^{mj} g_{nj} \right) - \frac{1}{4\pi} (\partial_m (\partial^n A^k)) F^{mj} g_{nj} - (\partial^n A^k) \frac{1}{c} j_n \\
&= \partial_m \left(-\frac{1}{4\pi} F^{kn} F^{mj} g_{nj} \right) - \frac{1}{4\pi} (\partial_m \partial_j A^k) F^{mj} - (\partial^a A^k) \frac{1}{c} j_a
\end{aligned} \tag{7.15}$$

Since the operator $F^{mj} \partial_m \partial_j$ is a product of symmetric and antisymmetric tensors (or operators), the middle term is zero, and we are left with

$$\partial_m \left(-\frac{1}{4\pi} F^{kn} F^{mj} g_{nj} + g^{mk} \frac{1}{16\pi} F_{ij} F^{ij} \right) = -\frac{1}{c} F^{ka} j_a \tag{7.16}$$

This provides the desired conservation relationship

$$\boxed{\begin{aligned}
\partial_m T^{mk} &= -\frac{1}{c} F^{ka} j_a \\
T^{mk} &= \frac{1}{4\pi} \left(-F^{mj} F^{kn} g_{nj} + \frac{g^{mk}}{4} F_{ij} F^{ij} \right)
\end{aligned}} \tag{7.17}$$

7.3 UNPACKING THE TENSOR

Energy term of the stress energy tensor

$$
\begin{aligned}
T^{00} &= -\frac{1}{4\pi} F^{0j} F^{0}{}_{j} + \frac{1}{16\pi} F^{ij} F_{ij} \\
&= -\frac{1}{4\pi} F^{0\alpha} F^{0}{}_{\alpha} + \frac{1}{16\pi} \left(F^{0j} F_{0j} + F^{\alpha j} F_{\alpha j} \right) \\
&= \frac{1}{4\pi} F^{0\alpha} F^{0\alpha} + \frac{1}{16\pi} \left(F^{0\alpha} F_{0\alpha} + F^{\alpha 0} F_{\alpha 0} + F^{\alpha\beta} F_{\alpha\beta} \right) \\
&= \frac{1}{4\pi} \mathbf{E}^2 + \frac{1}{16\pi} \left(-2\mathbf{E}^2 + F^{\alpha\beta} F^{\alpha\beta} \right)
\end{aligned}
\tag{7.18}
$$

The spatially indexed field tensor components are

$$
\begin{aligned}
F^{\alpha\beta} &= \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha} \\
&= -\partial_{\alpha} A^{\beta} + \partial_{\beta} A^{\alpha} \\
&= -\epsilon^{\sigma\alpha\beta} (\mathbf{B})^{\sigma},
\end{aligned}
\tag{7.19}
$$

so

$$
\begin{aligned}
F^{\alpha\beta} F^{\alpha\beta} &= \epsilon^{\sigma\alpha\beta} (\mathbf{B})^{\sigma} \epsilon^{\mu\alpha\beta} (\mathbf{B})^{\mu} \\
&= (2!) \delta^{\sigma\mu} (\mathbf{B})^{\sigma} (\mathbf{B})^{\mu} \\
&= 2\mathbf{B}^2
\end{aligned}
\tag{7.20}
$$

A final bit of assembly gives us T^{00}

$$
\boxed{T^{00} = \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) = \mathcal{E}}
\tag{7.21}
$$

Momentum terms of the stress energy tensor For the spatial T^{k0} components we have

$$\begin{aligned}
T^{\alpha 0} &= -\frac{1}{4\pi} F^{\alpha j} F^{0}{}_{j} + \frac{1}{16\pi} g^{\alpha 0} F^{ij} F_{ij} \\
&= -\frac{1}{4\pi} F^{\alpha j} F^{0}{}_{j} \\
&= -\frac{1}{4\pi} \left(F^{\alpha 0} F^{0}{}_{0} + F^{\alpha \beta} F^{0}{}_{\beta} \right) \\
&= \frac{1}{4\pi} F^{\alpha \beta} F^{0 \beta} \\
&= \frac{1}{4\pi} (-\epsilon^{\sigma \alpha \beta} (\mathbf{B})^{\sigma})(-(\mathbf{E})^{\beta}) \\
&= \frac{1}{4\pi} \epsilon^{\alpha \beta \sigma} (\mathbf{E})^{\beta} (\mathbf{B})^{\sigma}
\end{aligned} \tag{7.22}$$

So we have

$$T^{\alpha 0} = \frac{1}{4\pi} (\mathbf{E} \times \mathbf{B})^{\alpha} = \frac{\mathbf{S}^{\alpha}}{c}. \tag{7.23}$$

Symmetry It is simple to show that T^{km} is symmetric

$$\begin{aligned}
T^{mk} &= -\frac{1}{4\pi} F^{mj} F^{k}{}_{j} + \frac{1}{16\pi} g^{mk} F^{ij} F_{ij} \\
&= -\frac{1}{4\pi} F^{m}{}_{j} F^{kj} + \frac{1}{16\pi} g^{km} F^{ij} F_{ij} \\
&= T^{km}
\end{aligned} \tag{7.24}$$

Pressure and shear terms Let us now expand $T^{\beta\alpha}$, starting with the diagonal terms $T^{\alpha\alpha}$. Because this repeated index is not summed over, things get slightly irregular, so it is easier to drop the abstraction and just pick a specific α, say, $\alpha = 1$. Then we have

$$\begin{aligned}
T^{11} &= \frac{1}{4\pi} \left(-F^{1k} F^{1}{}_{k} - \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2) \right) \\
&= \frac{1}{4\pi} \left(-F^{10} F^{10} + F^{1\alpha} F^{1\alpha} - \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2) \right) \\
&= \frac{1}{4\pi} \left(-E_x^2 + F^{12} F^{12} + F^{13} F^{13} - \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2) \right)
\end{aligned} \tag{7.25}$$

For the magnetic components above we have for example

$$\begin{aligned} F^{12}F^{12} &= (\partial^1 A^2 - \partial^2 A^2)(\partial^1 A^2 - \partial^2 A^2) \\ &= (\partial_1 A^2 - \partial_2 A^2)(\partial_1 A^2 - \partial_2 A^2) \\ &= B_z^2 \end{aligned} \tag{7.26}$$

So we have

$$T^{11} = \frac{1}{4\pi}\left(-E_x^2 + B_y^2 + B_z^2 - \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2)\right) \tag{7.27}$$

Or

$$T^{11} = \frac{1}{8\pi}\left(-E_x^2 + E_y^2 + E_z^2 - B_x^2 + B_y^2 + B_z^2\right). \tag{7.28}$$

Clearly, the other diagonal terms follow the same pattern, and we can do a cyclic permutation of coordinates to find

$$\begin{aligned} T^{11} &= \frac{1}{8\pi}\left(-E_x^2 + E_y^2 + E_z^2 - B_x^2 + B_y^2 + B_z^2\right) \\ T^{22} &= \frac{1}{8\pi}\left(-E_y^2 + E_z^2 + E_x^2 - B_y^2 + B_z^2 + B_x^2\right) \\ T^{33} &= \frac{1}{8\pi}\left(-E_z^2 + E_x^2 + E_y^2 - B_z^2 + B_x^2 + B_y^2\right) \end{aligned} \tag{7.29}$$

For the off diagonal terms, let us pick T^{12} and expand that. We have

$$\begin{aligned} T^{12} &= \frac{1}{4\pi}\left(-F^{1k}F^{2}{}_{k} - \frac{1}{2}g^{12}(\mathbf{B}^2 - \mathbf{E}^2)\right) \\ &= \frac{1}{4\pi}\left(-F^{10}F^{20} + F^{1\alpha}F^{2\alpha}\right) \\ &= \frac{1}{4\pi}\left(-E_xE_y + \underbrace{\boxed{F^{11}}}_{=0}F^{21} + F^{12}\underbrace{\boxed{F^{22}}}_{=0} + F^{13}F^{23}\right) \\ &= \frac{1}{4\pi}\left(-E_xE_y + (-B_y)B_x\right) \end{aligned} \tag{7.30}$$

Again, with cyclic permutation of the coordinates we have

$$
\begin{aligned}
T^{12} &= -\frac{1}{4\pi}\left(E_x E_y + B_x B_y\right) \\
T^{23} &= -\frac{1}{4\pi}\left(E_y E_z + B_y B_z\right) \\
T^{31} &= -\frac{1}{4\pi}\left(E_z E_x + B_z B_x\right)
\end{aligned}
\tag{7.31}
$$

In class these were all written in the compact notation

$$
\boxed{T^{\alpha\beta} = -\frac{1}{4\pi}\left(E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}(\mathbf{E}^2 + \mathbf{B}^2)\right)}
\tag{7.32}
$$

Reading Covering lecture notes RelEMpp166-180.pdf.

7.4 RECAP

Last time we found that spacetime translation invariance led to the four conservation relations

$$
\partial_k T^{km} = 0
\tag{7.33}
$$

where

$$
T^{km} = \frac{1}{4\pi}\left(F^{kj} F^{ml} g_{jl} + \frac{1}{4} g^{km} F_{ij} F^{ij}\right)
\tag{7.34}
$$

last time we found for $m = 0$

$$
\frac{1}{c}\frac{\partial}{\partial t} T^{00} + \frac{\partial}{\partial x^\alpha} T^{\alpha 0} = 0
\tag{7.35}
$$

Here

$$
\begin{aligned}
T^{00} &= \tfrac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) &&= \text{energy density} \\
cT^{\alpha 0} &= \mathbf{S}^\alpha &&= \text{energy flux}
\end{aligned}
\tag{7.36}
$$

7.5 SPATIAL COMPONENTS OF T^{km}

Now for $m = 1, 2, 3$ we write

$$\partial_k T^{k\alpha} = 0 \tag{7.37}$$

so we write

$$\frac{\partial}{\partial t}\frac{\mathbf{S}^\alpha}{c^2} + \partial_\beta T^{\beta\alpha} = 0 \tag{7.38}$$

Recall that we argued that

$$\frac{\mathbf{S}}{c^2} = \text{momentum density} \tag{7.39}$$

(it also comes from Noether's theorem).

$$\frac{\partial}{\partial t}\left(\frac{T^{0\alpha}}{c}\right) + \frac{\partial}{\partial x^\beta} T^{\beta\alpha} = 0 \tag{7.40}$$

or

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{S}^\alpha}{c^2}\right) + \frac{\partial}{\partial x^\beta} T^{\beta\alpha} = 0. \tag{7.41}$$

Integrating over V we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_V d^3\mathbf{x}\left(\frac{\mathbf{S}^\alpha}{c^2}\right) &= -\int_V d^3\mathbf{x}\frac{\partial}{\partial x^\beta} T^{\beta\alpha} \\ &= -\int_V d^3\mathbf{x}\boldsymbol{\nabla}\cdot(\mathbf{e}_\beta T^{\beta\alpha}) \\ &= -\int_{\partial V} d^2\sigma\mathbf{n}\cdot\mathbf{e}_\beta)T^{\beta\alpha} \\ &\equiv -\int_{\partial V} d^2\sigma^\beta T^{\beta\alpha}\end{aligned} \tag{7.42}$$

We write this as

$$\frac{\partial}{\partial t}\left(\text{momentum of EM fields in V}\right)^\alpha = -\int_{\partial V} d^2\sigma^\beta T^{\beta\alpha} \tag{7.43}$$

and describe our spatial tensor components as

$$T^{\beta\alpha} = \text{flux of } \alpha\text{-th momentum through a unit area} \perp \beta, \tag{7.44}$$

where

$$\begin{aligned}
T^{\alpha\beta} &= \frac{1}{4\pi}\left(-F^{\alpha j}F^{\beta m}g_{mj} + \frac{1}{4}g^{\alpha\beta}F^{ij}F_{ij}\right) \\
&= \frac{1}{4\pi}\left(-F^{\alpha 0}F^{\beta 0} + F^{\alpha\sigma}F^{\beta\sigma} - \frac{1}{4}\delta^{\alpha\beta}2(\mathbf{B}^2 - \mathbf{E}^2)\right) \\
&= \frac{1}{4\pi}\left(-E^\alpha E^\beta + \sum_\sigma (\epsilon^{\mu\alpha\sigma}B^\mu)(\epsilon^{\nu\beta\sigma}B^\nu) - \frac{1}{2}\delta^{\alpha\beta}(\mathbf{B}^2 - \mathbf{E}^2)\right) \\
&= \frac{1}{4\pi}\left(-E^\alpha E^\beta + \sum_{\mu,\nu} \overbrace{\boxed{\delta^{\mu\alpha}_{[\nu\beta]}B^\mu B^\nu}}^{(\delta^{\alpha\beta}\delta^{\mu\nu} - \delta^{\alpha\nu}\delta^{\beta\mu})B^\mu B^\nu = \delta^{\alpha\beta}\mathbf{B}^2 - B^\alpha B^\beta} + \frac{1}{2}\delta^{\alpha\beta}(\mathbf{E}^2 - \mathbf{B}^2)\right) \\
&= -\frac{1}{4\pi}\left(E^\alpha E^\beta + B^\alpha B^\beta + \delta^{\alpha\beta}\left(-\frac{\mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2} - \mathbf{B}^2\right)\right) \\
&= -\frac{1}{4\pi}\left(E^\alpha E^\beta + B^\alpha B^\beta - \frac{\delta^{\alpha\beta}}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\right)
\end{aligned} \tag{7.45}$$

We define

$$\sigma^{\alpha\beta} = -T^{\alpha\beta} = \frac{1}{4\pi}\left(E^\alpha E^\beta + B^\alpha B^\beta - \frac{\delta^{\alpha\beta}}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\right) \tag{7.46}$$

This is the Maxwell stress tensor. Maxwell apparently derived this without any use of four vectors or symmetry arguments. I had be curious what his arguments were and how he related this to the Lorentz force?

FIXME: latex: hard to layout this in gigantic matrix form without it wrapping. Can an equation be rotated in latex?

$$\begin{aligned}
T^{0j} &= \left[\tfrac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) \quad \tfrac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^x \quad \tfrac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^y \quad \tfrac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^z\right] \\
T^{1j} &= \left[. \quad -\tfrac{1}{4\pi}\left(E_x^2 + B_x^2 - \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\right) \quad -\tfrac{1}{4\pi}\left(E_xE_y + B_xB_y\right) \quad -\tfrac{1}{4\pi}\left(E_xE_z + B_xB_z\right)\right] \\
T^{2j} &= \left[. \quad . \quad -\tfrac{1}{4\pi}\left(E_y^2 + B_y^2 - \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\right) \quad -\tfrac{1}{4\pi}\left(E_yE_z + B_yB_z\right)\right] \\
T^{3j} &= \left[. \quad . \quad . \quad -\tfrac{1}{4\pi}\left(E_z^2 + B_z^2 - \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\right)\right]
\end{aligned} \tag{7.47}$$

In words this matrix is

$$\begin{bmatrix} \text{energy density} & \frac{1}{c}(\text{energy flux in } \hat{\mathbf{x}}) & \frac{1}{c}(\text{energy flux in } \hat{\mathbf{y}}) & \cdots \\ c \times (\text{momentum density})^x & (\text{momentum})^x \text{flux in } \hat{\mathbf{x}} & (\text{momentum})^x \text{flux in } \hat{\mathbf{y}} & \cdots \\ c \times (\text{momentum density})^y & (\text{momentum})^y \text{flux in } \hat{\mathbf{x}} & (\text{momentum})^y \text{flux in } \hat{\mathbf{y}} & \cdots \\ c \times (\text{momentum density})^z & (\text{momentum})^z \text{flux in } \hat{\mathbf{x}} & (\text{momentum})^z \text{flux in } \hat{\mathbf{y}} & \cdots \end{bmatrix} \tag{7.48}$$

7.6 ON THE GEOMETRY

PICTURE: rectangular area with normal $\hat{\boldsymbol{\alpha}}$, and area $d^2\sigma^\alpha \perp \hat{\boldsymbol{\alpha}}$.

$T^{\alpha\beta}$ is the amount of P^β that goes through unit area $\perp \hat{\boldsymbol{\alpha}}$ in unit time. $d^2\sigma^\alpha T^{\alpha\beta}$ (no sum) is the amount of P^β through $d^2\sigma^\alpha$ in unit time.

For a general surface element

PICTURE: normal $\mathbf{n}$ decomposed into perpendicular components $\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}$, with respective area elements $d^2\sigma^\alpha$ and $d^2\sigma^\beta$.

PICTURE: triangulated area element decomposed into three perpendicular areas with their respective normals.

We have

$$\int d^3\mathbf{x} \frac{\partial}{\partial x^\alpha} T^{\alpha\beta} = \int d^3\mathbf{x} \boldsymbol{\nabla} \cdot (\mathbf{e}_\beta T^{\alpha\beta}) = \int d^2\sigma (\mathbf{n} \cdot \mathbf{e}_\beta) T^{\alpha\beta} \tag{7.49}$$

Write

$$d^2\boldsymbol{\sigma} = d^2\sigma\mathbf{n} = \sum_\alpha d^2\sigma n^\alpha \mathbf{e}_\alpha, \tag{7.50}$$

where $\mathbf{n} = (n^1, n^2, n^3)$. The amount of β momentum that goes through $d^2\sigma$ in unit time is

$$\sum_\alpha d^2\sigma^\alpha T^{\alpha\beta} \tag{7.51}$$

If this is greater than zero, this is a flow in the $\mathbf{n}$ direction, whereas if less than zero the momentum flows in the $-\mathbf{n}$ direction.

If $d^2\boldsymbol{\sigma}$ is at the surface of the body, the rate of flow of $(\text{momentum})^\beta$ through $d^2\boldsymbol{\sigma}$ is the $(\text{force})^\beta$ that acts on this element.

PICTURE: arbitrary surface depicted with an inwards normal $\mathbf{n}$.

For this surface with the inwards normal we can write

$$df^\beta = \sum_\alpha d^2\sigma^\alpha T^{\alpha\beta} \tag{7.52}$$

The (force)$^\beta$ acting on the $d^2\boldsymbol{\sigma}$ surface element. With an outwards normal we can write this in terms of the Maxwell stress tensor, which has an inverted sign

$$df^\beta = \sum_\alpha d^2\sigma^\alpha \sigma^{\alpha\beta} \tag{7.53}$$

To find the force on the body we want

$$F^\beta = \oint_{\text{surface of body with inwards normal orientation}} d^2\sigma^\alpha T^{\alpha\beta} \tag{7.54}$$

We can calculate the EM force on any body. We need to know $T^{\alpha\beta}$ on the surface, so we need the EM field on this boundary.

Example 7.1: Wall absorbing all radiation hitting it

With propagation direction $\mathbf{p}$ along the $\hat{\mathbf{x}}$ direction, and mutually perpendicular $\mathbf{E}$ and $\mathbf{B}$.

$$cT^{xx} = \text{amount of } P^x \text{ going in } \hat{\mathbf{x}} \text{ unit area} \perp \hat{\mathbf{x}} \text{ in unit time} \tag{7.55}$$

$$\begin{aligned} df^x &= T^{xx} d^2\sigma^x \\ df^y &= T^{yy} d^2\sigma^y \end{aligned} \tag{7.56}$$

with $cp = \omega$, our fields are

$$\begin{aligned} E_y &= p\beta \sin(cpt - px) \\ B_z &= p\beta \sin(cpt - px) \end{aligned} \tag{7.57}$$

$$\begin{aligned} T^{xx} &= -\frac{1}{4\pi}\left(\cancel{(E^x)^2} + \cancel{(B^x)^2} - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\right) \\ &= \frac{1}{8\pi}\left((E^y)^2 + (B^z)^2\right) \\ &= \frac{p^2\beta^2}{8\pi}\sin^2(cpt - px) \end{aligned} \tag{7.58}$$

$$T^{yx} = -\frac{1}{4\pi}\left(\cancel{E^x}E^y + \cancel{B^x B^y}\right) = 0 \tag{7.59}$$

The off diagonal $T^{\alpha\beta}$ components vanish since we have no non-zero pair of $E_\alpha E_\beta$ or $B_\alpha B_\beta$. Our other two diagonal terms are also zero

$$\begin{aligned} T^{yy} &= -\frac{1}{4\pi}\left((E^y)^2 + \cancel{(B^y)^2} - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\right) \\ &= -\frac{1}{4\pi}p^2\beta^2\sin^2(cpt - px)\left(1 - \frac{1}{2} - \frac{1}{2}\right) \\ &= 0 \end{aligned} \tag{7.60}$$

$$\begin{aligned} T^{yy} &= -\frac{1}{4\pi}\left(+\cancel{(E^z)^2}(B^z)^2 - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\right) \\ &= -\frac{1}{4\pi}p^2\beta^2\sin^2(cpt - px)\left(1 - \frac{1}{2} - \frac{1}{2}\right) \\ &= 0 \end{aligned} \tag{7.61}$$

For non-perpendicular reflection we have the same deal.
PICTURE: reflection off of a wall, with reflection coefficient R.

7.7 PROBLEMS

Exercise 7.1 Angular momentum of EM fields

(This was a worked problem covered in tutorial 5).

Long solenoid of radius R, n turns per unit length, current I. Coaxial with with solenoid are two long cylindrical shells of length l and (radius, charge) of (a, Q), and $(b, -Q)$ respectively, where $a < b$.

When current is gradually reduced what happens?

To determined this, compute the

a. initial Magnetic field,

b. initial Electric field,

c. Poynting vector before the current changes,

d. momentum density of the EM fields,

e. induced electric field after the current is changed, and

f. the torque and angular momentum induced by the fields.

Answer for Exercise 7.1

Part a. Initial Magnetic field For the initial static conditions where we have only a (constant) magnetic field, the Maxwell-Ampere equation takes the form

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} \tag{7.62}$$

On the name of this equation . In notes from one of the lectures I had this called Maxwell-Faraday equation, despite the fact that this is not the one that Maxwell made his displacement current addition. Did the Professor call it that, or was this my addition? In [15] Faraday's law is also called the Maxwell-Faraday equation. [2] calls this the Ampere-Maxwell equation, which makes more sense.

Put into integral form by integrating over an open surface we have

$$\int_A (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{a} = \frac{4\pi}{c} \int_A \mathbf{j} \cdot d\mathbf{a} \tag{7.63}$$

The current density passing through the surface is defined as the enclosed current, circulating around the bounding loop

$$I_{\text{enc}} = \int_A \mathbf{j} \cdot d\mathbf{a}. \tag{7.64}$$

This is a sensible definition. Consider a little bit of that current

$$dI_{\text{enc}} = \frac{dQ}{dV}\mathbf{v} \cdot d\mathbf{a}. \tag{7.65}$$

If we consider the charge density volume $dV = dadl$, where $da = \hat{\mathbf{v}} \cdot d\mathbf{a}$, we have

$$dI_{\text{enc}} = \frac{dQ}{dl}\frac{dl}{dt} = \frac{dQ}{dt} \tag{7.66}$$

At least dimensionally, this is a sensible quantity to define.

Motivation aside, by Stokes Theorem, we can therefore write the circulation of the magnetic field in terms of this enclosed current

$$\int_{\partial A} \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c} I_{\text{enc}} \tag{7.67}$$

Now consider separately the regions inside and outside the cylinder. Inside we have

$$\int_{\partial A} B \cdot d\mathbf{l} = \frac{4\pi I}{c} = 0, \tag{7.68}$$

Outside of the cylinder we have the equivalent of n loops, each with current I, so we have

$$\int \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi n I L}{c} = BL. \tag{7.69}$$

Our magnetic field is constant while I is constant, and in vector form this is

$$\mathbf{B} = \frac{4\pi n I}{c} \hat{\mathbf{z}} \tag{7.70}$$

Part b. Initial Electric field How about the electric fields?

For $r < a$, and $r > b$ we have $\mathbf{E} = 0$ since there is no charge enclosed by any Gaussian surface that we choose.

Between a and b we have, for a Gaussian surface of height l (assuming that $l \gg a$)

$$E(2\pi r)l = 4\pi(+Q), \tag{7.71}$$

so we have

$$\mathbf{E} = \frac{2Q}{rl} \hat{\mathbf{r}}. \tag{7.72}$$

Part c. Poynting vector before the current changes Our Poynting vector, the energy flux per unit time, is

$$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B}) \tag{7.73}$$

This is non-zero only in the region both between the solenoid and the enclosing cylinder (radius b) since that is the only place where both $\mathbf{E}$ and $\mathbf{B}$ are non-zero. That is

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi}(\mathbf{E} \times \mathbf{B}) \\ &= \frac{c}{4\pi}\frac{2Q}{rl}\frac{4\pi n I}{c}\hat{\mathbf{r}} \times \hat{\mathbf{z}} \\ &= -\frac{2QnI}{rl}\hat{\boldsymbol{\phi}} \end{aligned} \tag{7.74}$$

(since $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$, so $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$ after cyclic permutation)

A motivational aside: Momentum density Suppose $|\mathbf{E}| = |\mathbf{B}|$, then our Poynting vector is

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} = \frac{c\hat{\mathbf{k}}}{4\pi}\mathbf{E}^2, \tag{7.75}$$

but

$$\mathcal{E} = \text{energy density} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} = \frac{\mathbf{E}^2}{4\pi}, \tag{7.76}$$

so

$$\mathbf{S} = c\hat{\mathbf{k}}\mathcal{E} = \mathbf{v}\mathcal{E}. \tag{7.77}$$

Now recall the between (relativistic) mechanical momentum $\mathbf{p} = \gamma m \mathbf{v}$ and energy $\mathcal{E} = \gamma m c^2$

$$\mathbf{p} = \frac{\mathbf{v}}{c^2}\mathcal{E}. \tag{7.78}$$

This justifies calling the quantity

$$\mathbf{P}_{\text{EM}} = \frac{\mathbf{S}}{c^2}, \tag{7.79}$$

the momentum density.

Part d. Momentum density of the EM fields So we label our scaled Poynting vector the momentum density for the field

$$\mathbf{P}_{\text{EM}} = -\frac{2QnI}{c^2 rl}\hat{\boldsymbol{\phi}}, \tag{7.80}$$

and can now compute an angular momentum density in the field between the solenoid and the outer cylinder prior to changing the currents

$$\begin{aligned}\mathbf{L}_{\text{EM}} &= \mathbf{r} \times \mathbf{P}_{\text{EM}} \\ &= r\hat{\mathbf{r}} \times \mathbf{P}_{\text{EM}}\end{aligned} \tag{7.81}$$

This gives us

$$\mathbf{L}_{\text{EM}} = -\frac{2QnI}{c^2 l}\hat{\mathbf{z}} = \text{constant}. \tag{7.82}$$

Note that this is the angular momentum density in the region between the solenoid and the inner cylinder, between $z = 0$ and $z = l$. Outside of this region, the angular momentum density is zero.

Part e. Induced electric field after the current is changed When we turn off (or change) I, some of the magnetic field $\mathbf{B}$ will be converted into electric field $\mathbf{E}$ according to Faraday's law

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{7.83}$$

In integral form, utilizing an open surface, this is

$$\begin{aligned}\int_A (\boldsymbol{\nabla} \times \mathbf{l}) \cdot \hat{\mathbf{n}} dA &= \int_{\partial A} \mathbf{E} \cdot d\mathbf{l} \\ &= -\frac{1}{c}\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \\ &= -\frac{1}{c}\frac{\partial \Phi_B(t)}{\partial t},\end{aligned} \tag{7.84}$$

where we introduce the magnetic flux

$$\Phi_B(t) = \int_A \mathbf{B} \cdot d\mathbf{A}. \tag{7.85}$$

We can utilizing a circular surface cutting directly across the cylinder perpendicular to $\hat{\mathbf{z}}$ of radius r. Recall that we have the magnetic field eq. (7.70) only inside the solenoid. So for $r < R$ this flux is

$$\begin{aligned}\Phi_B(t) &= \int_A \mathbf{B} \cdot d\mathbf{A} \\ &= (\pi r^2) \frac{4\pi n I(t)}{c}.\end{aligned} \tag{7.86}$$

For $r > R$ only the portion of the surface with radius $r \leq R$ contributes to the flux

$$\begin{aligned}\Phi_B(t) &= \int_A \mathbf{B} \cdot d\mathbf{A} \\ &= \left(\pi R^2\right) \frac{4\pi n I(t)}{c}.\end{aligned} \tag{7.87}$$

We can now compute the circulation of the electric field

$$\int_{\partial A} \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c} \frac{\partial \Phi_B(t)}{\partial t}, \tag{7.88}$$

by taking the derivatives of the magnetic flux. For $r > R$ this is

$$\begin{aligned}\int_{\partial A} \mathbf{E} \cdot d\mathbf{l} &= (2\pi r) E \\ &= -\left(\pi R^2\right) \frac{4\pi n \dot{I}(t)}{c^2}.\end{aligned} \tag{7.89}$$

This gives us the magnitude of the induced electric field

$$\begin{aligned}E &= -\left(\pi R^2\right) \frac{4\pi n \dot{I}(t)}{2\pi r c^2} \\ &= -\frac{2\pi R^2 n \dot{I}(t)}{r c^2}.\end{aligned} \tag{7.90}$$

Similarly for $r < R$ we have

$$E = -\frac{2\pi r n \dot{I}(t)}{c^2} \tag{7.91}$$

Summarizing we have

$$\mathbf{E} = \begin{cases} -\frac{2\pi r n \dot{I}(t)}{c^2} \hat{\boldsymbol{\phi}} & \text{For } r < R \\ -\frac{2\pi R^2 n \dot{I}(t)}{r c^2} \hat{\boldsymbol{\phi}} & \text{For } r > R \end{cases} \tag{7.92}$$

Part f. Torque and angular momentum induced by the fields Our torque $\mathbf{N} = \mathbf{r} \times \mathbf{F} = d\mathbf{L}/dt$ on the outer cylinder (radius b) that is induced by changing the current is

$$\begin{aligned} \mathbf{N}_b &= (b\hat{\mathbf{r}}) \times (-Q\mathbf{E}_{r=b}) \\ &= bQ\frac{2\pi R^2 n\dot{I}(t)}{bc^2}\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} \\ &= \frac{1}{c^2}2\pi R^2 nQ\dot{I}\hat{\mathbf{z}}. \end{aligned} \tag{7.93}$$

This provides the induced angular momentum on the outer cylinder

$$\begin{aligned} \mathbf{L}_b &= \int dt\mathbf{N}_b = \frac{2\pi nR^2 Q}{c^2}\int_I^0 \frac{dI}{dt}dt \\ &= -\frac{2\pi nR^2 Q}{c^2}I. \end{aligned} \tag{7.94}$$

This is the angular momentum of b induced by changing the current or changing the magnetic field.

On the inner cylinder we have

$$\begin{aligned} \mathbf{N}_a &= (a\hat{\mathbf{r}}) \times (Q\mathbf{E}_{r=a}) \\ &= aQ\left(-\frac{2\pi}{c}na\dot{I}\right)\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} \\ &= -\frac{2\pi na^2 Q\dot{I}}{c^2}\hat{\mathbf{z}}. \end{aligned} \tag{7.95}$$

So our induced angular momentum on the inner cylinder is

$$\mathbf{L}_a = \frac{2\pi na^2 QI}{c^2}\hat{\mathbf{z}}. \tag{7.96}$$

The total angular momentum in the system has to be conserved, and we must have

$$\mathbf{L}_a + \mathbf{L}_b = -\frac{2nIQ}{c^2}\pi\left(R^2 - a^2\right)\hat{\mathbf{z}}. \tag{7.97}$$

At the end of the tutorial, this sum was equated with the field angular momentum density $\mathbf{L}_{\mathrm{EM}}$, but this has different dimensions. In fact, observe that the volume in which this angular momentum density is non-zero is the difference between the volume of the solenoid and the inner cylinder

$$V = \pi R^2 l - \pi a^2 l, \tag{7.98}$$

so if we are to integrate the angular momentum density eq. (7.82) over this region we have

$$\int \mathbf{L}_{\mathrm{EM}} dV = -\frac{2QnI}{c^2}\pi\left(R^2 - a^2\right)\hat{\mathbf{z}} \tag{7.99}$$

which does match with the sum of the mechanical angular momentum densities eq. (7.97) as expected.

Exercise 7.2 Force exerted on a wall from which an incident plane EM wave is reflected.

This is problem 1 from §47 of the text [11], which was covered in tutorial with very non subtle hints about how important this is (i.e. for the exam).

Determine the force exerted on a wall from which an incident plane EM wave is reflected (w/ reflection coefficient R) and incident angle θ.

Solution from the book

$$f_\alpha = -\sigma_{\alpha\beta} n_\beta - \sigma'_{\alpha\beta} n_\beta \tag{7.100}$$

Here $\sigma_{\alpha\beta}$ is the Maxwell stress tensor for the incident wave, and $\sigma'_{\alpha\beta}$ is the Maxwell stress tensor for the reflected wave, and n_β is normal to the wall.

Show this.

Answer for Exercise 7.2

On the signs of the force per unit area The signs in eq. (7.100) require a bit of thought. We have for the rate of change of the α component of the field momentum

$$\frac{d}{dt}\int d^3\mathbf{x}\left(\frac{S^\alpha}{c^2}\right) = -\int d^2\sigma^\beta T^{\beta\alpha} \tag{7.101}$$

where $d^2\sigma^\beta = d^2\sigma \mathbf{n}\cdot\mathbf{e}_\beta$, and $\mathbf{n}$ is the outwards unit normal to the surface. This is the rate of change of momentum for the field, the force on the field. For the force on the wall per unit area, we wish to invert this, giving

$$df^\alpha_{\text{on the wall, per unit area}} = (\mathbf{n}\cdot\mathbf{e}_\beta)T^{\beta\alpha} = -(\mathbf{n}\cdot\mathbf{e}_\beta)\sigma_{\beta\alpha} \tag{7.102}$$

Returning to the tutorial notes Simon writes

$$
\begin{aligned}
f_\perp &= -\sigma_{\perp\perp} - {\sigma'}_{\perp\perp} \\
f_\parallel &= -\sigma_{\parallel\perp} - {\sigma'}_{\parallel\perp}
\end{aligned}
\tag{7.103}
$$

and then says stating this solution is very non-trivial, because $\sigma_{\alpha\beta}$ is non-linear in $\mathbf{E}$ and $\mathbf{B}$. This non-triviality is a good point. Without calculating it, I find the results above to be pulled out of a magic hat. The point of the tutorial discussion was to work through this in detail.

Working out the tensor PICTURE: ...

The Reflection coefficient can be defined in this case as

$$
R = \frac{|\mathbf{E}'|^2}{|\mathbf{E}|^2},
\tag{7.104}
$$

a ratio of the powers of the reflected wave power to the incident wave power (which are proportional to $\mathbf{E}'^2$ and $\mathbf{E}^2$ respectively.

Suppose we pick the following orientation for the incident fields

$$
\begin{aligned}
E_x &= E \sin\theta \\
E_y &= -E \cos\theta \\
B_z &= E,
\end{aligned}
\tag{7.105}
$$

With the reflected assumed to be in some still perpendicular orientation (with this orientation picked for convenience)

$$
\begin{aligned}
E'_x &= E' \sin\theta \\
E'_y &= E' \cos\theta \\
B'_z &= E'.
\end{aligned}
\tag{7.106}
$$

Here

$$
\begin{aligned}
E &= E_0 \cos(\mathbf{p}\cdot\mathbf{x} - \omega t) \\
E' &= \sqrt{R} E_0 \cos(\mathbf{p}'\cdot\mathbf{x} - \omega t)
\end{aligned}
\tag{7.107}
$$

Observe that while the propagation directions are difference for the incident and the reflected waves, these differences in phase are incorporated into the E and E' variables that we will work with below. In the very end when the forces are computed, averages will be taken, but until then we will see that these phase differences do not effect the physics explicitly. As Simon pointed out this makes good physical sense since we can form a picture of these things as just momentum and energy fields hitting an object. We could even incorporate an additional constant phase difference into the reflected wave (which may also make physical sense), but it would not change the pressure that the radiation applies to the surface.

$$\sigma_{\alpha\beta} = -T^{\alpha\beta} = \frac{1}{4\pi}\left(\mathcal{E}^\alpha\mathcal{E}^\beta + \mathcal{B}^\alpha\mathcal{B}^\beta - \frac{1}{2}\delta^{\alpha\beta}(\vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2)\right) \tag{7.108}$$

Aside: On the geometry, and the angle of incidence According to wikipedia [17] the angle of incidence is measured from the normal.

Let us use complex numbers to get the orientation of the electric and propagation direction fields right. We have for the incident propagation direction

$$-\hat{\mathbf{p}} \sim e^{i(\pi+\theta)} \tag{7.109}$$

or

$$\hat{\mathbf{p}} \sim e^{i\theta} \tag{7.110}$$

If we pick the electric field rotated negatively from that direction, we have

$$\begin{aligned}\hat{\mathbf{E}} &\sim -ie^{i\theta} \\ &= -i(\cos\theta + i\sin\theta) \\ &= -i\cos\theta + \sin\theta\end{aligned} \tag{7.111}$$

Or

$$\begin{aligned}E_x &\sim \sin\theta \\ E_y &\sim -\cos\theta\end{aligned} \tag{7.112}$$

For the reflected direction we have

$$\hat{\mathbf{p}}' \sim e^{i(\pi-\theta)} = -e^{-i\theta} \tag{7.113}$$

rotating negatively for the electric field direction, we have

$$\begin{aligned}\hat{\mathbf{E}}' &\sim -i(-e^{-i\theta}) \\ &= i(cos\theta - i\sin\theta) \\ &= icos\theta + \sin\theta\end{aligned} \tag{7.114}$$

Or

$$\begin{aligned}E'_x &\sim \sin\theta \\ E'_y &\sim \cos\theta\end{aligned} \tag{7.115}$$

Back to the problem (again) Where $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ are the total EM fields.

Aside: Why the fields are added in this fashion was not clear to me, but I guess this makes sense. Even if the propagation directions differ, the total field at any point is still just a superposition.

$$\begin{aligned}\vec{\mathcal{E}} &= \mathbf{E} + \mathbf{E}' \\ \vec{\mathcal{B}} &= \mathbf{B} + \mathbf{B}'\end{aligned} \tag{7.116}$$

Get

$$\begin{aligned}\sigma_{33} &= \frac{1}{4\pi}(\overbrace{B_z B_z}^{= \vec{\mathcal{B}}^2} - \frac{1}{2}(\vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2)) = 0 \\ \sigma_{31} &= 0 = \sigma_{32} \\ \sigma_{11} &= \frac{1}{4\pi}\left((\mathcal{E}^1)^2 - \frac{1}{2}(\vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2)\right)\end{aligned} \tag{7.117}$$

$$\begin{aligned}\vec{\mathcal{B}}^2 &= (B_z + B'_z)^2 = (E + E')^2 \\ \vec{\mathcal{E}}^2 &= (\mathbf{E} + \mathbf{E}')^2\end{aligned} \tag{7.118}$$

so

$$\begin{aligned}
\sigma_{11} &= \frac{1}{4\pi}\left((\mathcal{E}^1)^2 - \frac{1}{2}((\mathcal{E}^1)^2 + (\mathcal{E}^2)^2 + (E + E')^2\right) \\
&= \frac{1}{8\pi}\left((\mathcal{E}^1)^2 - (\mathcal{E}^2)^2 - (E + E')^2\right) \\
&= \frac{1}{8\pi}\left((E + E')^2 \sin^2\theta - (E' - E)^2 \cos^2\theta - (E + E')^2\right) \\
&= \frac{1}{8\pi}\left(E^2(\sin^2\theta - \cos^2\theta - 1)\right) \\
&+ \frac{1}{8\pi}\left((E')^2(\sin^2\theta - \cos^2\theta - 1) + 2EE'(\sin^2\theta + \cos^2\theta - 1)\right) \\
&= \frac{1}{8\pi}\left(-2E^2\cos^2\theta - 2(E')^2\cos^2\theta\right) \\
&= -\frac{1}{4\pi}(E^2 + (E')^2)\cos^2\theta \\
&= \sigma_{\parallel} + \sigma'_{\parallel}
\end{aligned} \tag{7.119}$$

This last bit I did not get. What is $\sigma_{\parallel}$ and $\sigma'_{\parallel}$. Are these parallel to the wall or parallel to the normal to the wall. It turns out that this appears to mean parallel to the normal. We can see this by direct calculation

$$\begin{aligned}
\sigma_{xx}^{\text{incident}} &= \frac{1}{4\pi}\left(E_x^2 - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\right) \\
&= \frac{1}{4\pi}\left(E^2\sin^2\theta - \frac{1}{2}2E^2\right) \\
&= -\frac{1}{4\pi}E^2\cos^2\theta
\end{aligned} \tag{7.120}$$

$$\begin{aligned}
\sigma_{xx}^{\text{reflected}} &= \frac{1}{4\pi}\left({E'_x}^2 - \frac{1}{2}(\mathbf{E}'^2 + \mathbf{B}'^2)\right) \\
&= \frac{1}{4\pi}\left(E'^2\sin^2\theta - \frac{1}{2}2E'^2\right) \\
&= -\frac{1}{4\pi}E'^2\cos^2\theta
\end{aligned} \tag{7.121}$$

So by comparison we see that we have

$$\sigma_{11} = \sigma_{xx}^{\text{incident}} + \sigma_{xx}^{\text{reflected}} \tag{7.122}$$

Moving on, for our other component on the x, y place σ_{12} we have

$$\begin{aligned}
\sigma_{12} &= \frac{1}{4\pi}\mathcal{E}^1\mathcal{E}^2 \\
&= \frac{1}{4\pi}(E + E')\sin\theta(-E + E')\cos\theta \\
&= \frac{1}{4\pi}((E')^2 - E^2)\sin\theta\cos\theta
\end{aligned} \tag{7.123}$$

Again we can compare to the sums of the reflected and incident tensors for this x, y component. Those are

$$\begin{aligned}
\sigma_{12}^{\text{incident}} &= \frac{1}{4\pi}(E^1 E^2) \\
&= -\frac{1}{4\pi}E^2\sin\theta\cos\theta,
\end{aligned} \tag{7.124}$$

and

$$\begin{aligned}
\sigma_{12}^{\text{reflected}} &= \frac{1}{4\pi}({E'}^1 {E'}^2) \\
&= \frac{1}{4\pi}{E'}^2\sin\theta\cos\theta
\end{aligned} \tag{7.125}$$

Which demonstrates that we have

$$\sigma_{12} = \sigma_{12}^{\text{incident}} + \sigma_{12}^{\text{reflected}} \tag{7.126}$$

Summarizing, for the components in the x, y plane we have found that we have

$$\sigma_{\alpha\beta}^{\text{total}} n_\beta = \sigma_{\alpha 1}^{\text{total}} = \sigma_{\alpha 1} + {\sigma'}_{\alpha 1} \tag{7.127}$$

(where $n_\beta = \delta^{\beta 1}$)

This result, assumed in the text, was non-trivial to derive. It is also not generally true. We have

$$
\begin{aligned}
\sigma_{22} &= \frac{1}{4\pi}\left((\mathcal{E}^y)^2 - \frac{1}{2}(\vec{\mathcal{E}}^2 + \vec{\mathcal{B}}^2)\right) \\
&= \frac{1}{8\pi}\left((\mathcal{E}^y)^2 - (\mathcal{E}^x)^2 - \vec{\mathcal{B}}^2)\right) \\
&= \frac{1}{8\pi}\left((E' - E)^2 \cos^2\theta - (E + E')^2 \sin^2\theta - (E + E')^2\right) \\
&= \frac{1}{8\pi}\left(E^2(-1 + \cos^2\theta - \sin^2\theta)\right) \\
&+ \frac{1}{8\pi}\left(+E'^2(-1 + \cos^2\theta - \sin^2\theta) + 2EE'(-\cos^2\theta - \sin^2\theta - 1)\right) \\
&= -\frac{1}{4\pi}\left(E^2 \sin^2\theta + (E')^2 \sin^2\theta + 2EE'\right)
\end{aligned}
\tag{7.128}
$$

If we compare to the incident and reflected tensors we have

$$
\begin{aligned}
\sigma_{yy}^{\text{incident}} &= \frac{1}{4\pi}\left((E^y)^2 - \frac{1}{2}E^2\right) \\
&= \frac{1}{4\pi}E^2(\cos^2\theta - 1) \\
&= -\frac{1}{4\pi}E^2 \sin^2\theta
\end{aligned}
\tag{7.129}
$$

and

$$
\begin{aligned}
\sigma_{yy}^{\text{reflected}} &= \frac{1}{4\pi}\left((E'^y)^2 - \frac{1}{2}E'^2\right) \\
&= \frac{1}{4\pi}E'^2(\cos^2\theta - 1) \\
&= -\frac{1}{4\pi}E'^2 \sin^2\theta
\end{aligned}
\tag{7.130}
$$

There is a cross term that we can not have summing the two, so we have, in general

$$
\sigma_{22}^{\text{total}} \neq \sigma_{yy}^{\text{incident}} + \sigma_{yy}^{\text{reflected}}
\tag{7.131}
$$

Force per unit area?

$$
f_\alpha = n^x \sigma_{x\alpha}
\tag{7.132}
$$

Averaged

$$
\begin{aligned}
\langle \sigma_{xx} \rangle &= -\frac{1}{8\pi} E_0^2 (1+R) \cos^2\theta \\
\left\langle \sigma_{xy} \right\rangle &= -\frac{1}{8\pi} E_0^2 (1-R) \sin\theta \cos\theta
\end{aligned}
\tag{7.133}
$$

$$
\begin{aligned}
\langle \mathbf{S} \rangle &= -\frac{c}{8\pi} E_0^2 \hat{\mathbf{n}} \\
\langle \mathbf{S}' \rangle &= -\frac{c}{8\pi} E_0^2 \hat{\mathbf{n}}'
\end{aligned}
\tag{7.134}
$$

$$
\langle |\mathbf{S}| \rangle = \text{Work} = W
\tag{7.135}
$$

$$
\begin{aligned}
f_x &= n^x \sigma_{xx} = W(1+R)\cos^2\theta \\
f_y &= n^y \sigma_{xy} = W(1-R)\sin\theta\cos\theta \\
f_z &= 0
\end{aligned}
\tag{7.136}
$$

Exercise 7.3 Force per unit area for a Infinite parallel plate capacitor.

Find the forces per unit area $\sigma_{\alpha\beta}$ for a Infinite parallel plate capacitor.

Answer for Exercise 7.3

$$
\begin{aligned}
\mathbf{B} &= 0 \\
\mathbf{E} &= -\frac{\sigma}{\epsilon_0} \mathbf{e}_z
\end{aligned}
\tag{7.137}
$$

FIXME: derive this. Observe that we have no distance dependence in the field because it is an infinite plate.

$$
\begin{aligned}
\sigma_{11} &= \left(-\frac{1}{2} \delta^{11} \left(\frac{-\sigma}{\epsilon_0} \right)^2 \right) = -\frac{\sigma^2}{2\epsilon_0^2} = \sigma_{22} \\
\sigma_{33} &= \left((E^3)^2 - \frac{1}{2} \mathbf{E}^2 \right) = -\frac{1}{2} \mathbf{E}^2 = -\sigma_{22}
\end{aligned}
\tag{7.138}
$$

Force per unit area is then

$$\begin{aligned} f_\alpha &= n_\beta \sigma_{\alpha\beta} \\ &= n_3 \sigma_{\alpha 3} \end{aligned} \tag{7.139}$$

So

$$\begin{aligned} f_1 &= 0 = f_2 \\ f_3 &= \sigma_{33} = -\frac{\sigma^2}{2\epsilon_0^2} \end{aligned} \tag{7.140}$$

$$\mathbf{f} = -\frac{\sigma^2}{2\epsilon_0^2}\mathbf{e}_z \tag{7.141}$$

Exercise 7.4 Fields generated by an arbitrarily moving charge

Show that for a particle moving on a worldline parametrized by $(ct, \mathbf{x}_c(t))$, the retarded time t_r with respect to an arbitrary space time point $(ct, \mathbf{x})$, defined in class as:

$$|\mathbf{x} - \mathbf{x}_c(t_r)| = c(t - t_r) \tag{7.142}$$

obeys

$$\boldsymbol{\nabla} t_r = -\frac{\mathbf{x} - \mathbf{x}_c(t_r)}{c|\mathbf{x} - \mathbf{x}_c(t_r)| - \mathbf{v}_c(t_r) \cdot (\mathbf{x} - \mathbf{x}_c(t_r))} \tag{7.143}$$

and

$$\frac{\partial t_r}{\partial t} = \frac{c|\mathbf{x} - \mathbf{x}_c(t_r)|}{c|\mathbf{x} - \mathbf{x}_c(t_r)| - \mathbf{v}_c(t_r) \cdot (\mathbf{x} - \mathbf{x}_c(t_r))} \tag{7.144}$$

a. Then, use these to derive the expressions for **E** and **B** given in the book (and in the class notes).
b. Finally, re-derive the already familiar expressions for the EM fields of a particle moving with uniform velocity.

Answer for Exercise 7.4

Gradient and time derivatives of the retarded time function Let us use notation something like our text [11], where the solution to this problem is outlined in §63, and write

$$\begin{aligned}\mathbf{R}(t_r) &= \mathbf{x} - \mathbf{x}_c(t_r) \\ R &= |\mathbf{R}|\end{aligned} \tag{7.145}$$

where

$$\frac{\partial \mathbf{R}}{\partial t_r} = -\mathbf{v}_c. \tag{7.146}$$

From $R^2 = \mathbf{R} \cdot \mathbf{R}$ we also have

$$2R\frac{\partial R}{\partial t_r} = 2\mathbf{R} \cdot \frac{\partial \mathbf{R}}{\partial t_r}, \tag{7.147}$$

so if we write

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R}, \tag{7.148}$$

we have

$$R'(t_r) = -\hat{\mathbf{R}} \cdot \mathbf{v}_c. \tag{7.149}$$

Proceeding in the manner of the text, we have

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t_r}\frac{\partial t_r}{\partial t} = -\hat{\mathbf{R}} \cdot \mathbf{v}_c \frac{\partial t_r}{\partial t}. \tag{7.150}$$

From eq. (7.142) we also have

$$R = |\mathbf{x} - \mathbf{x}_c(t_r)| = c(t - t_r), \tag{7.151}$$

so

$$\frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t_r}{\partial t}\right). \tag{7.152}$$

This and eq. (7.150) gives us

$$\boxed{\frac{\partial t_r}{\partial t} = \frac{1}{1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c}}} \tag{7.153}$$

For the gradient we operate on the implicit equation eq. (7.151) again. This gives us

$$\boldsymbol{\nabla} R = \boldsymbol{\nabla}(ct - ct_r) = -c\boldsymbol{\nabla} t_r. \tag{7.154}$$

However, we can also use the spatial definition of $R = |\mathbf{x} - \mathbf{x}_c(t')|$. Note that this distance $R = R(t_r)$ is a function of space and time, since $t_r = t_r(\mathbf{x}, t)$ is implicitly a function of the spatial and time positions at which the retarded time is to be measured.

$$\begin{aligned}
\boldsymbol{\nabla} R &= \boldsymbol{\nabla} \sqrt{(\mathbf{x} - \mathbf{x}_c(t_r))^2} \\
&= \frac{1}{2R}\boldsymbol{\nabla}(\mathbf{x} - \mathbf{x}_c(t_r))^2 \\
&= \frac{1}{R}(x^\beta - x_c^\beta)\mathbf{e}_\alpha\partial_\alpha(x^\beta - x_c^\beta(t_r)) \\
&= \frac{1}{R}(\mathbf{R})_\beta\mathbf{e}_\alpha({\delta_\alpha}^\beta - \partial_\alpha x_c^\beta(t_r))
\end{aligned} \tag{7.155}$$

We have only this bit $\partial_\alpha x_c^\beta(t_r)$ to expand, but that is just going to require a chain rule expansion. This is easier to see in a more generic form

$$\frac{\partial f(g)}{\partial x^\alpha} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x^\alpha}, \tag{7.156}$$

so we have

$$\frac{\partial x_c^\beta(t_r)}{\partial x^\alpha} = \frac{\partial x_c^\beta(t_r)}{\partial t_r}\frac{\partial t_r}{\partial x^\alpha}, \tag{7.157}$$

which gets us close to where we want to be

$$\begin{aligned}
\boldsymbol{\nabla} R &= \frac{1}{R}\left(\mathbf{R} - (\mathbf{R})_\beta\frac{\partial x_c^\beta(t_r)}{\partial t_r}\mathbf{e}_\alpha\frac{\partial t_r}{\partial x^\alpha}\right) \\
&= \frac{1}{R}\left(\mathbf{R} - \mathbf{R}\cdot\frac{\partial \mathbf{x}_c^\beta(t_r)}{\partial t_r}\boldsymbol{\nabla} t_r\right)
\end{aligned} \tag{7.158}$$

Putting the pieces together we have only minor algebra left since we can now equate the two expansions of $\boldsymbol{\nabla} R$

$$-c\boldsymbol{\nabla} t_r = \hat{\mathbf{R}} - \hat{\mathbf{R}} \cdot \mathbf{v}_c(t_r)\boldsymbol{\nabla} t_r. \tag{7.159}$$

This is given in the text, but these in between steps are left for us and for our homework assignments! From this point we can rearrange to find the desired result

$$\boxed{\boldsymbol{\nabla} t_r = -\frac{1}{c}\frac{\hat{\mathbf{R}}}{1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c}} = -\frac{\hat{\mathbf{R}}}{c}\frac{\partial t_r}{\partial t}} \tag{7.160}$$

Part a.

Computing the EM fields from the Lienard-Wiechert potentials Now we are ready to derive the values of **E** and **B** that arise from the Lienard-Wiechert potentials. We have for the electric field.

We will evaluate

$$\begin{aligned}
\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi \\
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{B}
\end{aligned} \tag{7.161}$$

For the electric field we will use the chain rule on the vector potential

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial t_r}{\partial t}\frac{\partial \mathbf{A}}{\partial t_r}. \tag{7.162}$$

Similarly for the gradient of the scalar potential we have

$$\begin{aligned}
\boldsymbol{\nabla}\phi &= \mathbf{e}_\alpha \frac{\partial \phi}{\partial x^\alpha} \\
&= \mathbf{e}_\alpha \frac{\partial \phi}{\partial t_r}\frac{\partial t_r}{\partial x^\alpha} \\
&= \frac{\partial \phi}{\partial t_r}\boldsymbol{\nabla} t_r \\
&= -\frac{\partial \phi}{\partial t_r}\frac{\hat{\mathbf{R}}}{c}\frac{\partial t_r}{\partial t}.
\end{aligned} \tag{7.163}$$

Our electric field is thus

$$\mathbf{E} = -\frac{\partial t_r}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t_r} - \frac{\hat{\mathbf{R}}}{c}\frac{\partial \phi}{\partial t_r}\right) \tag{7.164}$$

For the magnetic field we have

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{A} &= \mathbf{e}_\alpha \times \frac{\partial \mathbf{A}}{\partial x^\alpha} \\ &= \mathbf{e}_\alpha \times \frac{\partial \mathbf{A}}{\partial t_r}\frac{\partial t_r}{\partial x^\alpha}.\end{aligned} \tag{7.165}$$

The magnetic field will therefore be found by evaluating

$$\mathbf{B} = (\boldsymbol{\nabla} t_r) \times \frac{\partial \mathbf{A}}{\partial t_r} = -\frac{\partial t_r}{\partial t}\frac{\hat{\mathbf{R}}}{c} \times \frac{\partial \mathbf{A}}{\partial t_r} \tag{7.166}$$

Let us compare this to $\hat{\mathbf{R}} \times \mathbf{E}$

$$\begin{aligned}\hat{\mathbf{R}} \times \mathbf{E} &= \hat{\mathbf{R}} \times \left(-\frac{\partial t_r}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t_r} - \frac{\hat{\mathbf{R}}}{c}\frac{\partial \phi}{\partial t_r}\right)\right) \\ &= \hat{\mathbf{R}} \times \left(-\frac{\partial t_r}{\partial t}\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t_r}\right)\end{aligned} \tag{7.167}$$

This equals eq. (7.166), verifying that we have

$$\mathbf{B} = \hat{\mathbf{R}} \times \mathbf{E}, \tag{7.168}$$

something that we can determine even without fully evaluating $\mathbf{E}$.

We are now left to evaluate the retarded time derivatives found in eq. (7.164). Our potentials are

$$\begin{aligned}\phi(\mathbf{x}, t) &= \frac{e}{R(t_r)}\frac{\partial t_r}{\partial t} \\ \mathbf{A}(\mathbf{x}, t) &= \frac{e\mathbf{v}_c(t_r)}{cR(t_r)}\frac{\partial t_r}{\partial t}\end{aligned} \tag{7.169}$$

It is clear that the quantity $\partial t_r/\partial t$ is going to show up all over the place, so let us label it γ_{t_r}. This is justified by comparing to a particle's boosted rest frame worldline

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} ct \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma ct \\ -\gamma\beta ct \end{bmatrix}, \tag{7.170}$$

where we have $\partial t'/\partial t = \gamma$, so for the remainder of this part of this problem we will write

$$\gamma_{t_r} \equiv \frac{\partial t_r}{\partial t} = \frac{1}{1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c}}. \tag{7.171}$$

Using primes to denote partial derivatives with respect to the retarded time t_r we have

$$\begin{aligned}
\phi' &= e\left(-\frac{R'}{R^2}\gamma_{t_r} + \frac{\gamma'_{t_r}}{R}\right) \\
\mathbf{A}' &= e\frac{\mathbf{v}_c}{c}\left(-\frac{R'}{R^2}\gamma_{t_r} + \frac{\gamma'_{t_r}}{R}\right) + e\frac{\mathbf{a}_c}{c}\frac{\gamma_{t_r}}{R},
\end{aligned} \tag{7.172}$$

so the electric field is

$$\begin{aligned}
\mathbf{E} &= -\gamma_{t_r}\left(\frac{1}{c}\mathbf{A}' - \frac{\hat{\mathbf{R}}}{c}\phi'\right) \\
&= -\frac{e\gamma_{t_r}}{c}\left(\frac{\mathbf{v}_c}{c}\left(-\frac{R'}{R^2}\gamma_{t_r} + \frac{\gamma'_{t_r}}{R}\right) + \frac{\mathbf{a}_c}{c}\frac{\gamma_{t_r}}{R} - \hat{\mathbf{R}}\left(-\frac{R'}{R^2}\gamma_{t_r} + \frac{\gamma'_{t_r}}{R}\right)\right) \\
&= -\frac{e\gamma_{t_r}}{c}\left(\frac{\mathbf{v}_c}{c}\left(\frac{c}{R^2}\gamma_{t_r} + \frac{\gamma'_{t_r}}{R}\right) + \frac{\mathbf{a}_c}{c}\frac{\gamma_{t_r}}{R} - \hat{\mathbf{R}}\left(\frac{c}{R^2}\gamma_{t_r} + \frac{\gamma'_{t_r}}{R}\right)\right) \\
&= -\frac{e\gamma_{t_r}}{cR}\left(\gamma_{t_r}\left(\frac{\mathbf{a}_c}{c} + \frac{\mathbf{v}_c}{R} - \frac{\hat{\mathbf{R}}c}{R}\right) + \gamma'_{t_r}\left(\frac{\mathbf{v}_c}{c} - \hat{\mathbf{R}}\right)\right).
\end{aligned} \tag{7.173}$$

Here is where things get slightly messy.

$$\begin{aligned}
\gamma'_{t_r} &= \frac{\partial}{\partial t_r}\frac{1}{1 - \frac{\mathbf{v}_c}{c} \cdot \hat{\mathbf{R}}} \\
&= -\gamma_{t_r}^2 \frac{\partial}{\partial t_r}\left(1 - \frac{\mathbf{v}_c}{c} \cdot \hat{\mathbf{R}}\right) \\
&= \gamma_{t_r}^2\left(\frac{\mathbf{a}_c}{c} \cdot \hat{\mathbf{R}} + \frac{\mathbf{v}_c}{c} \cdot \hat{\mathbf{R}}'\right),
\end{aligned} \tag{7.174}$$

and messier

$$\begin{aligned}
\hat{\mathbf{R}}' &= \frac{\partial}{\partial t_r}\frac{\mathbf{R}}{R} \\
&= \frac{\mathbf{R}'}{R} - \frac{\mathbf{R}R'}{R^2} \\
&= -\frac{\mathbf{v}_c}{R} - \frac{\hat{\mathbf{R}}(-c)}{R} \\
&= \frac{1}{R}\left(-\mathbf{v}_c + c\hat{\mathbf{R}}\right),
\end{aligned} \tag{7.175}$$

then a bit unmessier

$$\begin{aligned}\gamma'_{t_r} &= \gamma_{t_r}^2 \left(\frac{\mathbf{a}_c}{c} \cdot \hat{\mathbf{R}} + \frac{\mathbf{v}_c}{c} \cdot \hat{\mathbf{R}}' \right) \\ &= \gamma_{t_r}^2 \left(\frac{\mathbf{a}_c}{c} \cdot \hat{\mathbf{R}} + \frac{\mathbf{v}_c}{cR} \cdot (-\mathbf{v}_c + c\hat{\mathbf{R}}) \right) \\ &= \gamma_{t_r}^2 \left(\hat{\mathbf{R}} \cdot \left(\frac{\mathbf{a}_c}{c} + \frac{\mathbf{v}_c}{R} \right) - \frac{\mathbf{v}_c^2}{cR} \right).\end{aligned} \tag{7.176}$$

Now we are set to plug this back into our electric field expression and start grouping terms

$$\begin{aligned}\mathbf{E} &= -\frac{e\gamma_{t_r}^2}{cR} \left(\frac{\mathbf{a}_c}{c} + \frac{\mathbf{v}_c}{R} - \frac{\hat{\mathbf{R}}c}{R} + \gamma_{t_r} \left(\hat{\mathbf{R}} \cdot \left(\frac{\mathbf{a}_c}{c} + \frac{\mathbf{v}_c}{R} \right) - \frac{\mathbf{v}_c^2}{cR} \right) \left(\frac{\mathbf{v}_c}{c} - \hat{\mathbf{R}} \right) \right) \\ &= -\frac{e\gamma_{t_r}^3}{cR} \left(\left(\frac{\mathbf{a}_c}{c} + \frac{\mathbf{v}_c}{R} - \frac{\hat{\mathbf{R}}c}{R} \right) \left(1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c} \right) + \left(\hat{\mathbf{R}} \cdot \left(\frac{\mathbf{a}_c}{c} + \frac{\mathbf{v}_c}{R} \right) - \frac{\mathbf{v}_c^2}{cR} \right) \left(\frac{\mathbf{v}_c}{c} - \hat{\mathbf{R}} \right) \right) \\ &= -\frac{e\gamma_{t_r}^3}{c^2R} \left(\mathbf{a}_c \left(1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c} \right) + \hat{\mathbf{R}} \cdot \mathbf{a}_c \left(\frac{\mathbf{v}_c}{c} - \hat{\mathbf{R}} \right) \right) \\ &\quad - \frac{e\gamma_{t_r}^3}{cR} \left(\left(\frac{\mathbf{v}_c}{R} - \frac{\hat{\mathbf{R}}c}{R} \right) \left(1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c} \right) + \left(\hat{\mathbf{R}} \cdot \left(\frac{\mathbf{v}_c}{R} \right) - \frac{\mathbf{v}_c^2}{cR} \right) \left(\frac{\mathbf{v}_c}{c} - \hat{\mathbf{R}} \right) \right)\end{aligned} \tag{7.177}$$

Using

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{7.178}$$

We can verify that

$$\begin{aligned}-\left(\mathbf{a}_c \left(1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c} \right) + \hat{\mathbf{R}} \cdot \mathbf{a}_c \left(\frac{\mathbf{v}_c}{c} - \hat{\mathbf{R}} \right) \right) &= -\mathbf{a}_c + \mathbf{a}\hat{\mathbf{R}} \cdot \frac{\mathbf{v}}{c} - \hat{\mathbf{R}} \cdot \mathbf{a}_c \frac{\mathbf{v}_c}{c} + \hat{\mathbf{R}} \cdot \mathbf{a}_c \hat{\mathbf{R}} \\ &= \hat{\mathbf{R}} \times \left(\left(\hat{\mathbf{R}} - \frac{\mathbf{v}_c}{c} \right) \times \mathbf{a}_c \right),\end{aligned} \tag{7.179}$$

which gets us closer to the desired end result

$$\begin{aligned}\mathbf{E} &= \frac{e\gamma_{t_r}^3}{c^2R} \hat{\mathbf{R}} \times \left(\left(\hat{\mathbf{R}} - \frac{\mathbf{v}_c}{c} \right) \times \mathbf{a}_c \right) \\ &\quad - \frac{e\gamma_{t_r}^3}{cR^2} \left(\left(\mathbf{v}_c - \hat{\mathbf{R}}c \right) \left(1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c} \right) + \left(\hat{\mathbf{R}} \cdot \mathbf{v}_c - \frac{\mathbf{v}_c^2}{c} \right) \left(\frac{\mathbf{v}_c}{c} - \hat{\mathbf{R}} \right) \right).\end{aligned} \tag{7.180}$$

It is also easy to show that the remaining bit reduces nicely, since all the dot product terms conveniently cancel

$$-\left(\left(\mathbf{v}_c - \hat{\mathbf{R}} c\right)\left(1 - \hat{\mathbf{R}} \cdot \frac{\mathbf{v}_c}{c}\right) + \left(\hat{\mathbf{R}} \cdot \mathbf{v}_c - \frac{\mathbf{v}_c^2}{c}\right)\left(\frac{\mathbf{v}_c}{c} - \hat{\mathbf{R}}\right)\right) = c\left(1 - \frac{\mathbf{v}_c^2}{c^2}\right)\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right) \tag{7.181}$$

This completes the exercise, leaving us with

$$\begin{aligned} \mathbf{E} &= \frac{e\gamma_{t_r}^3}{c^2 R} \hat{\mathbf{R}} \times \left(\left(\hat{\mathbf{R}} - \frac{\mathbf{v}_c}{c}\right) \times \mathbf{a}_c\right) + \frac{e\gamma_{t_r}^3}{R^2}\left(1 - \frac{\mathbf{v}_c^2}{c^2}\right)\left(\hat{\mathbf{R}} - \frac{\mathbf{v}_c}{c}\right) \\ \mathbf{B} &= \hat{\mathbf{R}} \times \mathbf{E}. \end{aligned} \tag{7.182}$$

Looking back to eq. (7.171) where γ_{t_r} was defined, we see that this compares to (63.8-9) in the text.

Part b.

EM fields from a uniformly moving source For a uniform source moving in space at constant velocity

$$\mathbf{x}_c(t) = \mathbf{v} t, \tag{7.183}$$

our retarded time measured from the spacetime point $(ct, \mathbf{x})$ is defined implicitly by

$$R = |\mathbf{x} - \mathbf{x}_c(t_r)| = c(t - t_r). \tag{7.184}$$

Squaring this we have

$$\mathbf{x}^2 + \mathbf{v}^2 t_r^2 - 2 t_r \mathbf{x} \cdot \mathbf{v} = c^2 t^2 + c^2 t_r^2 - 2 c t t_r, \tag{7.185}$$

or

$$(c^2 - \mathbf{v}^2) t_r^2 + 2 t_r (-ct + \mathbf{x} \cdot \mathbf{v}) = \mathbf{x}^2 - c^2 t^2. \tag{7.186}$$

Rearranging to complete the square we have

$$\begin{aligned}
&\left(\sqrt{c^2 - \mathbf{v}^2} t_r - \frac{tc^2 - \mathbf{x} \cdot \mathbf{v}}{\sqrt{c^2 - \mathbf{v}^2}} \right)^2 \\
&= \mathbf{x}^2 - c^2 t^2 + \frac{(tc^2 - \mathbf{x} \cdot \mathbf{v})^2}{c^2 - \mathbf{v}^2} \\
&= \frac{(\mathbf{x}^2 - c^2 t^2)(c^2 - \mathbf{v}^2) + (tc^2 - \mathbf{x} \cdot \mathbf{v})^2}{c^2 - \mathbf{v}^2} \\
&= \frac{\mathbf{x}^2 c^2 - \mathbf{x}^2 \mathbf{v}^2 - \cancel{c^4 t^2} + c^2 t^2 \mathbf{v}^2 + \cancel{t^2 c^4} + (\mathbf{x} \cdot \mathbf{v})^2 - 2tc^2 (\mathbf{x} \cdot \mathbf{v})}{c^2 - \mathbf{v}^2} \\
&= \frac{c^2 (\mathbf{x}^2 + t^2 \mathbf{v}^2 - 2t(\mathbf{x} \cdot \mathbf{v})) - \mathbf{x}^2 \mathbf{v}^2 + (\mathbf{x} \cdot \mathbf{v})^2}{c^2 - \mathbf{v}^2} \\
&= \frac{c^2 (\mathbf{x} - \mathbf{v} t)^2 - (\mathbf{x} \times \mathbf{v})^2}{c^2 - \mathbf{v}^2}
\end{aligned} \tag{7.187}$$

Taking roots (and keeping the negative so that we have $t_r = t - |\mathbf{x}|/c$ for the $\mathbf{v} = 0$ case, we have

$$\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} c t_r = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \left(ct - \mathbf{x} \cdot \frac{\mathbf{v}}{c} - \sqrt{ (\mathbf{x} - \mathbf{v} t)^2 - \left(\mathbf{x} \times \frac{\mathbf{v}}{c} \right)^2 } \right), \tag{7.188}$$

or with $\boldsymbol{\beta} = \mathbf{v}/c$, this is

$$c t_r = \frac{1}{1 - \boldsymbol{\beta}^2} \left(ct - \mathbf{x} \cdot \boldsymbol{\beta} - \sqrt{ (\mathbf{x} - \mathbf{v} t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2 } \right). \tag{7.189}$$

What is our retarded distance $R = ct - ct_r$? We get

$$R = \frac{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v} t) + \sqrt{(\mathbf{x} - \mathbf{v} t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2}}{1 - \boldsymbol{\beta}^2}. \tag{7.190}$$

For the vector distance we get (with $\boldsymbol{\beta} \cdot (\mathbf{x} \wedge \boldsymbol{\beta}) = (\boldsymbol{\beta} \cdot \mathbf{x}) \boldsymbol{\beta} - \mathbf{x} \boldsymbol{\beta}^2$)

$$\mathbf{R} = \frac{\mathbf{x} - \mathbf{v} t + \boldsymbol{\beta} \cdot (\mathbf{x} \wedge \boldsymbol{\beta}) + \boldsymbol{\beta} \sqrt{(\mathbf{x} - \mathbf{v} t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2}}{1 - \boldsymbol{\beta}^2}. \tag{7.191}$$

For the unit vector $\hat{\mathbf{R}} = \mathbf{R}/R$ we have

$$\hat{\mathbf{R}} = \frac{\mathbf{x} - \mathbf{v}t + \boldsymbol{\beta} \cdot (\mathbf{x} \wedge \boldsymbol{\beta}) + \boldsymbol{\beta}\sqrt{(\mathbf{x} - \mathbf{v}t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2}}{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t) + \sqrt{(\mathbf{x} - \mathbf{v}t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2}}. \tag{7.192}$$

The acceleration term in the electric field is zero, so we are left with just

$$\mathbf{E} = \frac{e\gamma_{t_r}^3}{R^2}\left(1 - \frac{\mathbf{v}_c^2}{c^2}\right)\left(\hat{\mathbf{R}} - \frac{\mathbf{v}_c}{c}\right). \tag{7.193}$$

Leading to γ_{t_r}, we have

$$\hat{\mathbf{R}} \cdot \boldsymbol{\beta} = \frac{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t + R^*\boldsymbol{\beta})}{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t) + R^*}, \tag{7.194}$$

where, following §38 of the text we write

$$R^* = \sqrt{(\mathbf{x} - \mathbf{v}t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2} \tag{7.195}$$

This gives us

$$\gamma_{t_r} = \frac{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t) + R^*}{R^*(1 - \boldsymbol{\beta}^2)}. \tag{7.196}$$

Observe that this equals one when $\boldsymbol{\beta} = 0$ as expected.
We can also compute

$$\begin{aligned}
\hat{\mathbf{R}} - \boldsymbol{\beta} &= \frac{\mathbf{x} + \boldsymbol{\beta} \cdot (\mathbf{x} \wedge \boldsymbol{\beta}) - \mathbf{v}t + \boldsymbol{\beta}\sqrt{(\mathbf{x} - \mathbf{v}t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2}}{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t) + \sqrt{(\mathbf{x} - \mathbf{v}t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2}} - \boldsymbol{\beta} \\
&= \frac{(\mathbf{x} - \mathbf{v}t)(1 - \boldsymbol{\beta}^2)}{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t) + \sqrt{(\mathbf{x} - \mathbf{v}t)^2 - (\mathbf{x} \times \boldsymbol{\beta})^2}}.
\end{aligned} \tag{7.197}$$

Our long and messy expression for the field is therefore

$$\begin{aligned}
\mathbf{E} &= e\gamma_{t_r}^3 \frac{1}{R^2}(1 - \boldsymbol{\beta}^2)(\hat{\mathbf{R}} - \boldsymbol{\beta}) \\
&= e\left(\frac{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t) + R^*}{R^*(1 - \boldsymbol{\beta}^2)}\right)^3 \frac{(1 - \boldsymbol{\beta}^2)^2}{(\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t) + R^*)^2}(1 - \boldsymbol{\beta}^2)\frac{(\mathbf{x} - \mathbf{v}t)(1 - \boldsymbol{\beta}^2)}{\boldsymbol{\beta} \cdot (\mathbf{x} - \mathbf{v}t) + R^*}
\end{aligned} \tag{7.198}$$

This gives us our final result

$$\mathbf{E} = e\frac{1}{(R^*)^3}(1-\boldsymbol{\beta}^2)(\mathbf{x}-\mathbf{v}t) \tag{7.199}$$

As a small test we observe that we get the expected result

$$\mathbf{E} = e\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{7.200}$$

for the $\boldsymbol{\beta} = 0$ case.

When $\mathbf{v} = V\mathbf{e}_1$ this also recovers equation (38.6) from the text as desired, and if we switch to primed coordinates

$$\begin{aligned}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
(1-\beta^2)r'^2 &= (x-vt)^2 + (y^2+z^2)(1-\beta^2),
\end{aligned} \tag{7.201}$$

we recover the field equation derived twice before in previous problem sets

$$\mathbf{E} = \frac{e}{(r')^3}(x', \gamma y', \gamma z') \tag{7.202}$$

EM fields from a uniformly moving source along x axis Initially I had errors in the vector treatment above, so tried with the simpler case using uniform velocity v along the x axis instead. Comparison of the two showed where my errors were in the vector algebra, and that is now also fixed up.

Performing all the algebra to solve for t_r in

$$|\mathbf{x} - vt_r\mathbf{e}_1| = c(t - t_r), \tag{7.203}$$

I get

$$ct_r = \frac{ct - x\beta - \sqrt{(x-vt)^2 + (y^2+z^2)(1-\beta^2)}}{1-\beta^2} = -\gamma(\beta x' + r') \tag{7.204}$$

This matches the vector expression from eq. (7.189) with the special case of $\mathbf{v} = v\mathbf{e}_1$ so we at least started off on the right foot.

For the retarded distance $R = ct - ct_r$ we get

$$R = \frac{\beta(x - vt) + \sqrt{(x - vt)^2 + (y^2 + z^2)(1 - \beta^2)}}{1 - \beta^2} = \gamma(\beta x' + r') \tag{7.205}$$

This also matches eq. (7.190), so things still seem okay with the vector approach. What is our vector retarded distance

$$\begin{aligned}
\mathbf{R} &= \mathbf{x} - \beta c t_r \mathbf{e}_1 \\
&= (x - \beta c t_r, y, z) \\
&= \left(\frac{x - vt + \beta \sqrt{(x - vt)^2 + (y^2 + z^2)(1 - \beta^2)}}{1 - \beta^2}, y, z \right) \\
&= \left(\gamma(x' + \beta r'), y', z' \right)
\end{aligned} \tag{7.206}$$

So

$$\begin{aligned}
\hat{\mathbf{R}} &= \frac{1}{\gamma(\beta x' + r')} \left(\gamma(x' + \beta r'), y', z' \right) \\
&= \frac{1}{\beta x' + r'} \left(x' + \beta r', \frac{y'}{\gamma}, \frac{z'}{\gamma} \right)
\end{aligned} \tag{7.207}$$

$$\begin{aligned}
\hat{\mathbf{R}} - \boldsymbol{\beta} &= \frac{1}{\gamma(\beta x' + r')} \left(\gamma(x' + \beta r'), y', z' \right) - (\beta, 0, 0) \\
&= \frac{1}{\beta x' + r'} \left(x'(1 - \beta^2), \frac{y'}{\gamma}, \frac{z'}{\gamma} \right) \\
&= \frac{1}{\gamma(\beta x' + r')} (x - vt, y, z)
\end{aligned} \tag{7.208}$$

For $\partial t_r / \partial t$, using $\hat{\mathbf{R}}$ calculated above, or from eq. (7.204) calculating directly I get

$$\frac{\partial t_r}{\partial t} = \frac{r' + \beta x'}{r'(1 - \beta^2)} = \frac{\gamma(r' + \beta x')}{R^*}, \tag{7.209}$$

where, as in §38 of the text, we write

$$R^* = \sqrt{(x - vt)^2 + (y^2 + z^2)(1 - \beta^2)}. \tag{7.210}$$

Putting all the pieces together I get

$$\mathbf{E} = e(1-\beta^2)\frac{(x - vt, y, z)}{\cancel{\gamma(\beta x' + r')}}\left(\frac{\cancel{\gamma(r' + \beta x')}}{R^*}\right)^3 \frac{1}{\cancel{\gamma^2(\beta x' + r')^2}} \tag{7.211}$$

so we have

$$\mathbf{E} = e\frac{1-\beta^2}{(R^*)^3}(x - vt, y, z) \tag{7.212}$$

This matches equation (38.6) in the text.

Exercise 7.5 Energy-momentum tensor and electromagnetic forces

In class, it was argued that in the absence of charges and currents, the energy-momentum tensor (or the "stress-energy" tensor) of the electromagnetic field

$$T^{km} = -\frac{1}{4\pi}F^{kj}F^{m}{}_{j} + \frac{1}{16\pi}g^{km}F^{ij}F_{ij}, \tag{7.213}$$

is conserved:

$$\partial_k T^{km} = 0. \tag{7.214}$$

In this problem, you will study the fate of eq. (7.213), the law of energy and momentum conservation in the presence of charged particles and currents given by a 4-vector current j^l.

a. Conservation relation in the presence of sources
Use the equations of motion in the presence of sources, $\partial_l F^{lk} = \frac{4\pi}{c} j^m$, the fact that $F^{lk} = \partial^l A^m - \partial^m A^l$, and appropriate index gymnastics to show that eq. (7.214) is now replaced by

$$\partial_k T^{km} = -\frac{1}{c}F^{ml}j_l. \tag{7.215}$$

b. Timelike component of the conservation relation
Consider the $m = 0$ components of eq. (7.215). Show that it implies the energy conservation equation already discussed in class (see notes pp. 125-127):

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{j}. \tag{7.216}$$

Recall the physical interpretation of the various terms in this equation.

c. Spacelike component of the conservation relation
Consider the $m = \alpha$ components of eq. (7.215). Show that it implies that:

$$\frac{\partial}{\partial t}\left(\frac{S^\alpha}{c^2}\right) + \frac{\partial}{\partial x^\beta} T^{\beta\alpha} = -\left(\rho E^\alpha + \frac{1}{c}\left(\mathbf{j}\times\mathbf{B}\right)^\alpha\right) \equiv -f^\alpha \tag{7.217}$$

Give a physical interpretation of f^α.

d. Integrated over a volume
Integrate eq. (7.217) over a closed volume V and use integration by parts to obtain

$$\frac{\partial}{\partial t}\int_V d^3\mathbf{x}\frac{S^\alpha}{c^2} = -\int_{\partial V=S} d^2\sigma^\beta T^{\beta\alpha} - \int_V d^3\mathbf{x} f^\alpha \tag{7.218}$$

Give a physical interpretation of eq. (7.218) as expressing momentum conservation. In particular, explain how, if the volume V is that of a body (made of charged particles – bound or otherwise), this implies that:

$$\begin{aligned}&\frac{d}{dt}\left(\mathbf{p}_{\text{EM field in } V} + \mathbf{p}_{\text{charged particles in } V}\right)^\alpha \\ &\quad= \int_{\text{surface of body}} \left((\text{surface force})^\alpha \text{on body due to shears and pressures}\right)\end{aligned} \tag{7.219}$$

(Note that here $d^2\sigma^\beta$ is an outward normal vector to the surface of the body, so the surface has a relative minus signs w.r.t the one from class, where an inward normal was used.)

e. Pressure and shear of linearly polarized EM wave
Imagine that a place linearly polarized electromagnetic wave is falling on a flat surface at an angle of incidence α, and is completely absorbed by the body. Find the pressure and shear on a unit area of the surface using the Maxwell stress tensor.

Answer for Exercise 7.5

Part a. Diving straight in, a contraction of the coordinates of the four gradient with the stress energy tensor appears to produce most of the desired result

$$
\begin{aligned}
\partial_k T^{km} &= \frac{1}{4\pi}\left(-\partial_k (F^{kj} F^m{}_j) + \frac{1}{4} g^{km} \partial_k (F^{ij} F_{ij})\right) \\
&= \frac{1}{4\pi}\left(-\partial^k (F_{kj} F^{mj}) + \frac{1}{2} F_{ij} \partial^m F^{ij}\right) \\
&= \frac{1}{4\pi}\left(-F^{mj} \overbrace{\boxed{\partial^k F_{kj}}}^{=4\pi j_j/c} - \underbrace{\boxed{F_{kj}\partial^k F^{mj}}}_{\text{rename } k \to i} + \frac{1}{2} F_{ij} \partial^m F^{ij}\right) \\
&= -\frac{1}{c} F^{ma} j_a + \frac{F_{ij}}{4\pi}\left(-\partial^i F^{mj} + \frac{1}{2} \partial^m F^{ij}\right)
\end{aligned}
\tag{7.220}
$$

To complete the task, it only remains to show that this second term is zero. First let us get rid of the $1/2$ by writing $1 = 1/2 + 1/2$ using the index swapping trick

$$
\begin{aligned}
F_{ij} \partial^i F^{mj} &= \frac{1}{2} F_{ij} \partial^i F^{mj} + \frac{1}{2} F_{ji} \partial^j F^{mi} \\
&= \frac{1}{2} F_{ij} \left(\partial^i F^{mj} - \partial^j F^{mi}\right).
\end{aligned}
\tag{7.221}
$$

This gives us for the second term

$$
\begin{aligned}
&\frac{F_{ij}}{4\pi}\left(-\partial^i F^{mj} + \frac{1}{2} \partial^m F^{ij}\right) \\
&\quad= \frac{F_{ij}}{8\pi}\left(\partial^i F^{jm} + \partial^j F^{mi} + \partial^m F^{ij}\right) \\
&\quad= \frac{F_{ij}}{8\pi}\left(\partial^i \partial^j A^m - \partial^i \partial^m A^j + \partial^j \partial^m A^i - \partial^j \partial^i A^m + \partial^m \partial^i A^j - \partial^m \partial^j A^i\right).
\end{aligned}
\tag{7.222}
$$

By commuting derivatives, assuming the typical sufficient continuity of the fields, all of these six terms in braces cancel. This completes this portion of the exercise.

Part b. The goal is to express the four divergence

$$
\partial_k T^{k0} = -\frac{1}{c} F^{0a} j_a,
\tag{7.223}
$$

explicitly utilizing a space time split from some stationary frame where the fields and currents are observed as **E**, **B**, **j**, and ρ. On the RHS, because $F^{00} = 0$ the summation is reduced to three indices

$$F^{0a} j_a = F^{0\alpha} j_\alpha = -F^{0\alpha} (\mathbf{j})^\alpha. \tag{7.224}$$

In this the tensor factor is

$$\begin{aligned} F^{0\alpha} &= \partial^0 A^\alpha - \partial^\alpha A^0 \\ &= \frac{1}{c}\partial_t A^\alpha + \partial_\alpha A^0 \\ &= -(\mathbf{E})^\alpha, \end{aligned} \tag{7.225}$$

and the RHS of eq. (7.223) is reduced to

$$-\frac{1}{c} F^{0a} j_a = -\frac{1}{c}\mathbf{E}\cdot\mathbf{j}. \tag{7.226}$$

Now let us expand the LHS. Recall that

$$\begin{aligned} T^{00} &= \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = \mathcal{E} \\ T^{\alpha 0} &= \frac{1}{4\pi}(\mathbf{E}\times\mathbf{B})^\alpha = \frac{\mathbf{S}^\alpha}{c}. \end{aligned} \tag{7.227}$$

With $\partial_0 = \partial_t / c$, our equation becomes

$$\partial_k T^{k0} = \frac{1}{c}\frac{\partial}{\partial t}\mathcal{E} + \frac{\partial}{\partial x^\alpha}\frac{\mathbf{S}^\alpha}{c} = -\frac{1}{c}\mathbf{E}\cdot\mathbf{j}. \tag{7.228}$$

Multiplying through by c recovers eq. (7.216) as desired.

Part c. The goal is to expand

$$\partial_k T^{k\alpha} = -\frac{1}{c} F^{\alpha l} j_l. \tag{7.229}$$

On the RHS is

$$
\begin{aligned}
-\frac{1}{c}F^{\alpha l}j_l &= -\frac{1}{c}\left(F^{\alpha 0}j_0 + F^{\alpha\beta}j_\beta\right) \\
&= -\mathbf{E}^\alpha\rho - \frac{1}{c}(-\epsilon^{\sigma\alpha\beta}\mathbf{B}^\sigma)(-\mathbf{j}^\beta) \\
&= -\mathbf{E}^\alpha\rho - \frac{1}{c}\epsilon^{\alpha\beta\sigma}\mathbf{B}^\sigma\mathbf{j}^\beta \\
&= -(\rho\mathbf{E} + \frac{\mathbf{j}}{c}\times\mathbf{B})^\alpha.
\end{aligned}
\tag{7.230}
$$

For the LHS of eq. (7.229), using

$$
T^{0\alpha} = \frac{\mathbf{S}^\alpha}{c}. \tag{7.231}
$$

Putting the pieces together leaves us with the desired relationship

$$
\frac{1}{c}\frac{\partial}{\partial t}\frac{S^\alpha}{c} + \frac{\partial T^{\beta\alpha}}{\partial x^\beta} = -\left(\rho\mathbf{E} + \frac{\mathbf{j}}{c}\times\mathbf{B}\right)^\alpha. \tag{7.232}
$$

The RHS can be seen to be the (negated) Lorentz force per unit volume. Introducing discrete charge and current densities utilizing delta functions and integrating, gives us exactly the spatial (non-energy) components of the Lorentz force equation (this is done in detail in the next portion of this problem below).

This is a rather interesting result. In §33 of [11] the energy momentum tensor was found to be closely related to the spacetime translation symmetries for the charge and current free Lagrangian density for the field (although this produced a non-symmetric tensor and a special value of zero had to be added to get it into symmetric form). So without any requirement to perform variation of the interaction action

$$
S = -mc\int ds - \frac{e}{c}\int ds u^i A_i, \tag{7.233}
$$

one still ends up with all the components of the Lorentz force equation! Only the Lagrangian density for the field was required to obtain the result (which was also indirectly used to obtain the relation of the field to the charge and current densities). The interaction action (and thus the Lorentz force equation itself) seems to be almost redundant. What it does provide, however, is excellent motivation for the labeling of

$$
\frac{S^\alpha}{c^2}, \tag{7.234}
$$

as momentum density for the EM field. In class when the Poynting vector **S** was introduced, and a dimensional analysis motivation was presented, we were told a more satisfying identification of $\mathbf{S}/c^2$ with the momentum density would be forthcoming and here it is. With force per volume on the RHS and the time derivative of a "something" S^α/c^2 on the LHS, one is forced to conclude that this "something" is a momentum density. Not just by dimensions, but by context in its use in a force like equation.

Part d. Integrating eq. (7.217) over a closed volume V gives

$$\begin{aligned}
0 &= \int_V d^3\mathbf{x}\frac{\partial}{\partial t}\left(\frac{S^\alpha}{c^2}\right) + \int_V d^3\mathbf{x}\frac{\partial}{\partial x^\beta}T^{\beta\alpha} + \int_V d^3\mathbf{x}\rho E^\alpha + \frac{1}{c}\left(\mathbf{j}\times\mathbf{B}\right)^\alpha \\
&= \frac{\partial}{\partial t}\int_V d^3\mathbf{x}\frac{S^\alpha}{c^2} + \int_V d^3\mathbf{x}\boldsymbol{\nabla}\cdot(\mathbf{e}_\beta T^{\beta\alpha}) \\
&+ \sum_b q_b \int_V d^3\mathbf{x}\left(E^\alpha + \frac{1}{c}\left(\mathbf{v}_b(t)\times\mathbf{B}\right)^\alpha\right)\delta^3(\mathbf{x}-\mathbf{x}_b(t)) \\
&= \frac{\partial}{\partial t}\int_V d^3\mathbf{x}\frac{S^\alpha}{c^2} + \int_{\partial V} d^2\sigma(\mathbf{n}\cdot\mathbf{e}_\beta)T^{\beta\alpha} + \sum_b q_b\left(E^\alpha(\mathbf{x}_b) + \left(\frac{\mathbf{v}_b(t)}{c}\times\mathbf{B}(\mathbf{x}_b)\right)^\alpha\right).
\end{aligned} \tag{7.235}$$

In the first integral, the integration and time derivative operational order was exchanged. In the second integral the contraction was written as a spatial divergence $\partial_\beta T^{\beta\alpha} = \boldsymbol{\nabla}\cdot(\mathbf{e}_\beta T^{\beta\alpha})$, so that Stokes theorem could be used to express this integral as the integral over the boundary of the surface, with outward normal **n**. In the last, the charge and current densities were expressed in terms of discrete particles

$$\begin{aligned}
\rho &= \sum_b q_b \delta^3(\mathbf{x}-\mathbf{x}_b(t)) \\
\mathbf{j} &= \sum_b q_b \mathbf{v}_b(t)\delta^3(\mathbf{x}-\mathbf{x}_b(t)).
\end{aligned} \tag{7.236}$$

So with the surface area element $d^2\sigma$, and the outward normal **n** on that surface, an indexed normal area element can be introduced as in the problem statement

$$d^2\sigma^\beta \equiv d^2\sigma(\mathbf{n}\cdot\mathbf{e}_\beta). \tag{7.237}$$

So our integrated conservation relationship is left in the form

$$\frac{\partial}{\partial t}\int_V d^3\mathbf{x}\frac{S^\alpha}{c^2} + \int_{\partial V} d^2\sigma^\beta T^{\beta\alpha} = -\sum_b q_b\left(\mathbf{E}(\mathbf{x}_b) + \left(\frac{\mathbf{v}_b(t)}{c}\times\mathbf{B}(\mathbf{x}_b)\right)\right)^\alpha. \tag{7.238}$$

Observe that the RHS is the α component of the (negated) Lorentz force f_α on the particles from the field, so the RHS represents the force of the charge distribution on the field. Looking at the LHS of the equation where the time derivative of $\int d^3\mathbf{x} S^\alpha/c^2$ appears, there is finally an excellent justification for calling S^α/c^2 the momentum density.

Once this Lorentz force is expressed as a rate of change of momentum

$$\frac{d}{dt}\mathbf{p}_{\text{charges}} = \sum_b q_b \left(\mathbf{E}(\mathbf{x}_b) + \frac{\mathbf{v}_b(t)}{c} \times \mathbf{B}(\mathbf{x}_b) \right), \tag{7.239}$$

and the field momentum is also expressed in terms of the momentum density

$$\mathbf{p}_{\text{EM field}} = \int d^3\mathbf{x} \frac{\mathbf{S}}{c^2}, \tag{7.240}$$

the desired result is produced

$$\frac{d}{dt} \left(\mathbf{p}_{\text{EM field}} + \mathbf{p}_{\text{charges}} \right)^\alpha = -\int_{\partial V} d^2\sigma^\beta T^{\beta\alpha}. \tag{7.241}$$

Any change in the momentum of the field or the charges acted on by the field in a volume, is found to equal a force per unit area, acting on the surface of that volume. Those components of this force that are normal to the surface can be called pressure, and just as in mechanics, the portion of this force per unit area acting tangentially along the surface, can be called shear.

Part e. In class we found a Coulomb gauge solution for the linearly polarized EM wave to be

$$\begin{aligned} \mathbf{E} &= k\boldsymbol{\beta} \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) \\ \mathbf{B} &= \hat{\mathbf{k}} \times \mathbf{E} \\ c^2\mathbf{k}^2 &= \omega^2 \\ \boldsymbol{\beta} \cdot \mathbf{k} &= 0, \end{aligned} \tag{7.242}$$

where $\hat{\mathbf{k}}$ is the propagation direction. For this problem, let us align $\mathbf{k}$ along the z-axis, and $\boldsymbol{\beta}$ along the x-axis. The fields are then just

$$\begin{aligned} \mathbf{E} &= k\beta \sin(\omega t - kz)\mathbf{e}_1 \\ \mathbf{B} &= k\beta \sin(\omega t - kz)\mathbf{e}_2. \end{aligned} \tag{7.243}$$

Computation of the stress energy tensor components becomes straightforward.

$$T^{00} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) = \frac{k^2\beta^2}{4\pi}\sin^2(\omega t - \mathbf{k}\cdot\mathbf{x}). \tag{7.244}$$

The Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \frac{ck^2\beta^2}{4\pi}\sin^2(\omega t - kz)\mathbf{e}_3, \tag{7.245}$$

determines the energy flux components of the tensor $T^{0\alpha} = S^\alpha/c$

$$\begin{aligned}
T^{01} &= T^{10} = 0 \\
T^{02} &= T^{20} = 0 \\
T^{03} &= T^{30} = \frac{k^2\beta^2}{4\pi}\sin^2(\omega t - kz).
\end{aligned} \tag{7.246}$$

The stress and shear components are left. All the off diagonal components are zero

$$\begin{aligned}
T^{21} &= T^{12} = -\frac{1}{4\pi}(E_x \cancel{E_y} + \cancel{B_x} B_y) = 0 \\
T^{31} &= T^{13} = -\frac{1}{4\pi}(E_x \cancel{E_z} + \cancel{B_x B_z}) = 0 \\
T^{32} &= T^{23} = -\frac{1}{4\pi}(\cancel{E_y E_z} + B_y \cancel{B_z}) = 0
\end{aligned} \tag{7.247}$$

Two of our diagonal stress components are also zero

$$\begin{aligned}
T^{11} &= \frac{1}{4\pi}\left(E_x^2 + \cancel{B_y^2} - \frac{1}{2}(E_x^2 + B_y^2)\right) = 0 \\
T^{22} &= \frac{1}{4\pi}\left(\cancel{E_y^2} + B_y^2 - \frac{1}{2}(E_x^2 + B_y^2)\right) = 0
\end{aligned} \tag{7.248}$$

(since $E_x^2 = B_y^2 = k^2\beta^2\sin^2(\omega t - kz)$). We are left with just the T^{33} term

$$T^{33} = -\frac{1}{4\pi}\left(\cancel{E_z^2} + \cancel{B_z^2} - \frac{1}{2}(E_x^2 + B_y^2)\right) = \frac{1}{4\pi}k^2\beta^2\sin^2(\omega t - kz) \tag{7.249}$$

In matrix form this is

$$\left\| T^{ab} \right\| = \frac{k^2 \beta^2}{4\pi} \sin^2(\omega t - kz) \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{7.250}$$

Check: The trace should be zero:

$$T^i{}_i = T^{00} - T^{33} = 0. \tag{7.251}$$

Continuing: From $\partial_a T^{ab} = 0$ we have

$$\frac{\partial}{\partial t}\left(\frac{S_z}{c^2}\right) + \frac{\partial}{\partial x^\beta} T^{\beta 3} = 0, \tag{7.252}$$

which is what can be used to compute the force. Integrating this we have

$$\begin{aligned} \frac{\partial}{\partial t} \int_V d^3\mathbf{x} \frac{S_z}{c^2} &= -\int_{\partial V} d^2\sigma (\mathbf{n} \cdot \mathbf{e}_\beta) T^{\beta 3} \\ &= -\int_{\partial V} d^2\sigma (\mathbf{n} \cdot \mathbf{e}_3) T^{33} \end{aligned} \tag{7.253}$$

On the RHS, the RHS of the EM field momentum, is the force that the field applies to the volume it passes through. Let us align the wall that absorbs the light tilted at an angle α from the vertical. Temporarily utilizing complex numbers with $\mathbf{e}_3 \sim 1$ and $\mathbf{e}_1 \sim i$ to compute the rotated coordinates we have

$$\begin{aligned} \mathbf{n} &\sim i e^{i(\pi/2 - \alpha)} \\ &= i^2 e^{-i\alpha} \\ &= -\cos\alpha + i \sin\alpha \\ &\sim -\mathbf{e}_3 \cos\alpha + \mathbf{e}_1 \sin\alpha \end{aligned} \tag{7.254}$$

PICTURE: ...

The dot product is thus

$$\mathbf{n} \cdot \mathbf{e}_3 = -\cos\alpha \tag{7.255}$$

If we create a volume bounded by an area ΔA on the surface, passing into the wall, the stress energy tensor is only non-zero on the outwards facing surface, so the force on that surface is

$$\begin{aligned}\mathbf{F} &= -\int_{\partial V} d^2\sigma(\mathbf{n}\cdot\mathbf{e}_3)T^{33}\mathbf{e}_3 \\ &= -\int_{\partial V} d^2\sigma(-\cos\alpha)\frac{k^2\beta^2}{4\pi}\sin^2(\omega t - kz)\mathbf{e}_3 \\ &= \int_{\partial V} d^2\sigma\cos\alpha\frac{k^2\beta^2}{8\pi}(1-\cos(2(\omega t - kz))\mathbf{e}_3\end{aligned} \tag{7.256}$$

Averaged over one period $T = 2\pi/\omega$, or one wave length $\lambda = 2\pi/k$, we find that the average momentum transferred to the wall per unit time is

$$\langle\mathbf{F}\rangle = \Delta A\cos\alpha\frac{k^2\beta^2}{8\pi}\mathbf{e}_3 \tag{7.257}$$

This can be resolved into a component normal to the absorbing wall (the pressure) and a component tangential to the wall. The normal component is just the inwards normal

$$-\mathbf{n} = \mathbf{e}_3\cos\alpha - \mathbf{e}_1\sin\alpha \tag{7.258}$$

Tangent to this is

$$\mathbf{t} = \mathbf{e}_1\cos\alpha + \mathbf{e}_3\sin\alpha. \tag{7.259}$$

Dotting with the time averaged force per unit area above we have the pressure and shear respectively

$$\begin{aligned}\text{Pressure} &= \cos^2\alpha\frac{k^2\beta^2}{8\pi} \\ \text{Shear} &= \cos\alpha\sin\alpha\frac{k^2\beta^2}{8\pi}\end{aligned} \tag{7.260}$$

Check: A sanity check with $\alpha = 0$, we see that the pressure is maximized when the light is perpendicular to the wall, and we have zero shear at that angle as expected. For $\alpha = \pi/2$ we see that both the pressure and shear drop to zero, also a good sanity check.

Disclaimer FIXME: One mark was lost in the calculation of the non-diagonal terms of the Maxwell stress tensor. Believe that one of those must have been non-zero. Go re-calculate.

Exercise 7.6 Monochromatic stress energy tensor

a. Show that the energy momentum tensor of a plane monochromatic wave with 4-vector

$$k^i = \left(\frac{\omega}{c}, \mathbf{k}\right), \tag{7.261}$$

and energy density $\mathcal{E}$ can be written as

$$T^{ij} = \frac{\mathcal{E}c^2}{\omega^2} k^i k^j. \tag{7.262}$$

b. Can one conclude now that $\frac{\mathcal{E}c^2}{\omega^2}$ for a plane wave is a Lorentz scalar?

Answer for Exercise 7.6

Part a. Determining the stress energy tensor. In the Coulomb gauge we used Fourier methods to find that the potential had the form

$$\begin{aligned}
\phi &= 0 \\
\mathbf{A} &= \boldsymbol{\beta} \cos(\omega t - \mathbf{k} \cdot \mathbf{x}) \\
c^2 \mathbf{k}^2 &= \omega^2 \\
\boldsymbol{\beta} \cdot \mathbf{k} &= 0.
\end{aligned} \tag{7.263}$$

For this problem it appears that working in the Lorentz gauge is required, and we want solutions of the form

$$A^m = D^m \cos(k_a x^a). \tag{7.264}$$

First, observe that the Lorentz gauge condition $\partial_m A^m = 0$ requires

$$-D^m k_m \sin(k_a x^a) = 0. \tag{7.265}$$

Application of the wave equation operator

$$\partial_b \partial^b A^m = 0, \tag{7.266}$$

gives us

$$-D^m k_b k^b \cos(k_a x^a) = 0, \tag{7.267}$$

providing the lightlike constraint on k. All told our four potential with constraints is

$$\begin{aligned} A^m &= D^m \cos(k_a x^a) \\ k^a k_a &= 0 \\ D^m k_m &= 0. \end{aligned} \tag{7.268}$$

We could also arrive at this point using 4D Fourier methods, which would be fun, but a bit more time consuming, and a little overkill given that the problem only requires us to tackle the linear monochromatic case.

On to the problem. We now need our electromagnetic tensor components.

$$\begin{aligned} F^{ij} &= \partial^i A^j - \partial^j A^i \\ &= D^j \partial^i \cos(k^a x_a) - D^i \partial^j \cos(k^a x_a) \\ &= \sin(k^a x_a)(D^i k^j - D^j k^i) \end{aligned} \tag{7.269}$$

Our stress energy tensor is

$$\begin{aligned} T^{ij} &= \frac{1}{4\pi}\left(-F^{ia} F_{ba} g^{bj} + \frac{1}{4} g^{ij} F_{ab} F^{ab}\right) \\ &= \frac{1}{4\pi}\left(-F^{ai} F_{ab} g^{bj} + \frac{1}{4} g^{ij} F_{ab} F^{ab}\right) \end{aligned} \tag{7.270}$$

Let us now expand the product of tensors

$$\begin{aligned} F_{ab} F^{ai} &= \sin^2(k^a x_a)(D_a k_b - D_b k_a)(D^a k^i - D^i k^a) \\ &= \sin^2(k^a x_a)(D_a k_b D^a k^i - D_a k_b D^i k^a - D_b k_a D^a k^i + D_b k_a D^i k^a) \\ &= \sin^2(k^a x_a)(D_a D^a k_b k^i - \cancel{D_a k^a} k_b D^i - D_b \cancel{k_a D^a} k^i + D_b D^i \cancel{k_a k^a}) \\ &= \sin^2(k^a x_a) D_a D^a k_b k^i \end{aligned} \tag{7.271}$$

We see from this that our action term is zero

$$F_{ab} F^{ab} = \sin^2(k^a x_a) D_a D^a \cancel{k_b k^b}, \tag{7.272}$$

so the stress energy tensor is reduced to

$$
\begin{aligned}
T^{ij} &= -\frac{1}{4\pi} \sin^2(k^a x_a) D_a D^a k_b k^i g^{jb} \\
&= -\frac{1}{4\pi} \sin^2(k^a x_a) D_a D^a k^j k^i
\end{aligned}
\tag{7.273}
$$

The energy density term of the stress energy tensor encapsulates most of these terms

$$
T^{00} = -\frac{1}{4\pi} \sin^2(k^a x_a) D_a D^a \frac{\omega^2}{c^2} = \mathcal{E}, \tag{7.274}
$$

so we can write

$$
T^{ij} = \mathcal{E} \frac{c^2}{\omega^2} k^i k^j, \tag{7.275}
$$

which completes the first part of this problem.

Part b. On the question of the Lorentz scalar. Yes, one can now conclude that $\frac{\mathcal{E}c^2}{\omega^2}$ for a plane wave is a Lorentz scalar.

Observe that the $k^i k^j$ transforms as a rank 2 tensor, as does T^{ij}. Because the product $\mathcal{E}c^2/\omega^2$ and $k^i k^j$ must transform as a rank 2 tensor, this can only mean that the $\mathcal{E}c^2/\omega^2$ portion transforms as a Lorentz scalar.

8 RADIATION REACTION

Reading Covering chapter 5 §37, and chapter 8 §65 material from the text [11], and lecture notes RelEMpp181-195.pdf.

8.1 A CLOSED SYSTEM OF CHARGED PARTICLES

Consider a closed system of charged particles (m_a, q_a) and imagine there is a frame where they are non-relativistic $v_a/c \ll 1$. In this case we can describe the dynamics using a Lagrangian only for particles. i.e.

$$\mathcal{L} = \mathcal{L}(\mathbf{x}_1, \cdots, \mathbf{x}_N, \mathbf{v}_1, \cdots, \mathbf{v}_N) \tag{8.1}$$

If we work t order $(v/c)^2$.
If we try to go to $O((v/c)^3$, it is difficult to only use $\mathcal{L}$ for particles.
This can be inferred from

$$P = \frac{2}{3}\frac{e^2}{c^3}\left|\ddot{\mathbf{d}}\right|^2 \tag{8.2}$$

because at this order, due to radiation effects, we need to include EM field as dynamical.

8.2 START SIMPLE

Start with a system of (non-relativistic) free particles

$$\begin{aligned} S &= \sum_a -m_a c \int_{\text{a-th particle worldline}} ds \\ &= \sum_a -m_a c^2 \int_{t_1}^{t_2} dt \sqrt{1 - \mathbf{v}_a^2/c^2} \\ &\approx \sum_a -m_a c^2 \int_{t_1}^{t_2} dt \left(1 - \frac{1}{2}\frac{\mathbf{v}^2}{c^2} - \frac{1}{8}\frac{\mathbf{v}_a^4}{c^4} \right) \\ &= \sum_a \int_{t_1}^{t_2} dt \left(\cancel{-m_a c^2} + \frac{1}{2} m_a \mathbf{v}^2 + \frac{1}{8} m_a \mathbf{v}_a^2 \frac{\mathbf{v}_a^2}{c^2} \right) \end{aligned} \tag{8.3}$$

So in the non-relativistic limit, after dropping the constant term that does not effect the dynamics, our Lagrangian is

$$\mathcal{L}(\mathbf{x}_a, \mathbf{v}_a) = \frac{1}{2}\sum_a m_a \mathbf{v}_a^2 + \frac{1}{8}\frac{m_a \mathbf{v}_a^4}{c^2} \tag{8.4}$$

The first term is $O((v/c)^0)$ where the second is $O((v/c)^2)$.

Next include the fact that particles are charged.

$$\mathcal{L}_{\text{interaction}} = \sum_a \left(\cancel{q_a \frac{\mathbf{v}_a}{c} \cdot \mathbf{A}(\mathbf{x}_a, t)} - q_a \phi(\mathbf{x}_a, t) \right) \tag{8.5}$$

Here, working to $O((v/c)^0)$, where we consider the particles moving so slowly that we have only a Coulomb potential ϕ, not $\mathbf{A}$.

HERE: these are NOT 'EXTERNAL' potentials. They are caused by all the charged particles.

$$\partial_i F^{il} = \frac{4\pi}{c} j^l = 4\pi\rho \tag{8.6}$$

For $l = \alpha$ we have have $4\pi\rho\mathbf{v}/c$, but we will not do this today (tomorrow).

To leading order in v/c, particles only created Coulomb fields and they only "feel" Coulomb fields. Hence to $O((v/c)^0)$, we have

$$\mathcal{L} = \sum_a \frac{m_a \mathbf{v}_a^2}{2} - q_a \phi(\mathbf{x}_a, t) \tag{8.7}$$

What is the $\phi(\mathbf{x}_a, t)$, the Coulomb field created by all the particles.

How to find?

$$\partial_i F^{i0} = \frac{4\pi}{c} = 4\pi\rho \tag{8.8}$$

or

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho = -\boldsymbol{\nabla}^2 \phi \tag{8.9}$$

where

$$\rho(\mathbf{x}, t) = \sum_a q_a \delta^3(\mathbf{x} - \mathbf{x}_a(t)) \tag{8.10}$$

This is a Poisson equation

$$\Delta\phi(\mathbf{x}) = \sum_a q_a 4\pi\delta^3(\mathbf{x}-\mathbf{x}_a) \tag{8.11}$$

(where the time dependence has been suppressed). This has solution

$$\phi(\mathbf{x},t) = \sum_b \frac{q_b}{|\mathbf{x}-\mathbf{x}_b(t)|} \tag{8.12}$$

This is the sum of instantaneous Coulomb potentials of all particles at the point of interest. Hence, it appears that $\phi(\mathbf{x}_a,t)$ should be evaluated in eq. (8.12) at $\mathbf{x}_a$?

However eq. (8.12) becomes infinite due to contributions of the a-th particle itself. Solution to this is to drop the term, but let us discuss this first.

Let us talk about the electrostatic energy of our system of particles.

$$\begin{aligned}\mathcal{E} &= \frac{1}{8\pi}\int d^3\mathbf{x}\left(\mathbf{E}^2 + \cancel{\mathbf{B}^2}\right)\\ &= \frac{1}{8\pi}\int d^3\mathbf{x}\mathbf{E}\cdot(-\nabla\phi)\\ &= \frac{1}{8\pi}\int d^3\mathbf{x}\left(\nabla\cdot(\mathbf{E}\phi) - \phi\nabla\cdot\mathbf{E}\right)\\ &= -\frac{1}{8\pi}\oint d^2\boldsymbol{\sigma}\cdot\mathbf{E}\phi + \frac{1}{8\pi}\int d^3\mathbf{x}\phi\nabla\cdot\mathbf{E}\end{aligned} \tag{8.13}$$

The first term is zero since $\mathbf{E}\phi$ for a localized system of charges $\sim 1/r^3$ or higher as $V\to\infty$. In the second term

$$\nabla\cdot\mathbf{E} = 4\pi\sum_a q_a\delta^3(\mathbf{x}-\mathbf{x}_a(t)) \tag{8.14}$$

So we have

$$\sum_a \frac{1}{2}\int d^3\mathbf{x} q_a\delta^3(\mathbf{x}-\mathbf{x}_a)\phi(\mathbf{x}) \tag{8.15}$$

for

$$\mathcal{E} = \frac{1}{2}\sum_a q_a\phi(\mathbf{x}_a) \tag{8.16}$$

Now substitute eq. (8.12) into eq. (8.16) for

$$\mathcal{E} = \frac{1}{2}\sum_a \frac{q_a^2}{|\mathbf{x} - \mathbf{x}_a|} + \frac{1}{2}\sum_{a\neq b} \frac{q_a q_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{8.17}$$

or

$$\mathcal{E} = \frac{1}{2}\sum_a \frac{q_a^2}{|\mathbf{x} - \mathbf{x}_a|} + \sum_{a<b} \frac{q_a q_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{8.18}$$

The first term is the sum of the electrostatic self energies of all particles. The source of this infinite self energy is in assuming a point like nature of the particle. i.e. We modeled the charge using a delta function instead of using a continuous charge distribution.

Recall that if you have a charged sphere of radius r

PICTURE: total charge q, radius r, our electrostatic energy is

$$\mathcal{E} \sim \frac{q^2}{r} \tag{8.19}$$

Stipulate that rest energy $m_e c^2$ is all of electrostatic origin $\sim e^2/r_e$ we get that

$$r_e \sim \frac{e^2}{m_e c^2} \tag{8.20}$$

This is called the classical radius of the electron, and is of a very small scale 10^{-13}cm.

As a matter of fact the applicability of classical electrodynamics breaks down much sooner than this scale since quantum effects start kicking in.

Our Lagrangian is now

$$\mathcal{L}_a = \frac{1}{2} m_a \mathbf{v}_a^2 - q_a \phi(\mathbf{x}_a, t) \tag{8.21}$$

where ϕ is the electrostatic potential due to all other particles, so we have

$$\mathcal{L}_a = \frac{1}{2} m_a \mathbf{v}_a^2 - \frac{1}{2}\sum_{a\neq b} \frac{q_a q_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{8.22}$$

and for the system

$$\mathcal{L} = \frac{1}{2}\sum_a m_a \mathbf{v}_a^2 - \sum_{a<b} \frac{q_a q_b}{|\mathbf{x}_a - \mathbf{x}_b|} \tag{8.23}$$

This is THE Lagrangian for electrodynamics in the non-relativistic case, starting with the relativistic action.

8.3 WHAT IS NEXT?

We continue to the next order of v/c tomorrow.

Reading Covering chapter 8 §65 material from the text [11], and lecture notes RelEMpp181-195.pdf.

8.4 RECAP

Last time we started with our relativistic Lagrangian for a single particle

$$\mathcal{L}_a = -mc^2 \sqrt{1 - \frac{\mathbf{v}_a^2}{c^2}} - \frac{q_a}{c} \frac{dx^i}{dt} A_i \tag{8.24}$$

and found that to the first order in v/c we had

$$\mathcal{L}_a = \frac{1}{2} m_a \mathbf{v}_a^2 - q_a \phi(\mathbf{x}_a, t). \tag{8.25}$$

Here the potential was approximated by Taylor expansion to contain just

$$\phi(\mathbf{x}_a, t) = \frac{1}{2} \sum_{a \neq b} \frac{q_b}{|\mathbf{x}_a - \mathbf{x}_b|} + \frac{q_a}{\text{``}\mathbf{x}_a - \mathbf{x}_a''}. \tag{8.26}$$

The second term is something that no sane person would write, and represents the infinite electrostatic self energy of a charge. This is infinite because we have assumed (by virtue of using a delta function for the current and charge distribution) that the charge is pointlike. The "solution" to this problem was to omit this self energy term completely, essentially treating the charge of the electron as distributed. We avoid looking specifically where it is located.

The logic here is that this does not affect the motion (i.e. The Euler Lagrange equations) for the particle, provided it is viewed from afar, with distances $\gg$ size of particle.

We made an estimate of the scale for which our Lagrangian does not apply. Namely

$$\frac{e^2}{r_e} \sim m_e c^2, \tag{8.27}$$

so we were able to conclude that the "classical radius of the electron", something that does not really exist, was of the scale

$$r_e \sim \frac{e^2}{m_e c^2} \sim 10^{-13}\text{cm} \tag{8.28}$$

(We do see this quantity arise in physics, but it is not a radius in the classical sense).

If this estimate was right, we would calculate that classical EM is value at $r \gg r_e \sim 10^{-13}$cm. In reality, classical electrodynamics breaks down at much larger distances.

NOTE: LHC is probing $\sim 10^{-16}$cm.

Our strategy here is to focus on the structure that can be observed. We do not have a way to probe to the small scale distances where the structure of the electron is relevant, so our description avoids that small range.

FIXME: I can not honestly say that I grasp the logic used to drop this self energy term. This was compared to the concept of mass renormalization from Quantum field theory, where if I recall correctly, certain infinities were avoided by carefully avoiding points of singularity where there was nothing observable. This is definitely something to revisit. If this shows up even in classical electrodynamics, it is going to be even harder to understand later with the complexity of Quantum field theory tossed into the mix.

8.5 MOVING ON TO THE NEXT ORDER IN (v/c)

Recall that we dropped terms from the original Lagrangian, which was

$$\mathcal{L}_a = -m_a c^2 \sqrt{1 - \frac{\mathbf{v}_a^2}{c^2}} - q_a \phi(\mathbf{x}_a, t) + q_a \frac{\mathbf{v}_a}{c} \cdot \mathbf{A}(\mathbf{x}_a, t). \tag{8.29}$$

We expanded the square root previously keeping only the first order term in $(v/c)^2$. Now we will do one more. Recall that our fractional binomial series expansion is

$$(1 + x)^n = 1 + \frac{n}{1!} x + + \frac{n(n-1)}{2!} x^2 + + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots \tag{8.30}$$

so the square root in the Lagrangian expands as

$$\begin{aligned} -m_a c^2 &\sqrt{1 - \frac{\mathbf{v}_a^2}{c^2}} \\ &= -m_a c^2 \left(1 + \frac{1}{2^1 1!} \left(-\frac{\mathbf{v}_a^2}{c^2} \right) + \frac{1(-1)}{2^2 2!} \left(-\frac{\mathbf{v}_a^2}{c^2} \right)^2 + \frac{1(-1)(-3)}{2^3 3!} \left(-\frac{\mathbf{v}_a^2}{c^2} \right)^3 + \cdots \right) \\ &= -m_a c^2 + m_a \frac{\mathbf{v}_a^2}{2} + m_a \frac{\mathbf{v}_a^4}{8 c^2} + \cdots \end{aligned} \tag{8.31}$$

Thus to the next order the single particle Lagrangian is

$$\mathcal{L}_a = \frac{1}{2} m_a \mathbf{v}_a^2 + \frac{m_a}{8} \frac{\mathbf{v}_a^4}{c^2} - q_a \phi(\mathbf{x}_a, t) + q_a \frac{\mathbf{v}_a}{c} \cdot \mathbf{A}(\mathbf{x}_a, t). \tag{8.32}$$

Goal: Calculate $\phi(\mathbf{x}_a)$, $\mathbf{A}(\mathbf{x}_a)$ due to all other particles in a $\mathbf{v}/c$ expansion.

We write

$$\phi(\mathbf{x}_a, t) = \phi^{(0)}(\mathbf{x}_a, t) + \phi^{(1)}(\mathbf{x}_a, t) + \phi^{(2)}(\mathbf{x}_a, t). \tag{8.33}$$

Last time we found that the zeroth order term in this approximation was

$$\phi^{(0)}(\mathbf{x}_a, t) = \sum_{b \neq a} \frac{q_b}{|\mathbf{x}_a(t) - \mathbf{x}_b(t)|}, \tag{8.34}$$

and we wish to calculate the next term in the expansion.

We also want to a first order approximation of the vector potential

$$\mathbf{A}(\mathbf{x}_a, t) = \cancel{\mathbf{A}^{(0)}(\mathbf{x}_a, t)} + \mathbf{A}^{(1)}(\mathbf{x}_a, t) + \cancel{\mathbf{A}^{(2)}(\mathbf{x}_a, t)} \tag{8.35}$$

There is no zero order term and we do not need the second order term (today).

Because

$$\Box \mathbf{A} \sim \frac{\rho \mathbf{v}}{c} \tag{8.36}$$

We know the charge and current distributions

$$\phi(\mathbf{x}, t) = \int d^3\mathbf{x} \frac{\rho\left(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c\right)}{|\mathbf{x} - \mathbf{x}'|} \tag{8.37}$$

$$\begin{aligned} \rho(\mathbf{x}, t) &= \sum_b q_b \delta^3(\mathbf{x} - \mathbf{x}_b(t)) \\ \mathbf{j}(\mathbf{x}, t) &= \sum_b q_b \mathbf{v}_b(t) \delta^3(\mathbf{x} - \mathbf{x}_b(t)) \end{aligned} \tag{8.38}$$

We will use the fact that particles have $v \ll c$. The typical time where the charge distribution will change significantly is of order $\frac{r_{ab}}{v} \gg \frac{r_{ab}}{c}$. (Here $r_a b/c$ is the time that it takes light to cross the interval, whereas $r_a b/v$ is the time that it takes the particle to do the same).

In other words, in time $|\mathbf{x} - \mathbf{x}'|/c \sim r_{ab}/c$, ρ will not change much.

$$\rho\left(\mathbf{x}', t - \left|\mathbf{x} - \mathbf{x}'\right|/c\right) \approx \rho(\mathbf{x}', t) - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \frac{\partial}{\partial t} \rho(\mathbf{x}', t) + \frac{1}{2} \left(\frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)^2 \frac{\partial^2}{\partial t^2} \rho(\mathbf{x}', t) \tag{8.39}$$

$$\phi(\mathbf{x}, t) = \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} - \frac{\partial}{\partial t} \int d^3\mathbf{x}' \frac{1}{c} \rho(\mathbf{x}', t) + \frac{1}{2c^2} \int d^3\mathbf{x} \left|\mathbf{x} - \mathbf{x}'\right| \frac{\partial^2}{\partial t^2} \rho(\mathbf{x}', t) \tag{8.40}$$

The second integral is the total charge $\times 1/c$, and does not change in time. So to first order our charge density is

$$\rho\left(\mathbf{x}', t - \left|\mathbf{x} - \mathbf{x}'\right|/c\right) \approx \rho(\mathbf{x}', t) = \sum_b \frac{q_b}{|\mathbf{x} - \mathbf{x}_b(t)|} \tag{8.41}$$

How about $\mathbf{A}$?

$$\mathbf{A}(\mathbf{x}_a, t) = \cancel{\mathbf{A}^{(0)}(\mathbf{x}_a, t)} + \mathbf{A}^{(1)}(\mathbf{x}_a, t) + \cancel{\mathbf{A}^{(2)}(\mathbf{x}_a, t)} \tag{8.42}$$

$$\begin{aligned} A^{(1)} &= \frac{1}{c} \int d^3\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \mathbf{j}\left(\mathbf{x}', t - \left|\mathbf{x} - \mathbf{x}'\right|/c\right) \\ &\approx \frac{1}{c} \int d^3\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \mathbf{j}(\mathbf{x}', t) \end{aligned} \tag{8.43}$$

Ah, this shows why it was written that there is no second order term. Because $\mathbf{j} \sim \mathbf{v}_a$, we necessarily have $\mathbf{v}_a/c$ dependence even in the zeroth order expansion about $t = 0$ in our retarded time expansion of $\mathbf{A}(\mathbf{x}', t_r)$.

Assembling all the results, we have

$$\mathcal{L}_a = \frac{1}{2} m_a \mathbf{v}_a^2 + \frac{m_a}{8} \frac{\mathbf{v}_a^4}{c^2} - q_a \phi^{(0)}(\mathbf{x}_a, t) - q_a \phi^{(2)}(\mathbf{x}_a, t) + q_a \frac{\mathbf{v}_a}{c} \cdot \mathbf{A}^{(1)}(\mathbf{x}_a, t) \tag{8.44}$$

$$\begin{aligned} \phi^{(2)}(\mathbf{x}, t) &= \frac{\partial}{\partial t} \left(\frac{1}{2c^2} \frac{\partial}{\partial t} \int d^3\mathbf{x}' \left|\mathbf{x} - \mathbf{x}'\right| \rho(\mathbf{x}', t) \right) \\ &= \frac{\partial}{\partial t} \left(\frac{1}{2c^2} \frac{\partial}{\partial t} \int d^3\mathbf{x}' \left|\mathbf{x} - \mathbf{x}'\right| \sum_b q_b \delta^3(\mathbf{x} - \mathbf{x}_b(t)) \right) \\ &= \frac{\partial}{\partial t} \left(\frac{1}{2c^2} \frac{\partial}{\partial t} \sum_b q_b |\mathbf{x} - \mathbf{x}_b(t)| \right) \end{aligned} \tag{8.45}$$

And

$$\begin{aligned} \mathbf{A}^{(1)}(\mathbf{x},t) &= \frac{1}{c}\int d^3\mathbf{x}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{j}(\mathbf{x},t) \\ &= \frac{1}{c}\int d^3\mathbf{x}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\sum_b q_b\mathbf{v}_b\delta^3(\mathbf{x}-\mathbf{x}_b) \\ &= \frac{1}{c}\sum_b q_b\mathbf{v}_b\frac{1}{|\mathbf{x}-\mathbf{x}_b|} \end{aligned} \tag{8.46}$$

Recall that $\phi^{(0)}$ was given by eq. (8.34).

8.6 A GAUGE TRANSFORMATION TO SIMPLIFY THINGS

Remember: Gauge transformation

$$\begin{aligned} \phi'(\mathbf{x},t) &= \phi(\mathbf{x},t) - \frac{1}{c}\frac{\partial f(\mathbf{x},t)}{\partial t} \\ \mathbf{A}'(\mathbf{x},t) &= \mathbf{A}(\mathbf{x},t) + \boldsymbol{\nabla} f(\mathbf{x},t) \end{aligned} \tag{8.47}$$

This will not change the physics. Take

$$f(\mathbf{x},t) = \sum_b \frac{q_b}{2c}\frac{\partial}{\partial t}|\mathbf{x}-\mathbf{x}_b(t)| \tag{8.48}$$

Then

$$\phi'^{(2)} = 0 \tag{8.49}$$

$$\mathbf{A}'^{(1)}(\mathbf{x},t) = \frac{1}{c}\sum_b \frac{q_b\mathbf{v}_b}{|\mathbf{x}-\mathbf{x}_b|} + \boldsymbol{\nabla}\sum_b \frac{q_b}{2c}\frac{\partial}{\partial t}|\mathbf{x}-\mathbf{x}_b|. \tag{8.50}$$

Inverting the order of time and space derivatives we find

$$\begin{aligned}
\boldsymbol{\nabla}\frac{\partial}{\partial t}|\mathbf{x}-\mathbf{x}_b(t)| &= \frac{\partial}{\partial t}\boldsymbol{\nabla}|\mathbf{x}-\mathbf{x}_b(t)| \\
&= \frac{\partial}{\partial t}e_\alpha\partial_\alpha((x^\beta - x_b^\beta(t))^2)^{1/2} \\
&= \frac{\partial}{\partial t}e_\alpha\frac{(x^\beta - x_b^\beta(t))\partial_\alpha(x^\beta - x_b^\beta(t))}{|\mathbf{x}-\mathbf{x}_b(t)|} \\
&= \frac{\partial}{\partial t}e_\alpha\frac{(x^\beta - x_b^\beta(t))\delta_\alpha^\beta}{|\mathbf{x}-\mathbf{x}_b(t)|} \\
&= \frac{\partial}{\partial t}\frac{\mathbf{x}-\mathbf{x}_b(t)}{|\mathbf{x}-\mathbf{x}_b(t)|}.
\end{aligned} \tag{8.51}$$

Let us write

$$\mathbf{n} \equiv \frac{\mathbf{x}-\mathbf{x}_b(t)}{|\mathbf{x}-\mathbf{x}_b|}, \tag{8.52}$$

for the unit vector in the direction pointing from $\mathbf{x}_b$ to $\mathbf{x}$. Evaluating the time derivative, we have

$$\begin{aligned}
\dot{\mathbf{n}} &= \frac{-\mathbf{v}_b(t)}{|\mathbf{x}-\mathbf{x}_b(t)|} + (\mathbf{x}-\mathbf{x}_b(t))\frac{\partial}{\partial t}\frac{1}{|\mathbf{x}-\mathbf{x}_b(t)|} \\
&= \frac{-\mathbf{v}_b(t)}{|\mathbf{x}-\mathbf{x}_b(t)|} + (\mathbf{x}-\mathbf{x}_b(t))\left(-\frac{1}{\not{2}}\right)\frac{\not{2}(x^\alpha - x_b^\alpha(t)(-v_b^\alpha(t))}{|\mathbf{x}-\mathbf{x}_b(t)|^3} \\
&= \frac{-\mathbf{v}_b(t)}{|\mathbf{x}-\mathbf{x}_b(t)|} + \frac{\mathbf{n}(\mathbf{n}\cdot\mathbf{v}_b)}{|\mathbf{x}-\mathbf{x}_b(t)|}.
\end{aligned} \tag{8.53}$$

Assembling all the results we have

$$\mathbf{A}'^{(1)}(\mathbf{x}, t) = \sum_b q_b\frac{\mathbf{v}_b + \mathbf{n}(\mathbf{n}\cdot\mathbf{v}_b)}{2c|\mathbf{x}-\mathbf{x}_b|}, \tag{8.54}$$

and the Lagrangian for our particle after the gauge transformation is

$$\mathcal{L}_a = \frac{1}{2}m_a\mathbf{v}_a^2 + \frac{m_a}{8}\frac{\mathbf{v}_a^4}{c^2} - \sum_{b\neq a}\frac{q_a q_b}{|\mathbf{x}_a(t)-\mathbf{x}_b(t)|} + \sum_b q_a q_b\frac{\mathbf{v}_a\cdot\mathbf{v}_b + (\mathbf{n}\cdot\mathbf{v}_a)(\mathbf{n}\cdot\mathbf{v}_b)}{2c^2|\mathbf{x}-\mathbf{x}_b|}. \tag{8.55}$$

Next time we will probably get to the Lagrangian for the entire system. It was hinted that this is called the Darwin Lagrangian (after Charles Darwin's grandson).

Reading Covering chapter 8 §65 material from the text [11], and lecture notes RelEMpp181-195.pdf.

Next week (last topic): attempt to go to the next order $(v/c)^3$ - radiation damping, the limitations of classical electrodynamics, and the relevant time/length/energy scales.

8.7 RECAP

A system of N charged particles $m_a, q_a; a \in [1, N]$ closed system and nonrelativistic, $v_a/c \ll 1$. In this case we can incorporate EM effects in a Lagrangian ONLY involving particles (EM field not a dynamical DOF). In general case, this works to $O((v/c)^2)$, because at $O((v/c))$ system radiation effects occur.

In a specific case, when

$$\frac{m_1}{q_1} = \frac{m_2}{q_2} = \frac{m_3}{q_3} = \cdots \tag{8.56}$$

we can do that (meaning use a Lagrangian with particles only) to $O((v/c)^4)$ because of specific symmetries in such a system.

The Lagrangian for our particle after the gauge transformation is

$$\mathcal{L}_a = \frac{1}{2} m_a \mathbf{v}_a^2 + \frac{m_a}{8} \frac{\mathbf{v}_a^4}{c^2} - \sum_{b \neq a} \frac{q_a q_b}{|\mathbf{x}_a(t) - \mathbf{x}_b(t)|} + \sum_b q_a q_b \frac{\mathbf{v}_a \cdot \mathbf{v}_b + (\mathbf{n} \cdot \mathbf{v}_a)(\mathbf{n} \cdot \mathbf{v}_b)}{2c^2 |\mathbf{x} - \mathbf{x}_b|}. \tag{8.57}$$

Next time we will probably get to the Lagrangian for the entire system. It was hinted that this is called the Darwin Lagrangian (after Charles Darwin's grandson).

We find for whole system

$$\mathcal{L} = \sum_a \mathcal{L}_a + \frac{1}{2} \sum_a \mathcal{L}_a(interaction) \tag{8.58}$$

$$\mathcal{L} = \frac{1}{2} \sum_a m_a \mathbf{v}_a^2 + \sum_a \frac{m_a}{8} \frac{\mathbf{v}_a^4}{c^2} - \sum_{a<b} \frac{q_a q_b}{|\mathbf{x}_a(t) - \mathbf{x}_b(t)|} + \sum_b q_a q_b \frac{\mathbf{v}_a \cdot \mathbf{v}_b + (\mathbf{n} \cdot \mathbf{v}_a)(\mathbf{n} \cdot \mathbf{v}_b)}{2c^2 |\mathbf{x} - \mathbf{x}_b|}. \tag{8.59}$$

This is the Darwin Lagrangian (also Charles). The Darwin Hamiltonian, from $H = \sum_a q_a p_a - \mathcal{L}$, which toggles the sign on all but the first term, is

$$\begin{aligned} H &= \sum_a \frac{p_a}{2m_a} \mathbf{v}_a^2 - \sum_a \frac{p_a^4}{8 m_a^3 c^2} \\ &+ \sum_{a<b} \frac{q_a q_b}{|\mathbf{x}_a(t) - \mathbf{x}_b(t)|} - \sum_{a<b} \frac{q_a q_b}{2c^2 m_a m_b} \frac{\mathbf{p}_a \cdot \mathbf{p}_b + (\mathbf{n}_{ab} \cdot \mathbf{p}_a)(\mathbf{n}_{ab} \cdot \mathbf{p}_b)}{|\mathbf{x} - \mathbf{x}_b|}. \end{aligned} \tag{8.60}$$

(note, this is also the result to be obtained in problem 2, §65 of the text.)

8.8 INCORPORATING RADIATION EFFECTS AS A FRICTION TERM

To $O((v/c)^3)$ obvious problem due to radiation (system not closed). We will incorporate radiation via a function term in the EOM

Again consider the dipole system

$$\begin{aligned} m\ddot{z} &= -kz \\ \omega^2 &= \frac{k}{m} \end{aligned} \tag{8.61}$$

or

$$m\ddot{z} = -\omega^2 mz \tag{8.62}$$

gives

$$\frac{d}{dt}\left(\frac{m}{2}\dot{z}^2 + \frac{m\omega^2}{2}z^2\right) = 0 \tag{8.63}$$

(because there is no radiation).

The energy radiated per unit time averaged per period is

$$P = \frac{2e^2}{3c^3}\left\langle \dddot{z}^2 \right\rangle \tag{8.64}$$

We will modify the EOM

$$m\ddot{z} = -\omega^2 mz + f_{\text{radiation}} \tag{8.65}$$

Employing an integration factor $\dot{z}$ we have

$$m\ddot{z}\dot{z} = -\omega^2 mz\dot{z} + f_{\text{radiation}}\dot{z} \tag{8.66}$$

or

$$\frac{d}{dt}\left(m\dot{z}^2 + \omega^2 mz^2\right) = f_{\text{radiation}}\dot{z} \tag{8.67}$$

Observe that the last expression, force times velocity, has the form of power

$$m\frac{d^2 z}{dt^2}\frac{dz}{dt} = \frac{d}{dt}\left(\frac{m}{2}\left(\frac{dz}{dt}\right)^2\right) \tag{8.68}$$

So we can make an identification with the time rate of energy lost by the system due to radiation

$$\frac{d}{dt}\left(m\dot{z}^2 + \omega^2 m z^2\right) \equiv \frac{d\mathcal{E}}{dt}. \tag{8.69}$$

Average over period both sides

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = \langle f_{\text{radiation}} \dot{z} \rangle = -\frac{2e^2}{3c^3}\left\langle \ddot{z}^2 \right\rangle \tag{8.70}$$

We demand this last equality, by requiring the energy change rate to equal that of the dipole power (but negative since it is a loss) that we previously calculated.

Claim:

$$f_{\text{radiation}} = \frac{2e^2}{3c^3}\dddot{z} \tag{8.71}$$

Proof: We need to show

$$\langle f_{\text{radiation}} \rangle = -\frac{2e^2}{3c^3}\left\langle \ddot{z}^2 \right\rangle \tag{8.72}$$

We have

$$\begin{aligned}\frac{2e^2}{3c^3}\langle \dddot{z}\dot{z} \rangle &= \frac{2e^2}{3c^3}\frac{1}{T}\int_0^T dt \dddot{z}\dot{z} \\ &= \frac{2e^2}{3c^3}\frac{1}{T}\int_0^T dt \cancel{\frac{d}{dt}(\ddot{z}\dot{z})} - \frac{2e^2}{3c^3}\frac{1}{T}\int_0^T dt(\ddot{z})^2\end{aligned} \tag{8.73}$$

We first used $(\ddot{z}\dot{z})' = \dddot{z}\dot{z} + (\ddot{z})^2$. The first integral above is zero since the derivative of $\ddot{z}\dot{z} = (-\omega^2 z_0 \sin \omega t)(\omega z_0 \cos \omega t) = -\omega^3 z_0^2 \sin(2\omega t)/2$ is also periodic, and vanishes when integrated over the interval.

$$\frac{2e^2}{3c^3}\langle \dddot{z}\dot{z} \rangle = -\frac{2e^2}{3c^3}\left\langle (\ddot{z})^2 \right\rangle \tag{8.74}$$

We can therefore write

$$m\ddot{z} = -m\omega^2 z + \frac{2e^2}{3c^3}\dddot{z} \tag{8.75}$$

Our "frictional" correction is the radiation reaction force proportional to the third derivative of the position.

Rearranging slightly, this is

$$\ddot{z} = -\omega^2 z + \frac{2}{3c}\left(\frac{e^2}{mc^2}\right)\dddot{z} = -\omega^2 z + \frac{2}{3c}\frac{r_e}{c}\dddot{z}, \tag{8.76}$$

where $r_e \sim 10^{-13}$cm is the "classical radius" of the electron. In our frictional term we have r_e/c, the time for light to cross the classical radius of the electron.

There are lots of problems with this. One of the easiest is with $\omega = 0$. Then we have

$$\ddot{z} = \frac{2}{3}\frac{r_e}{c}\dddot{z} \tag{8.77}$$

with solution

$$z \sim e^{\alpha t}, \tag{8.78}$$

where

$$\alpha \sim \frac{c}{r_e} \sim \frac{1}{\tau_e}. \tag{8.79}$$

This is a self accelerating system! Note that we can also get into this trouble with $\omega \neq 0$, but those examples are harder to find (see: [4]).

FIXME: borrow this text again to give that section a read.

The sensible point of view is that this third term (f_{rad}) should be taken seriously only if it is small compared to the first two terms.

Reading Some of this, at least the second order expansion, was covered in chapter 8 §65 material from the text [11].

Covering lecture notes RelEMpp181-195.pdf.

8.9 RADIATION REACTION FORCE

We previously obtained the radiation reaction force by adding a "frictional" force to the harmonic oscillator system. Now its time to obtain this by continuing the expansion of the potentials to the next order in $\mathbf{v}/c$.

Recall that our potentials are

$$\begin{aligned}\phi(\mathbf{x}, t) &= \int d^3\mathbf{x}' \frac{\rho\left(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c\right)}{|\mathbf{x} - \mathbf{x}'|} \\ \mathbf{A}(\mathbf{x}, t) &= \frac{1}{c} \int d^3\mathbf{x}' \frac{\mathbf{j}\left(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/c\right)}{|\mathbf{x} - \mathbf{x}'|}.\end{aligned} \tag{8.80}$$

We can expand in Taylor series about t. For the charge density this is

$$\begin{aligned}&\rho\left(\mathbf{x}', t - \left|\mathbf{x} - \mathbf{x}'\right|/c\right) \\ &\quad\approx \rho(\mathbf{x}', t) - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \frac{\partial}{\partial t}\rho(\mathbf{x}', t) \\ &\quad+ \frac{1}{2}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)^2 \frac{\partial^2}{\partial t^2}\rho(\mathbf{x}', t) - \frac{1}{6}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)^3 \frac{\partial^3}{\partial t^3}\rho(\mathbf{x}', t)\end{aligned}, \tag{8.81}$$

so that our scalar potential to third order is

$$\begin{aligned}\phi(\mathbf{x}, t) &= \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \frac{\partial}{\partial t} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \\ &\quad+ \frac{1}{2}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)^2 \frac{\partial^2}{\partial t^2} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{6}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)^3 \frac{\partial^3}{\partial t^3} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \\ &= \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} - \cancel{\frac{\partial}{\partial t} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \frac{|\mathbf{x} - \mathbf{x}'|}{c}} \\ &\quad+ \frac{1}{2}\frac{\partial^2}{\partial t^2} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)^2 - \frac{1}{6}\frac{\partial^3}{\partial t^3} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}\left(\frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)^3 \\ &= \phi^{(0)} + \phi^{(2)} + \phi^{(3)}\end{aligned} \tag{8.82}$$

Expanding the vector potential in Taylor series to second order we have

$$\begin{aligned}\mathbf{A}(\mathbf{x}, t) &= \frac{1}{c} \int d^3\mathbf{x}' \frac{\mathbf{j}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{c}\frac{|\mathbf{x} - \mathbf{x}'|}{c} \frac{\partial}{\partial t} \int d^3\mathbf{x}' \frac{\mathbf{j}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \\ &= \frac{1}{c} \int d^3\mathbf{x}' \frac{\mathbf{j}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{c^2}\frac{\partial}{\partial t} \int d^3\mathbf{x}' \mathbf{j}(\mathbf{x}', t) \\ &= \mathbf{A}^{(1)} + \mathbf{A}^{(2)}\end{aligned} \tag{8.83}$$

We have already considered the effects of the $\mathbf{A}^{(1)}$ term, and now move on to $\mathbf{A}^{(2)}$. We will write $\phi^{(3)}$ as a total derivative

$$\phi^{(3)} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int d^3\mathbf{x}'\rho(\mathbf{x}',t)\left|\mathbf{x}-\mathbf{x}'\right|^2\right) = \frac{1}{c}\frac{\partial}{\partial t}f^{(2)}(\mathbf{x},t) \tag{8.84}$$

and gauge transform it away as we did with $\phi^{(2)}$ previously.

$$\begin{aligned}\phi^{(3)\prime} &= \phi^{(3)} - \frac{1}{c}\frac{\partial f^{(2)}}{\partial t} = 0\\ \mathbf{A}^{(2)\prime} &= \mathbf{A}^{(2)} + \boldsymbol{\nabla} f^{(2)}\end{aligned} \tag{8.85}$$

$$\mathbf{A}^{(2)\prime} = -\frac{1}{c^2}\frac{\partial}{\partial t}\int d^3\mathbf{x}'\mathbf{j}(\mathbf{x}',t) - \frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int d^3\mathbf{x}'\rho(\mathbf{x}',t)\boldsymbol{\nabla}_{\mathbf{x}}\left|\mathbf{x}-\mathbf{x}'\right|^2 \tag{8.86}$$

Looking first at the first integral we can employ the trick of writing $\mathbf{e}_\alpha = \partial\mathbf{x}'/\partial x^{\alpha'}$, and then employ integration by parts

$$\begin{aligned}\int_V d^3\mathbf{x}'\mathbf{j}(\mathbf{x}',t) &= \int_V d^3\mathbf{x}'\mathbf{e}_\alpha j^\alpha(\mathbf{x}',t)\\ &= \int_V d^3\mathbf{x}'\frac{\partial\mathbf{x}'}{\partial x^{\alpha'}} j^\alpha(\mathbf{x}',t)\\ &= \int_V d^3\mathbf{x}'\frac{\partial}{\partial x^{\alpha'}}\left(\mathbf{x}' j^\alpha(\mathbf{x}',t)\right) - \int_V d^3\mathbf{x}'\mathbf{x}'\frac{\partial}{\partial x^{\alpha'}} j^\alpha(\mathbf{x}',t)\\ &= \int_{\partial V} d^2\boldsymbol{\sigma}\cdot\left(\mathbf{x}' j^\alpha(\mathbf{x}',t)\right) - \int d^3\mathbf{x}'\mathbf{x}' - \frac{\partial}{\partial t}\rho(\mathbf{x}',t)\\ &= \frac{\partial}{\partial t}\int d^3\mathbf{x}'\mathbf{x}'\rho(\mathbf{x}',t)\end{aligned} \tag{8.87}$$

For the second integral, we have

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{x}}\left|\mathbf{x}-\mathbf{x}'\right|^2 &= \mathbf{e}_\alpha\partial_\alpha(x^\beta - x^{\beta'})(x^\beta - x^{\beta'})\\ &= 2\mathbf{e}_\alpha\delta_{\alpha\beta}(x^\beta - x^{\beta'})\\ &= 2(\mathbf{x}-\mathbf{x}'),\end{aligned} \tag{8.88}$$

so our gauge transformed vector potential term is reduced to

$$\begin{aligned}\mathbf{A}^{(2)\prime} &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int d^3\mathbf{x}'\rho(\mathbf{x}',t)\left(\mathbf{x}' + \frac{1}{3}(\mathbf{x}-\mathbf{x}')\right)\\ &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int d^3\mathbf{x}'\rho(\mathbf{x}',t)\left(\frac{1}{3}\mathbf{x} + \frac{2}{3}\mathbf{x}'\right)\end{aligned} \tag{8.89}$$

Now we wish to employ a discrete representation of the charge density

$$\rho(\mathbf{x}', t) = \sum_{b=1}^{N} q_b \delta^3(\mathbf{x}' - \mathbf{x}_b(t)) \tag{8.90}$$

So that the second order vector potential becomes

$$\begin{aligned}
\mathbf{A}^{(2)'} &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int d^3\mathbf{x}'\left(\frac{1}{3}\mathbf{x} + \frac{2}{3}\mathbf{x}'\right)\sum_{b=1}^{N} q_b \delta^3(\mathbf{x}' - \mathbf{x}_b(t)) \\
&= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\sum_{b=1}^{N} q_b \left(\cancel{\frac{1}{3}\mathbf{x}} + \frac{2}{3}\mathbf{x}_b(t)\right) \\
&= -\frac{2}{3c^2}\sum_{b=1}^{N} q_b \ddot{\mathbf{x}}_b(t) \\
&= -\frac{2}{3c^2}\frac{d^2}{dt^2}\left(\sum_{b=1}^{N} q_b \mathbf{x}_b(t)\right).
\end{aligned} \tag{8.91}$$

We end up with a dipole moment

$$\mathbf{d}(t) = \sum_{b=1}^{N} q_b \mathbf{x}_b(t) \tag{8.92}$$

so we can write

$$\mathbf{A}^{(2)'} = -\frac{2}{3c^2}\ddot{\mathbf{d}}(t). \tag{8.93}$$

Observe that there is no magnetic field due to this contribution since there is no explicit spatial dependence

$$\boldsymbol{\nabla} \times \mathbf{A}^{(2)'} = 0 \tag{8.94}$$

we have also gauge transformed away the scalar potential contribution so have only the time derivative contribution to the electric field

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \cancel{\boldsymbol{\nabla}\phi} = \frac{2}{3c^2}\dddot{\mathbf{d}}(t). \tag{8.95}$$

To $O((v/c)^3)$ there is a homogeneous electric field felt by all particles, hence every particle feels a "friction" force

$$\mathbf{f}_{\text{rad}} = q\mathbf{E} = \frac{2q}{3c^3}\dddot{\mathbf{d}}(t). \tag{8.96}$$

Moral: $\mathbf{f}_{\text{rad}}$ arises in third order term $O((v/c)^3)$ expansion and thus should not be given a weight as important as the two other terms. i.e. Its consequences are less.

Example 8.1: Our dipole system

$$\begin{aligned} m\ddot{z} &= -m\omega^2 a + \frac{2e^2}{3c^3}\dddot{z} \\ &= -m\omega^2 a + \frac{2m}{3c}\frac{e^2}{mc^2}\dddot{z} \\ &= -m\omega^2 a + \frac{2m}{3}\frac{r_e}{c}\dddot{z} \end{aligned} \tag{8.97}$$

Here $r_e \sim 10^{-13}$cm is the classical radius of the electron. For periodic motion

$$\begin{aligned} z &\sim e^{i\omega t} z_0 \\ \ddot{z} &\sim \omega^2 z_0 \\ \dddot{z} &\sim \omega^3 z_0. \end{aligned} \tag{8.98}$$

The ratio of the last term to the inertial term is

$$\sim \frac{\omega^3 m (r_e/c) z_0}{m\omega^2 z_0} \sim \omega \frac{r_e}{c} \ll 1, \tag{8.99}$$

so

$$\begin{aligned} \omega &\ll \frac{c}{r_e} \\ &\sim \frac{1}{\tau_e} \\ &\sim \frac{10^{10}\text{cm/s}}{10^{-13}\text{cm}} \\ &\sim 10^{23}\text{Hz} \end{aligned} \tag{8.100}$$

So long as $\omega \ll 10^{23}$Hz, this approximation is valid.

8.10 LIMITS OF CLASSICAL ELECTRODYNAMICS

What sort of energy is this? At these frequencies QM effects come in

$$\hbar \sim 10^{-33}\mathrm{J}\cdot\mathrm{s} \sim 10^{-15}\mathrm{eV}\cdot\mathrm{s} \tag{8.101}$$

$$\hbar\omega_{max} \sim 10^{-15}\mathrm{eV}\cdot\mathrm{s}\times 10^{23}\frac{1}{\mathrm{s}} \sim 10^{8}\mathrm{eV} \sim 100\mathrm{MeV} \tag{8.102}$$

whereas the rest energy of the electron is

$$m_e c^2 \sim \frac{1}{2}\mathrm{MeV} \sim \mathrm{MeV}. \tag{8.103}$$

At these frequencies it is possible to create e^+ and e^- pairs. A theory where the number of particles (electrons and positrons) is NOT fixed anymore is required. An estimate of this frequency, where these effects have to be considered is possible.

PICTURE: different length scales with frequency increasing to the left and length scales increasing to the right.

- 10^{-13}cm, $r_e = e^2/mc^2$. LHC exploration.
- 137×10^{-13}cm, $\hbar/m_e c \sim \lambda/2\pi$, the Compton wavelength of the electron. QED and quantum field theory.
- $(137)^2 \times 10^{-13}$cm $\sim 10^{-10}$cm, Bohr radius. QM, and classical electrodynamics.

here

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137}, \tag{8.104}$$

is the fine structure constant.

Similar to the distance scale restrictions, we have field strength restrictions. A strong enough field (Electric) can start creating electron and positron pairs. This occurs at about

$$eE\lambda/2\pi \sim 2m_e c^2 \tag{8.105}$$

so the critical field strength is

$$\begin{aligned} E_{\text{crit}} &\sim \frac{m_e c^2}{\lambda/2\pi e} \\ &\sim \frac{m_e c^2}{\hbar e} m_e c \\ &\sim \frac{m_e^2 c^3}{\hbar e} \end{aligned} \tag{8.106}$$

Is this real? Yes, with a very heavy nucleus with some electrons stripped off, the field can be so strong that positron and electron pairs will be created. This can be observed in heavy ion collisions!

Part II

APPENDICES

A PROFESSOR POPPITZ'S HANDOUTS

The current path for Prof. Poppitz's handouts is http://www.physics.utoronto.ca/ poppitz/poppitz/PHY450.html. These were a valuable resource when I took the course. At the time of this writing (accessed Dec, 2014) the following files were available. The descriptions are from the 2011 versions of the files and may no longer match exactly.

- RelEM1-11.pdf

  space, time and Gallilean relativity (1-6); speed of light and Einsteins relativity principle (7-10); relativity of simultaneity (11).

- RelEM12-26.pdf

  spacetime, spacetime points, worldlines, interval (12-14) ; invariance of infinitesimal intervals (15-17); geometry of spacetime, lightlike, spacelike, timelike intervals, and worldlines (18-22); proper time (23-24); invariance of finite intervals (25-26).

- RelEM27-44.pdf

  analogy with rotations and derivation of Lorentz transformations (27-32); Minkowski space diagram of boosted frame (32.1) ; Using Minkowski diagram to see the perils of superluminal propagation (32.3); nonrelativistic limit of boosts (33); number of parameters of Lorentz transformations (34-35); introducing four-vectors, the metric tensor, the invariant "dot-product and SO(1,3) (36-40); the Poincare group (41); the convenience of "upper" and "lower"indices (42-43); tensors (44)

- RelEMpp52-56.pdf

  equation of motion, symmetries, and conserved quantities (energy-momentum 4 vector) from relativistic particle action.

- RelEMpp56.1-73.pdf

  comments on mass, energy, momentum, and massless particles (56.1-58); particles in external fields: Lorentz scalar field (59-62); reminder of a vector field under spatial rotations (63) and a Lorentz vector field (64-65) [Tuesday, Feb. 1]; the action for a relativistic particle in an external 4-vector field (65-66); the equation of motion of a relativistic particle in an external electromagnetic (4-vector) field (67,68,73) [Wednesday, Feb. 2]; mathematical interlude: (69-72): on 3x3 antisymmetric matrices, 3-vectors, and totally antisymmetric 3-index tensor - please read by yourselves, preferably by Wed., Feb. 2 class! (this is important, we will also soon need the 4-dimensional generalization)

- RelEMpp74-83.pdf

  gauge transformations in 3-vector language (74); energy of a relativistic particle in EM field (75); variational principle and equation of motion in 4-vector form (76-77); the field strength tensor (78-80); the fourth equation of motion (81) ; Lorentz transformation of the strength tensor (82) ; extra reading for the mathematically minded: gauge field, strength tensor, and gauge transformations in differential form language, not to be covered in class (83)

- RelEMpp84-102.pdf

  relativity, gauge invariance, and superposition principles and the action for the electromagnetic field coupled to charged particles (91-95); the 4-current and its physical interpretation (96-102), including a needed mathematical interlude on delta-functions of functions (98-100)

- RelEMpp103-113.pdf

  variational principle for the electromagnetic field and the relevant boundary conditions (103-105); the second set of Maxwell's equations from the variational principle (106-108); Maxwell's equations in vacuum and the wave equation in the non-relativistic Coulomb gauge (109-111)

- RelEMpp114-127.pdf

  reminder on wave equations (115); reminder on Fourier series and integral (115-117); Fourier expansion of the EM potential in Coulomb gauge and equation of motion for the spatial Fourier components (118-119); the general solution of Maxwell's equations in vacuum (120-121)

- RelEMpp114-127.pdf

  properties of monochromatic plane EM waves (122-124); energy and energy flux of the EM field and energy conservation from the equations of motion (125-127)

- RelEMpp128-135.pdf

  energy flux and momentum density of the EM wave (128-129); radiation pressure, its discovery and significance in physics (130-131); EM fields of moving charges: setting up the wave equation with a source (132-133); the convenience of Lorentz gauge in the study of radiation (134); reminder on Green's functions from electrostatics (135)

- RelEMpp136-146.pdf

  continued remainder of electrostatic Green's function (136); the retarded Green's function of the d'Alembert operator: derivation and properties (137-140); the solution of the d'Alembert equation with a source: retarded potentials (141-142); retarded time ; the Lienard-Wiechert potentials (143-146)

- RelEMpp147-165.pdf

  EM fields of a moving source (147-148+HW5); a particle at rest (148); a constant velocity particle (149-152); behavior of EM fields "at infinity" for a general-worldline source and radiation (152-153) ; radiated power (154); fields in the "wave zone" and discussions of approximations made (155-159); EM fields due to electric dipole radiation (160-163); Poynting vector, angular distribution, and power of dipole radiation (164-165)

- RelEMpp166-180.pdf

  spacetime translation invariance of the EM field action and the conservation of the energy-momentum tensor (170-172); properties of the energy-momentum tensor (172.1); the meaning of its components: energy ; the force on a surface element of a body (177-178); a plane wave example (179-180).

- RelEMpp181-195.pdf

  the Lagrangian for a system of non relativistic charged particles to zeroth order in (v/c): electrostatic energy of a system of charges and .mass renormalization ; (182-189) the EM potentials to order $(v/c)^2$ (190-193); the "Darwin Lagrangian. and Hamiltonian for a system of non-relativistic charged particles to order $(v/c)^2$ and its many uses in physics (194-195) ; (198.1-200) (last topic): attempt to go to the next order $(v/c)^3$ - radiation damping, the limitations of classical electrodynamics, and the relevant time/length/energy scales.

B

SOME TENSOR AND GEOMETRIC ALGEBRA COMPARISONS IN A SPACETIME CONTEXT

B.1 MOTIVATION

I have an ancient copy of the course text [8] from the library right now (mine is on order still) for my PHY450H1S course (relativistic electrodynamics). Given the transformation rule for a first rank tensor

$$A_i = \alpha_{im} A'_m, \tag{B.1}$$

they list the transformation rule for a second rank tensor as

$$A_{ik} = \alpha_{im} \alpha_{kl} A'_{ml}. \tag{B.2}$$

This is not motivated in any way. Let us compare to transformation of a bivector expressed in the Dirac basis, transformed by outermorphism. That is specifically a transformation of a antisymmetric tensor (once expressed in components anyways), but should provide some intuition.

It is also worthwhile to note that there are some old fashioned notational quirks in this text (at least the old version that I have currently borrowed). Specifically, they uses Latin indices four vectors with Greek indices for three vectors, completely opposite to what appears to be the current conventions. They also do not use upper and lower indices to keep track of bookkeeping. I will use the conventions I am used to for now.

B.2 NOTATION AND USE OF GEOMETRIC ALGEBRA HEREIN

I will use conventions from [1] using the Dirac basis, with a preference for index upper coordinates, and express a vector as

$$x = x^\alpha \gamma_\alpha = x_\alpha \gamma^\alpha, \tag{B.3}$$

Here the basis pairs $\{\gamma_\mu\}$ and $\{\gamma^\mu\}$ are reciprocal frames with $\gamma^\mu \cdot \gamma_\nu = \delta^\mu{}_\nu$. I will have no need for any specific metric convention here.

The dot and wedge products used will be defined in terms of their Clifford Algebra formulation

$$\begin{aligned} a \cdot b &= \frac{1}{2}(ab + ba) \\ a \wedge b &= \frac{1}{2}(ab - ba). \end{aligned} \tag{B.4}$$

The dot product between two bivectors A, B will also be used, defined as the scalar part of the product AB. In particular the identity for extraction of that scalar component from the dot product of two wedge products will be required

$$(a \wedge b) \cdot (c \wedge d) = (a(b \cdot c) - b(a \cdot c)) \cdot d = (a \cdot d)(b \cdot c) - (b \cdot d)(a \cdot c) \tag{B.5}$$

B.3 TRANSFORMATION OF THE COORDINATES

Let us assume our transformation is linear, and we will denote its action on vectors as follows

$$x' = L(x) = x^\alpha L(\gamma_\alpha). \tag{B.6}$$

Extracting coordinates for the transformed coordinates (assuming a non-moving frame where the unit vectors on both sides are the same), we have after dotting with γ^μ

$$x'^\mu = \left(x'^\alpha \gamma_\alpha\right) \cdot \gamma^\mu = x^\alpha \left(L(\gamma_\alpha) \cdot \gamma^\mu\right) \tag{B.7}$$

Now introduce a coordinate representation for the transformation L

$$L(\gamma_\alpha) \cdot \gamma^\mu = L_\alpha{}^\mu, \tag{B.8}$$

so our transformation rule for the four vector coordinates becomes

$$x'^\mu = x^\alpha L_\alpha{}^\mu. \tag{B.9}$$

We are now ready to look at the transformation of a bivector (a quantity having a rank two antisymmetric tensor representation in coordinates), and see how the coordinates transform.

Let us transform by outermorphism of the transformed vector factors the bivector

$$c = a \wedge b \rightarrow a' \wedge b'. \tag{B.10}$$

First we will need the coordinate representation of the bivector before transformation. We dot with $\gamma^\nu \wedge \gamma^\mu$ to pick up the desired term

$$\begin{aligned}
(a \wedge b) \cdot (\gamma_\nu \wedge \gamma_\mu) &= a^\alpha b^\beta (\gamma_\alpha \wedge \gamma_\beta) \cdot (\gamma^\nu \wedge \gamma^\mu) \\
&= a^\alpha b^\beta (\gamma_\alpha \delta_\beta{}^\nu - \gamma_\beta \delta_\alpha{}^\nu) \cdot \gamma^\mu \\
&= a^\alpha b^\beta (\delta_\alpha{}^\mu \delta_\beta{}^\nu - \delta_\beta{}^\mu \delta_\alpha{}^\nu) \\
&= a^\mu b^\nu - a^\nu b^\mu
\end{aligned} \tag{B.11}$$

If we introduce a rank two tensor now, say

$$T^{\mu\nu} = a^\mu b^\nu - a^\nu b^\mu, \tag{B.12}$$

we recover our bivector with

$$a \wedge b = \frac{1}{2} T^{\alpha\beta} \gamma_\alpha \wedge \gamma_\beta. \tag{B.13}$$

Now let us look at the coordinate representation of the transformed bivector. It will also be helpful to make use of the identity that can be observed above from the initial coordinate extraction

$$(\gamma_\alpha \wedge \gamma_\beta) \cdot (\gamma^\nu \wedge \gamma^\mu) = {\delta_\alpha}^\mu {\delta_\beta}^\nu - {\delta_\beta}^\mu {\delta_\alpha}^\nu \tag{B.14}$$

In coordinates our transformed bivector is

$$a' \wedge b' = a^\sigma {L_\sigma}^\alpha b^\pi {L_\pi}^\beta \gamma_\alpha \wedge \gamma_\beta, \tag{B.15}$$

and we can proceed with the coordinate extraction by taking dot products with $\gamma^\nu \wedge \gamma^\mu$ as before. This gives us

$$\begin{aligned}
(a' \wedge b') \cdot (\gamma^\nu \wedge \gamma^\mu) &= a^\sigma {L_\sigma}^\alpha b^\pi {L_\pi}^\beta \gamma_\alpha \wedge \gamma_\beta \\
&= a^\sigma {L_\sigma}^\alpha b^\pi {L_\pi}^\beta ({\delta_\alpha}^\mu {\delta_\beta}^\nu - {\delta_\beta}^\mu {\delta_\alpha}^\nu) \\
&= a^\sigma {L_\sigma}^\mu b^\pi {L_\pi}^\nu - a^\sigma {L_\sigma}^\nu b^\pi {L_\pi}^\mu \\
&= a^\sigma {L_\sigma}^\mu b^\pi {L_\pi}^\nu - a^\pi {L_\pi}^\nu b^\sigma {L_\sigma}^\mu \\
&= (a^\sigma b^\pi - a^\pi b^\sigma) {L_\sigma}^\mu {L_\pi}^\nu \\
&= T^{\sigma\pi} {L_\sigma}^\mu {L_\pi}^\nu
\end{aligned} \tag{B.16}$$

We are able to conclude that the bivector coordinates transform as

$$T^{\mu\nu} \rightarrow T^{\sigma\pi} {L_\sigma}^\mu {L_\pi}^\nu. \tag{B.17}$$

Except for the lowering index differences this verifies the rule eq. (B.2) from the text.

It would be reasonable seeming to impose such a tensor transformation rule on any antisymmetric rank 2 tensor, and in the text this is also imposed as the rule for transformation of symmetric rank 2 tensors. Do we have a simple example of a rank 2 symmetric tensor that can be expressed geometrically? The only one that comes to mind off the top of my head is the electrodynamic stress tensor, which is not exactly simple to work with.

B.4 LORENTZ TRANSFORMATION OF THE METRIC TENSORS

Following up on the previous thought, it is not hard to come up with an example of a symmetric tensor a whole lot simpler than the electrodynamic stress tensor. The metric tensor is probably the simplest symmetric tensor, and we get that by considering the dot product of two vectors. Taking the dot product of vectors a and b for example we have

$$a \cdot b = a^\mu b^\nu \gamma_\mu \cdot \gamma_\nu \tag{B.18}$$

From this, the metric tensors are defined as

$$\begin{aligned} g_{\mu\nu} &= \gamma_\mu \cdot \gamma_\nu \\ g^{\mu\nu} &= \gamma^\mu \cdot \gamma^\nu \end{aligned} \tag{B.19}$$

These are both symmetric and diagonal, and in fact equal (regardless of whether one picks a $+,-,-,-$ or $-,+,+,+$ signature for the space).

Let us look at the transformation of the dot product, utilizing the transformation of the four vectors being dotted to do so. By definition, when both vectors are equal, we have the (squared) spacetime interval, which based on the speed of light being constant, has been found to be an invariant under transformation.

$$a' \cdot b' = a^\mu b^\nu L(\gamma_\mu) \cdot L(\gamma_\nu) \tag{B.20}$$

We note that, like any other vector, the image $L(\gamma_\mu)$ of the Lorentz transform of the vector γ_μ can be written as

$$L(\gamma_\mu) = \left(L(\gamma_\mu) \cdot \gamma^\nu\right) \gamma_\nu \tag{B.21}$$

Similarly we can write any vector in terms of the reciprocal frame

$$\gamma_\nu = (\gamma_\nu \cdot \gamma_\mu)\gamma^\mu. \tag{B.22}$$

The dot product factor is a component of the metric tensor

$$g_{\nu\mu} = \gamma_\nu \cdot \gamma_\mu, \tag{B.23}$$

so we see that the dot product transforms as

$$a' \cdot b' = a^\mu b^\nu (L(\gamma_\mu) \cdot \gamma^\alpha)(L(\gamma_\nu) \cdot \gamma^\beta)\gamma_\alpha \cdot \gamma_\beta = a^\mu b^\nu {L_\mu}^\alpha {L_\nu}^\beta g_{\alpha\beta} \tag{B.24}$$

In particular, for $a = b$ where we have the invariant interval defined by the condition $a^2 = a'^2$, we must have

$$a^\mu a^\nu g_{\mu\nu} = a^\mu a^\nu L_\mu{}^\alpha L_\nu{}^\beta g_{\alpha\beta} \tag{B.25}$$

This implies that the symmetric metric tensor transforms as

$$g_{\mu\nu} = L_\mu{}^\alpha L_\nu{}^\beta g_{\alpha\beta} \tag{B.26}$$

Recall from eq. (B.17) that the coordinates representation of a bivector, an antisymmetric quantity transformed as

$$T^{\mu\nu} \rightarrow T^{\sigma\pi} L_\sigma{}^\mu L_\pi{}^\nu. \tag{B.27}$$

This is a very similar transformation, but differs from the bivector case where our free indices were upper indices. Suppose that we define an alternate set of coordinates for the Lorentz transformation. Let

$$L^\mu{}_\nu = L(\gamma^\mu) \cdot \gamma_\nu. \tag{B.28}$$

This can be related to the previous coordinate matrix by

$$L^\mu{}_\nu = g^{\mu\alpha} g_{\nu\beta} L_\alpha{}^\beta. \tag{B.29}$$

If we examine how the coordinates of x^2 transform in their lower index representation we find

$$x'^2 = x_\mu x_\nu L^\mu{}_\alpha L^\nu{}_\beta g^{\alpha\beta} = x^2 = x_\mu x_\nu g^{\mu\nu}, \tag{B.30}$$

and therefore find that the (upper index) metric tensor transforms as

$$g^{\mu\nu} \rightarrow g^{\alpha\beta} L^\mu{}_\alpha L^\nu{}_\beta. \tag{B.31}$$

Compared to *eq*. (B.27) we have almost the same structure of transformation. Are these the same? Does the notation I picked here introduce an apparent difference that does not actually exist? We really want to know if we have the identity

$$L(\gamma_\mu) \cdot \gamma^\nu \stackrel{?}{=} L(\gamma^\nu) \cdot \gamma_\mu, \tag{B.32}$$

If that were to be the case, then given the notation selected it would mean that $L_\mu{}^\nu = L^\nu{}_\mu$. If that were true it would justify a notational simplification $L_\mu{}^\nu = L^\nu{}_\mu = L^\nu_\mu$.

B.5 THE INVERSE LORENTZ TRANSFORMATION

To answer this question, let us consider a specific example, an x-axis boost of rapidity α. For that our Lorentz transformation takes the following form

$$L(x) = e^{-\sigma_1\alpha/2} x e^{\sigma_1\alpha/2}, \tag{B.33}$$

where $\sigma_k = \gamma_k\gamma_0$. Since σ_1 anticommutes with γ_0 and γ_1, but commutes with γ_2 and γ_3, we have

$$L(x) = (x^0\gamma_0 + x^1\gamma_1)e^{\sigma_1\alpha} + x^2\gamma_2 + x^3\gamma_3, \tag{B.34}$$

and after expansion this is

$$L(x) = \gamma_0(x^0\cosh\alpha - x^1\sinh\alpha) + \gamma_1(x^1\cosh\alpha - x^0\sinh\alpha) + \gamma_2 + \gamma_3. \tag{B.35}$$

Note that this is the first time a specific metric preference has been imposed, and $+, -, -, -$ has been used.

Observe that for the basis vectors themselves we have

$$\begin{bmatrix} L(\gamma_0) \\ L(\gamma_1) \\ L(\gamma_2) \\ L(\gamma_3) \end{bmatrix} = \begin{bmatrix} \gamma_0\cosh\alpha - \gamma_1\sinh\alpha \\ -\gamma_0\sinh\alpha + \gamma_1\cosh\alpha \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \tag{B.36}$$

Forming a matrix with μ indexing over rows and ν indexing over columns we have

$${L_\mu}^\nu = \begin{bmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{B.37}$$

Performing the same expansion for ${L^\nu}_\mu$, again with μ indexing over rows, we have

$${L^\nu}_\mu = \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{B.38}$$

This answers the question. We cannot assume that ${L_\mu}^\nu = {L^\nu}_\mu$. In fact, in this particular case, we have ${L^\nu}_\mu = ({L_\mu}^\nu)^{-1}$. Is that a general condition? Note that for the general case, we have to consider compounded transformations, where each can be a boost or rotation.

With my text still not here I have obtained a newer version of the course text from a different UofT library. In this newer version [9] (still not the 4th edition) it is at least updated with the "modern" upper and lower index formalism.

In this version they define a four-dimensional second rank tensor as the set of sixteen quantities

$$A^{\mu\nu}, \tag{B.39}$$

provided these transform under coordinate transformations like the products of components of two four vectors. They also provide raising and lowering rules that distinguish the quantities ${A^\mu}_\nu$, and ${A_\mu}^\nu$ by relating these to the raising and lowering operations so that, for example, ${A_0}^1 = A^{01}$, ${A^0}_1 = -A^{01}$. This is consistent with the notation I have used fairly blunderingly that seemed natural. This also highlights the difference between ${L_\mu}^\nu$, and ${L^\nu}_\mu$. We can relate both of these back to the index upper tensor representation

$$\begin{aligned} {L_\alpha}^\nu &= g_{\mu\alpha} L^{\mu\nu} \\ {L^\mu}_\alpha &= g_{\nu\alpha} L^{\mu\nu} \end{aligned} \tag{B.40}$$

This shows precisely how the two objects relate back to the original tensor $L^{\mu\nu}$, and why we cannot just write L_α^ν or L_α^μ respectively.

Note that in the third edition they still (somewhat surprisingly to me) continue to latin indices for $0, 1, 2, 3$ and greek for $1, 2, 3$ as in the original 1951 version.

B.6 DUALITY IN TENSOR FORM

Let us consider the subject of duality to antisymmetric forms. Within a geometric algebra context our duality is provided by multiplication by the pseudoscalar for the space.

For instance in $\mathbb{R}^3$ the dual to a bivector is the familiar cross product

$$\mathbf{a} \times \mathbf{b} = -I(\mathbf{a} \wedge \mathbf{b}), \tag{B.41}$$

where $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. In our spacetime context we use the pseudoscalar $I = \gamma_0\gamma_1\gamma_2\gamma_3$. Let us compute the coordinate representation of our vector, bivector, and trivector duals, which should compare with the tensor representation of the text.

In the text we have a statement that given an antisymmetric tensor $T^{\mu\nu}$, its dual is

$$\frac{1}{2}e^{\mu\nu}{}_{\alpha\beta}T^{\alpha\beta} \tag{B.42}$$

(I have adjusted the notation for the antisymmetric pseudotensor ϵ to retain free upper indices).

How does this compare the to Geometric Algebra bivector dual in spacetime? Let

$$T = \frac{1}{2}T^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = \sum_{\mu<\nu} T^{\mu\nu}\gamma_\mu \wedge \gamma_\nu. \tag{B.43}$$

We dot with $\gamma^\nu \wedge \gamma^\mu$ to extract the (tensor) coordinate representation

$$\begin{aligned}
T \cdot (\gamma^\nu \wedge \gamma^\mu) &= \frac{1}{2}T^{\alpha\beta}(\gamma_\alpha \wedge \gamma_\beta) \cdot (\gamma^\nu \wedge \gamma^\mu) \\
&= \frac{1}{2}T^{\alpha\beta}({\delta_\beta}^\nu {\delta_\alpha}^\mu - {\delta_\alpha}^\nu {\delta_\beta}^\mu) \\
&= \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) \\
&= T^{\mu\nu}.
\end{aligned} \tag{B.44}$$

The index manipulation gets a little hairy, but one can expand the dot products $(IT) \cdot (\gamma^\nu \wedge \gamma^\mu)$ to find that this dual has coordinates have the value,

$$(IT) \cdot (\gamma^\nu \wedge \gamma^\mu) = Ce^{\mu\nu}{}_{\alpha\beta}T^{\alpha\beta}, \tag{B.45}$$

where C is a constant multiplier that I messed up computing the actual value for.

It is also possible to verify that $(IT) \cdot T = 0$. Thus we can describe the duality of $T^{\mu\nu}$ and $e^{\mu\nu}{}_{\alpha\beta}T^{\alpha\beta}$ as the geometrical condition $T = ab$, $IT = cd$, where a, b, c, d are all mutually perpendicular.

Given a vector $x = x^\mu \gamma_\mu = x_\mu \gamma^\mu$ it is also possible to confirm that the coordinate representation of the Geometric Algebra vector dual has the form

$$Ix \sim e^{\sigma\pi\nu\mu}\gamma_\sigma\gamma_\sigma\gamma_\pi x_\nu \tag{B.46}$$

The coordinates of this product are a multiple of $\epsilon^{\sigma\pi\nu\mu}x_\mu$, which has the form specified in the text.

B.7 STOKES THEOREM

I once worked through the Geometric Algebra expression for Stokes Theorem. For a $k-1$ grade blade, the final result of that work was

$$\int (\nabla \wedge F) \cdot d^k x = \frac{1}{(k-1)!} \epsilon^{rs\cdots tu} \int da_u \frac{\partial F}{\partial a_u} \cdot (dx_r \wedge dx_s \wedge \cdots \wedge dx_t) \tag{B.47}$$

Let us expand this in coordinates to attempt to get the equivalent expression for an antisymmetric tensor of rank $k-1$.

Starting with the RHS of eq. (B.47) we have

$$\begin{aligned} F &= \frac{1}{(k-1)!} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \cdots \wedge \gamma^{\mu_{k-1}} \\ dx_r \wedge dx_s \wedge \cdots \wedge dx_t &= \frac{\partial x^{\nu_1}}{\partial a_r} \frac{\partial x^{\nu_2}}{\partial a_s} \cdots \frac{\partial x^{\nu_{k-1}}}{\partial a_t} \gamma_{\nu_1} \wedge \gamma_{\nu_2} \wedge \cdots \wedge \gamma_{\nu_{k-1}} da_r da_s \cdots da_t \end{aligned} \tag{B.48}$$

We need to expand the dot product of the wedges, for which we have

$$\begin{aligned} &\left(\gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \cdots \wedge \gamma^{\mu_{k-1}}\right) \cdot \left(\gamma_{\nu_1} \wedge \gamma_{\nu_2} \wedge \cdots \wedge \gamma_{\nu_{k-1}}\right) \\ &= {\delta^{\mu_{k-1}}}_{\nu_1} {\delta^{\mu_{k-2}}}_{\nu_2} \cdots {\delta^{\mu_1}}_{\nu_{k-1}} \epsilon^{\nu_1 \nu_2 \cdots \nu_{k-1}} \end{aligned} \tag{B.49}$$

Putting all the LHS bits together we have

$$\begin{aligned} &\frac{1}{((k-1)!)^2} \epsilon^{rs\cdots tu} \int da_u \frac{\partial}{\partial a_u} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \\ &\qquad {\delta^{\mu_{k-1}}}_{\nu_1} {\delta^{\mu_{k-2}}}_{\nu_2} \cdots {\delta^{\mu_1}}_{\nu_{k-1}} \epsilon^{\nu_1 \nu_2 \cdots \nu_{k-1}} \frac{\partial x^{\nu_1}}{\partial a_r} \frac{\partial x^{\nu_2}}{\partial a_s} \cdots \frac{\partial x^{\nu_{k-1}}}{\partial a_t} da_r da_s \cdots da_t \\ &= \frac{1}{((k-1)!)^2} \epsilon^{rs\cdots tu} \int da_u \frac{\partial}{\partial a_u} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \\ &\qquad \epsilon^{\mu_{k-1} \mu_{k-2} \cdots \mu_1} \frac{\partial x^{\mu_{k-1}}}{\partial a_r} \frac{\partial x^{\mu_{k-2}}}{\partial a_s} \cdots \frac{\partial x^{\mu_1}}{\partial a_t} da_r da_s \cdots da_t \\ &= \frac{1}{((k-1)!)^2} \epsilon^{rs\cdots tu} \int da_u \frac{\partial}{\partial a_u} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \left| \frac{\partial (x^{\mu_{k-1}}, x^{\mu_{k-2}}, \cdots, x^{\mu_1})}{\partial (a_r, a_s, \cdots, a_t)} \right| da_r da_s \cdots da_t \end{aligned} \tag{B.50}$$

Now, for the LHS of eq. (B.47) we have

$$\begin{aligned} \nabla \wedge F &= \gamma^\mu \wedge \partial_\mu F \\ &= \frac{1}{(k-1)!} \frac{\partial}{\partial x^{\mu_k}} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \gamma^{\mu_k} \wedge \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \cdots \wedge \gamma^{\mu_{k-1}} \end{aligned} \tag{B.51}$$

and the volume element of

$$d^k x = \frac{\partial x^{\nu_1}}{\partial a_1} \frac{\partial x^{\nu_2}}{\partial a_2} \cdots \frac{\partial x^{\nu_k}}{\partial a_k} \gamma_{\nu_1} \wedge \gamma_{\nu_2} \wedge \cdots \wedge \gamma_{\nu_k} da_1 da_2 \cdots da_k \tag{B.52}$$

Our dot product is

$$\begin{aligned}&\left(\gamma^{\mu_k} \wedge \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \cdots \wedge \gamma^{\mu_{k-1}}\right) \cdot \left(\gamma_{\nu_1} \wedge \gamma_{\nu_2} \wedge \cdots \wedge \gamma_{\nu_k}\right) \\ &\qquad = \delta^{\mu_{k-1}}{}_{\nu_1} \delta^{\mu_{k-2}}{}_{\nu_2} \cdots \delta^{\mu_1}{}_{\nu_{k-1}} \delta^{\mu_k}{}_{\nu_k} \epsilon^{\nu_1 \nu_2 \cdots \nu_k}\end{aligned} \tag{B.53}$$

The LHS of our k-form now evaluates to

$$\begin{aligned}(\gamma^\mu \wedge \partial_\mu F) \cdot d^k x &= \frac{1}{(k-1)!} \frac{\partial}{\partial x^{\mu_k}} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \\ &\quad \delta^{\mu_{k-1}}{}_{\nu_1} \delta^{\mu_{k-2}}{}_{\nu_2} \cdots \delta^{\mu_1}{}_{\nu_{k-1}} \delta^{\mu_k}{}_{\nu_k} \epsilon^{\nu_1 \nu_2 \cdots \nu_k} \frac{\partial x^{\nu_1}}{\partial a_1} \frac{\partial x^{\nu_2}}{\partial a_2} \cdots \frac{\partial x^{\nu_k}}{\partial a_k} da_1 da_2 \cdots da_k \\ &= \frac{1}{(k-1)!} \frac{\partial}{\partial x^{\mu_k}} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \\ &\quad \epsilon^{\mu_{k-1} \mu_{k-2} \cdots \mu_1 \mu_k} \frac{\partial x^{\mu_{k-1}}}{\partial a_1} \frac{\partial x^{\mu_{k-2}}}{\partial a_2} \cdots \frac{\partial x^{\mu_1}}{\partial a_{k-1}} \frac{\partial x^{\mu_k}}{\partial a_k} da_1 da_2 \cdots da_k \\ &= \frac{1}{(k-1)!} \frac{\partial}{\partial x^{\mu_k}} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \left| \frac{\partial(x^{\mu_{k-1}}, x^{\mu_{k-2}}, \cdots x^{\mu_1}, x^{\mu_k})}{\partial(a_1, a_2, \cdots, a_{k-1}, a_k)} \right| da_1 da_2 \cdots da_k\end{aligned} \tag{B.54}$$

Presuming no mistakes were made anywhere along the way (including in the original Geometric Algebra expression), we have arrived at Stokes Theorem for rank $k-1$ antisymmetric tensors F

$$\boxed{\begin{aligned}&\int \frac{\partial}{\partial x^{\mu_k}} F_{\mu_1 \mu_2 \cdots \mu_{k-1}} \left| \frac{\partial(x^{\mu_{k-1}}, x^{\mu_{k-2}}, \cdots x^{\mu_1}, x^{\mu_k})}{\partial(a_1, a_2, \cdots, a_{k-1}, a_k)} \right| da_1 da_2 \cdots da_k \\ &= \frac{1}{(k-1)!} \epsilon^{rs \cdots tu} \int da_u \frac{\partial}{\partial a_u} F_{\nu_1 \nu_2 \cdots \nu_{k-1}} \left| \frac{\partial(x^{\nu_{k-1}}, x^{\nu_{k-2}}, \cdots, x^{\nu_1})}{\partial(a_r, a_s, \cdots, a_t)} \right| da_r da_s \cdots da_t\end{aligned}} \tag{B.55}$$

The next task is to validate this, expanding it out for some specific ranks and hypervolume element types, and to compare the results with the familiar 3d expressions.

C

FREQUENCY FOUR VECTOR

Simon (our TA) stated eq. (3.52) without justification. Here's a little justification for the frequency four vector.

We know some of it from the QM context, and if we have been reading ahead know a bit of this from our text [11] (the energy momentum four vector relationships). Let us go back to the classical electromagnetism and recall what we know about the relation of frequency and wave numbers for continuous fields. We want solutions to Maxwell's equation in vacuum and can show that such solution also implies that our fields obey a wave equation

$$\frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} - \boldsymbol{\nabla}^2 \Psi = 0, \tag{C.1}$$

where Ψ is one of $\mathbf{E}$ or $\mathbf{B}$ or any component of either of these. There are other constraints imposed on the solutions by Maxwell's equations, but the electric and magnetic field components must obey eq. (C.1) in addition to those constraints.

A Fourier transform trial solution of the form

$$\Psi = (2\pi)^{-3/2} \int \tilde{\Psi}(\mathbf{k}, 0) e^{i(\omega t \pm \mathbf{k}\cdot\mathbf{x})} d^3\mathbf{k}. \tag{C.2}$$

can be applied to the wave equation, producing the constraint

$$\frac{1}{c^2}(i\omega)^2 \Psi - (\pm i\mathbf{k})^2 \Psi = 0. \tag{C.3}$$

So even in the continuous field domain (no QM), we have a relationship between frequency and wave number. We see that this also happens to have the form of a lightlike spacetime interval

$$\frac{\omega^2}{c^2} - \mathbf{k}^2 = 0. \tag{C.4}$$

Also recall that the photoelectric effect imposes an experimental constraint on photon energy, where we have

$$E = h\nu = \frac{h}{2\pi} 2\pi\nu = \hbar\omega \tag{C.5}$$

Therefore if we impose a mechanics like $P = (E/c, \mathbf{p})$ relativistic energy-momentum relationship on light, it then makes sense to form a nilpotent (lightlike) four vector for our photon energy. This combines our special relativistic expectations, with the constraints on the fields imposed by classical electromagnetism. We can then write for the photon four momentum

$$P = \left(\frac{\hbar\omega}{c}, \hbar k \right) \tag{C.6}$$

D

NON-INERTIAL (LOCAL) OBSERVERS

This was from the second tutorial.

D.1 BASIS CONSTRUCTION

Observations are made of either the three-vector, or the time like components of four-vectors, since these are the quantities that we can measure from our local observer frame. This is something that can be viewed in an approximate sense as being inertial, provided that we ignore the earth's rotation, the rotation around the solar system, the rotation of the solar system in the galaxy, the rotation of the galaxy in the local cluster, and so forth. Provided none of these are changing too fast relative to our measurements, we can make the inertial approximation.

Example. If we want to measure energy, it is the timelike component of the momentum.

$$E = cp^0 \tag{D.1}$$

PICTURE: Let us imagine a moving worldline in three dimensions. We can setup a frame and associated basis along the worldline of the particle, as well as a frame and basis for the stationary observer.

In class Simon used notation like $\{e^i_{\hat{o}}\}$, and $\{e^i_{\hat{a}}\}$, but also used $e^i_{\hat{0}}, e^i_{\hat{1}}, e^i_{\hat{2}}, e^i_{\hat{3}}$. It was fairly clear by the context what was meant, but lets avoid any more than one index at a time, and write $\{f_i\}$ for the frame moving along the worldline, and $\{e_i\}$ for the stationary frame.

Constructing a basis along the worldline For any timelike four-vector worldline we have a four-vector velocity of magnitude c, so we are free to define a timelike basis vector for our moving frame as

$$f_0 = u \tag{D.2}$$

going back to the first problem for u^i we have

$$f_0 = (\cosh(act), \sinh(act), 0, 0) \tag{D.3}$$

We are free to pick spatial unit vectors perpendicular to this, so for the y and z components it is natural to use

$$\begin{aligned} f_2 &= (0, 0, 1, 0) \\ f_3 &= (0, 0, 0, 1) \end{aligned} \tag{D.4}$$

We need one more, that is perpendicular to each of the above. By inspection one can pick

$$f_1 = (\sinh(act), \cosh(act), 0, 0) \tag{D.5}$$

Did Simon use any other principle to define this last beastie? I missed it if he did. I see that this happens to be the unit vector proportional to x^i.

Consider the stationary observer For a stationary observer, our worldline and four velocity respectively, for some constant $\mathbf{x}_0$ is

$$\begin{aligned} X &= (ct, \mathbf{x}_0) \\ \frac{dX}{ds} &= \frac{1}{c}\frac{dX}{d\tau} = (1, \mathbf{0}) \end{aligned} \tag{D.6}$$

Our time like unit vector is very simple

$$e_0 = \frac{dX}{ds} = (1, \mathbf{0}) \tag{D.7}$$

For the spatial unit vectors we have many choices. One would be aligned from the origin to the position vector

$$e_1 = \left(0, \frac{\mathbf{x}}{|\mathbf{x}|}\right), \tag{D.8}$$

with e_2 and e_3 oriented in any pair of mutually perpendicular spatial directions. Another option would be simply pick a e_α for each of the normal Euclidean basis directions

$$\begin{aligned} e_1 &= (0, 1, 0, 0) \\ e_2 &= (0, 0, 1, 0) \\ e_3 &= (0, 0, 0, 1) \end{aligned} \tag{D.9}$$

Observe, that we have (no sum) $e_\alpha \cdot e_\alpha = -1$ (and $e_0 \cdot e_0 = 1$).

Consider an inertial observer Now consider a slightly more complex case, where an observer is moving with some constant velocity $\mathbf{V} = c\boldsymbol{\beta}$. Our worldline is

$$X = (ct, \mathbf{x}_0 + \boldsymbol{\beta} ct). \tag{D.10}$$

Let us calculate the four velocity. We have

$$\frac{dX}{dt} = c(1, \boldsymbol{\beta}). \tag{D.11}$$

From this our proper time is

$$\tau = \frac{1}{c}\int_0^t c\sqrt{(1,\boldsymbol{\beta})^2} dt = \sqrt{1-\boldsymbol{\beta}^2} t. \tag{D.12}$$

Our worldline and four-velocity, parametrized in terms of proper time, with $\gamma = (1-\boldsymbol{\beta}^2)^{-1/2}$, are then

$$\begin{aligned} X &= (\gamma c\tau, \mathbf{x}_0 + \gamma\boldsymbol{\beta} c\tau) \\ u &= \gamma(1, \boldsymbol{\beta}) \end{aligned} \tag{D.13}$$

For this system, let us label the basis $\{h_k\}$. From above our time like unit vector is

$$h_0 = \gamma(1, \boldsymbol{\beta}) \tag{D.14}$$

We observe that this has the desired time like property, $(h_0)^2 = 1 > 0$.

Now, let us try Gram-Schmidt, subtracting the projection of h_0 on e_1 from e_1 and see what we get. Our projection is

$$\begin{aligned} \text{Proj}_{h_0}(e_1) &= \frac{e_1 \cdot h_0}{h_0 \cdot h_0} h_0 \\ &= (0,1,0,0) \cdot \gamma(1,\boldsymbol{\beta})\gamma(1,\boldsymbol{\beta}) \\ &= -\gamma^2 \beta_x (1, \boldsymbol{\beta}). \end{aligned} \tag{D.15}$$

We should have a space like vector normal to h_0 once we take the Gram-Schmidt difference

$$e_1 - \frac{e_1 \cdot h_0}{h_0 \cdot h_0} h_0 = (0,1,0,0) + \gamma^2 \beta_x (1, \boldsymbol{\beta}) \tag{D.16}$$

Let us compute the norm of this vector and verify that it is space like. We should also verify that it is normal to h_0 as expected. For the norm we have

$$
\begin{aligned}
-1 + \beta_x^2 + 2\beta_x\gamma^2(0,1,0,0)\cdot(1,\boldsymbol{\beta}) &= -1 + \beta_x^2 + 2\beta_x\gamma^2(-\beta_x) \\
&= \beta_x^2(1 - 2\gamma^2) - 1 \\
&= \beta_x^2 \frac{1 - \boldsymbol{\beta}^2 - 2}{1 - \boldsymbol{\beta}^2} - 1 \\
&= -\beta_x^2 \frac{1 + \boldsymbol{\beta}^2}{1 - \boldsymbol{\beta}^2} - 1
\end{aligned}
\tag{D.17}
$$

This is less than zero as we expect for a spacelike vector. Good. Our second spacelike unit vector is thus

$$
h_1 = \left(\beta_x^2 \frac{1 + \boldsymbol{\beta}^2}{1 - \boldsymbol{\beta}^2} + 1\right)^{-1/2} \Big((0,1,0,0) + \gamma^2\beta_x(1,\boldsymbol{\beta})\Big)
\tag{D.18}
$$

Let us verify that these two computed spacetime basis vectors are normal. Their dot product is proportional to

$$
\begin{aligned}
((0,1,0,0) + \gamma^2\beta_x(1,\boldsymbol{\beta}))\cdot(1,\boldsymbol{\beta}) &= -\beta_x + \gamma^2\beta_x(1 - \boldsymbol{\beta}^2) \\
&= -\beta_x + \beta_x \\
&= 0 \qquad \square
\end{aligned}
\tag{D.19}
$$

We could continue this, continuing the Gram-Schmidt iteration using e_2 and e_3 for the remainder of the initial spanning set.

Doing so, we would have

$$
h_2 \sim e_2 - \frac{e_2 \cdot h_1}{h_1 \cdot h_1} h_1 - \frac{e_2 \cdot h_0}{h_0 \cdot h_0} h_0.
\tag{D.20}
$$

After scaling so that $h_2 \cdot h_2 = -1$, we would then have

$$
h_3 \sim e_3 - \frac{e_3 \cdot h_2}{h_2 \cdot h_2} h_2 - \frac{e_3 \cdot h_1}{h_1 \cdot h_1} h_1 - \frac{e_3 \cdot h_0}{h_0 \cdot h_0} h_0.
\tag{D.21}
$$

Projections and the reciprocal basis Recall that for Euclidean space, when we had orthonormal vectors, we could simplify the Gram-Schmidt procedure from

$$e_{k+1} \sim f_{k+1} - \sum_{i=0}^{k} \frac{f_{k+1} \cdot e_i}{e_i \cdot e_i} e_i, \tag{D.22}$$

to

$$e_{k+1} \sim f_{k+1} - \sum_{i=0}^{k} \left(f_{k+1} \cdot e_i\right) e_i. \tag{D.23}$$

However, for our non-Euclidean space, we cannot do this. This suggests a nice intuitive motivation for the reciprocal basis. We can define, for any normalized basis $\{f^i\}$ in our Minkowski space (no sum)

$$e^i = \frac{e_i}{e_i \cdot e_i} \tag{D.24}$$

Now our Gram-Schmidt iteration becomes

$$e_{k+1} \sim f_{k+1} - \sum_{i=0}^{k} \left(f_{k+1} \cdot e_i\right) e^i, \tag{D.25}$$

and we identify, for a four vector b, the projection onto the chosen basis vector, as (no sum)

$$\text{Proj}_{e^i}(b) = (b \cdot e_i) e^i. \tag{D.26}$$

In particular, we have for the resolution of identity (now with summation implied again)

$$b = (b \cdot e_i) e^i. \tag{D.27}$$

This is nice and it allows us to work with four vectors in their entirety, instead of in coordinates. We have

$$x = x^i e_i = x_i e^i, \tag{D.28}$$

where

$$
\begin{aligned}
x^i &= x \cdot e^i \\
x_i &= x \cdot e_i
\end{aligned}
\tag{D.29}
$$

Also note that $e^\alpha = -e_\alpha$ and $e^0 = e_0$, just as the coordinates themselves vary sign with index raising and lowering dependent on whether they are time like or space like.

We have seen that the representation of the basis can be chosen to depend on the observer, and for the stationary observer, we had simply

$$
\begin{aligned}
e_0 &= (1, 0, 0, 0) \\
e_1 &= (0, 1, 0, 0) \\
e_2 &= (0, 0, 1, 0) \\
e_3 &= (0, 0, 0, 1),
\end{aligned}
\tag{D.30}
$$

with a reciprocal basis $e^i \cdot e_j = \delta^i{}_j$

$$
\begin{aligned}
e^0 &= (1, 0, 0, 0) \\
e^1 &= -(0, 1, 0, 0) \\
e^2 &= -(0, 0, 1, 0) \\
e^3 &= -(0, 0, 0, 1).
\end{aligned}
\tag{D.31}
$$

An alternate basis for the inertial frame Given the same h_0 as defined above for the inertial frame, let us define an alternate h_1, subtracting the timelike component from the worldline of the particle itself. Let

$$
\begin{aligned}
X &= (\gamma c\tau, \mathbf{x}_0 + \gamma \boldsymbol{\beta} c\tau) \\
h_0 &= \gamma(1, \boldsymbol{\beta}) \\
Y &= X - (X \cdot h_0) h^0
\end{aligned}
\tag{D.32}
$$

The dot product above is

$$
\begin{aligned}
X \cdot h_0 &= (\gamma c\tau, \mathbf{x}_0 + \gamma \boldsymbol{\beta} c\tau) \cdot \gamma(1, \boldsymbol{\beta}) \\
&= \gamma^2 c\tau - \gamma(\boldsymbol{\beta} \cdot \mathbf{x}_0) - \gamma^2 \boldsymbol{\beta}^2 c\tau \\
&= \gamma^2 c\tau(1 - \boldsymbol{\beta}^2) - \gamma(\boldsymbol{\beta} \cdot \mathbf{x}_0) \\
&= c\tau - \gamma(\boldsymbol{\beta} \cdot \mathbf{x}_0)
\end{aligned}
\tag{D.33}
$$

Our rejection of h_0 from X is then

$$\begin{aligned} Y &= (\gamma c\tau, \mathbf{x}_0 + \gamma\boldsymbol{\beta} c\tau) - (c\tau - \gamma(\boldsymbol{\beta}\cdot\mathbf{x}_0))\gamma(1,\boldsymbol{\beta}) \\ &= (\gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0), \mathbf{x}_0 + \gamma\boldsymbol{\beta} c\tau - c\tau\gamma\boldsymbol{\beta} + \gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)\boldsymbol{\beta}) \\ &= (\gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0), \mathbf{x}_0 + \gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)\boldsymbol{\beta}) \\ &= \gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)(1,\boldsymbol{\beta}) + (0,\mathbf{x}_0) \end{aligned} \tag{D.34}$$

We can verify that this is spacelike by computing the square

$$\begin{aligned} Y^2 &= \gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)^2 - \mathbf{x}_0^2 + 2\gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)(1,\boldsymbol{\beta})\cdot(0,\mathbf{x}_0) \\ &= \gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)^2 - \mathbf{x}_0^2 - 2\gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)^2 \\ &= -\gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)^2 - \mathbf{x}_0^2 \\ &< 0. \end{aligned} \tag{D.35}$$

A final normalization of this yields

$$h_1 = (\gamma^1(\boldsymbol{\beta}\cdot\mathbf{x}_0)^2 + \mathbf{x}_0^2)^{-1/2}\left(\gamma^2(\boldsymbol{\beta}\cdot\mathbf{x}_0)(1,\boldsymbol{\beta}) + (0,\mathbf{x}_0)\right) \tag{D.36}$$

It is easy enough to verify that we have $h_1 \cdot h_0 = 0$ as desired.

A followup note on the worldline basis Note that we can construct the spatial vector f^1 in eq. (D.5) systematically without use of any sort of intuition. We get this by Gram-Schmidt directly

$$\begin{aligned} f_1 &\sim e_1 - (e_1\cdot e_0)e^0 - \cancel{(e_1\cdot e_2)}e_2 - \cancel{(e_1\cdot e_3)}e_3 \\ &= (0,1,0,0) - (0,1,0,0)\cdot(\cosh(ac\tau), \sinh(ac\tau), 0, 0)e_0 \\ &= (0,1,0,0) + \sinh(ac\tau)(\cosh(ac\tau), \sinh(ac\tau), 0, 0) \\ &= (\sinh(ac\tau)\cosh(ac\tau), 1 + \sinh^2(ac\tau), 0, 0) \\ &= (\sinh(ac\tau)\cosh(ac\tau), \cosh^2(ac\tau), 0, 0) \\ &\sim (\sinh(ac\tau), \cosh(ac\tau), 0, 0) \qquad \square \end{aligned} \tag{D.37}$$

It is also noteworthy to observe that we have $f_i \cdot f_j = 0, i \neq j$, and $f_0 \cdot f_0 = 1$ and $f_\alpha \cdot f_\alpha = -1$, as desired.

Relating the Lorentz transformation and coordinate transformations We are familiar now with the tensor form of the Lorentz transformation. This takes coordinates to coordinates

$$x'^i = L_j{}^i x^j \tag{D.38}$$

Specifying just the coordinates and not the basis associated with the coordinates leaves out some valuable seeming information. For instance, is the basis associated with the pre and post transformed coordinates the same?

For example, suppose that our basis for the primed coordinates is $\{f_i\}$, construction of the four vector (in its entirety) out of its coordinates and this basis requires the sum

$$\begin{aligned} X &= x'^i f_i \\ &= (L_j{}^i f_i) x^j \end{aligned} \tag{D.39}$$

This interior sum $L_j{}^i f_i$ is a linear combination of the primed basis vectors, but we see that these are in fact a set of vectors, and can be considered the basis for the unprimed coordinates. We could for example write

$$e_i = L_j{}^i f_i. \tag{D.40}$$

With such a description, our Lorentz transformation becomes just a mechanism to map vectors in one basis into another. To make this clear, let us work in the opposite order, and suppose that we have a pair of bases $\{e_i\}$ and $\{f_i\}$. For any vector X we can calculate the coordinates utilizing the reciprocal frame.

$$X = (X \cdot e^i) e_i = (X \cdot f^j) f_j. \tag{D.41}$$

Writing

$$\begin{aligned} x^i &= X \cdot e^i \\ x'^i &= X \cdot f^i. \end{aligned} \tag{D.42}$$

This is

$$x'^k f_k = x^j e_j. \tag{D.43}$$

Dotting with f^i we have

$$x'^i = x^j (e_j \cdot f^i). \tag{D.44}$$

In this form we see explicitly that the Lorentz transformation is in fact the "direction cosines" associated with a change of basis. Specifically, we can write

$$L_j{}^i = e_j \cdot f^i \tag{D.45}$$

I like this as a way to view the Lorentz transformation, since the explicit inclusion of the basis sets involved makes the geometry clear.

Example D.1: A coordinate calculation example

I have gone to the effort of calculating some basis representations in a lot more detail than we covered in the tutorial, and explore some of the ideas further. This seemed important to get a feel for what we were discussing, and to see how the pieces fit together.

Let us do one more simple example, where we look at the coordinates of a four vector in the coordinate system where the time like direction is the proper velocity, and also eliminate the the y and z coordinates from the mix to simplify it further. For such a system we have only two choices for our spatial basis vector (we can alter the sign).

For our spacetime point, consider the worldline for a particle moving at a constant velocity. That is

$$X = (ct, p_0 + \beta ct). \tag{D.46}$$

As before our proper time is

$$\tau = \sqrt{1 - \beta^2} t, \tag{D.47}$$

allowing us to re-parametrize the worldline, and have a proper time parametrized velocity

$$\begin{aligned} X &= (\gamma c\tau, p_0 + \beta\gamma c\tau) \\ u &= \gamma(1, \beta) \end{aligned} \tag{D.48}$$

Let us utilize the standard basis for the stationary frame, and denote this $\{e_i\}$

$$\begin{aligned} e_0 &= (1, 0) \\ e_1 &= (0, 1) \end{aligned} \tag{D.49}$$

and calculate a basis $\{f_i\}$ for which $f_0 = u$ is the time like direction. By Gram-Schmidt, our space like basis vector is

$$\begin{aligned} f_1 &\sim e_1 - (e_1 \cdot f_0) f^0 \\ &= (0,1) - (0,1) \cdot \gamma(1,\beta)\gamma(1,\beta) \\ &= (0,1) - \gamma^2(-\beta)(1,\beta) \\ &= (\gamma^2\beta, 1 + \beta^2\gamma^2) \\ &= \left(\gamma^2\beta, \frac{1}{1-\beta^2}(1-\beta^2+\beta^2)\right) \\ &= \left(\gamma^2\beta, \gamma^2\right) \\ &\sim -\gamma(\beta, 1) \end{aligned} \tag{D.50}$$

The negative sign here is a bit of sneaky move and chosen only after calculating the coordinates of the vector in this frame, so that at speed $\beta = 0$, the coordinates in frames $\{e^i\}$ and $\{f^i\}$ are the same. Our basis is then

$$\begin{aligned} f_0 &= \gamma(1,\beta) \\ f_1 &= -\gamma(\beta, 1) \end{aligned} \tag{D.51}$$

One can quickly verify that $f_0 \cdot f_0 = 1$, $f_1 \cdot f_1 = -1$, and $f_0 \cdot f_1 = 0$. Our reciprocal frame, defined so that $f^i \cdot f_j = \delta^i{}_j$ is

$$\begin{aligned} f^0 &= \gamma(1,\beta) \\ f^1 &= \gamma(\beta, 1) \end{aligned} \tag{D.52}$$

With this basis our coordinate representation is

$$X = \underbrace{(X \cdot f^0)}_{x^0} f_0 + \underbrace{(X \cdot f^1)}_{x^1} f_1, \tag{D.53}$$

and we calculate our coordinates to be

$$\begin{aligned} x^0 &= c\tau - \gamma p_0 \beta \\ x^1 &= \gamma p_0 \end{aligned} \tag{D.54}$$

As a check one can verify that $X = x^0 f_0 + x^1 f_1$ as expected. So we see that in a frame for which the proper velocity is the time like basis vector, our particle is at rest (moving only in time).

Some interesting information can be extracted after making the coordinate calculation. It is interesting to note that the position $x^1 = \gamma p_0$ equals p_0 when $\beta = 0$. When the particle is observed at rest in one frame, it remains at rest in the frame for which its proper velocity is the time like direction (the particle's rest frame). Furthermore, when the particle is observed moving, the position in the particles rest frame is always greater than the observed position $x_0 \gamma \ge x_0$. In other words, the particle's position appears closer to the origin in the observer's frame than it is in the rest frame (it is position is contracted).

Also see that the rest frame time matches the observer frame time when the particle is observed at rest ($\beta = 0$). The time in the rest frame is always less than the time in the observer frame and by increasing *beta* we can shift the initial time position of the particle in its rest frame as far backwards as we like. Similarly, if the particle is observed moving backwards in the observer frame, the initial time position of the particle in the rest frame can be pushed as far forward in time as we like.

An initially confusing aspect of the given non-inertial worldline For the worldline

$$X = \frac{1}{a}(\sinh(ac\tau), \cosh(ac\tau)), \tag{D.55}$$

we calculated

$$\begin{aligned} u &= (\cosh(ac\tau), \sinh(ac\tau)) \\ f_0 = u &= (\cosh(ac\tau), \sinh(ac\tau)) \\ f_1 &= (\sinh(ac\tau), \cosh(ac\tau)). \end{aligned} \tag{D.56}$$

The curious thing about this basis is that when one calculates the rest frame coordinates

$$\begin{aligned} x^0 &= X \cdot f^0 = 0 \\ x^1 &= X \cdot f^1 = \frac{1}{a}, \end{aligned} \tag{D.57}$$

the timelike coordinate is zero uniformly? We can verify easily that the position four vector is recovered as expected from $X = x^0 f_0 + x^1 f_1$, but it still seems irregular that we have no timelike coordinate?

Oh! I see. This is a spacelike four vector. Look at the length

$$X^2 = \frac{1}{a^2}(\sinh^2(ac\tau) - \cosh^2(ac\tau)) = -\frac{1}{a^2} < 0. \tag{D.58}$$

Because it is spacelike in one frame, it can only be (just) spacelike in its rest frame.

D.2 SPLIT OF ENERGY AND MOMENTUM (VERY ROUGH NOTES)

Disclaimer. At the very end of the tutorial Simon jotted some very quick notes, and I have included what I got of those without editing below. I have yet to go through these and make something coherent of them.

In a coordinate representation, the timelike component of our momentum was obtained by extracting the first coordinate

$$p^0 = (p^0, p^1, p^2, p^3) \cdot (1, 0, 0, 0). \tag{D.59}$$

This was (after scaling) was our energy term $E = cp^0$, and we can extract this in the observer frame by dotting with our observer frame timelike basis vector e^0

$$E_{\text{observer}} = cp \cdot e^0 \equiv cp^0 \tag{D.60}$$

In the observers reference frame, where $u^i = (1, 0, 0, 0)$, and $p^i = mcu^i$, we have

$$p^i = (mc, 0, 0, 0) \tag{D.61}$$

$$u'^i_{\text{observed}} = \gamma((1, v/c, 0, 0) \tag{D.62}$$

$$u^i_{\text{observer}} = (1, 0, 0, 0) \tag{D.63}$$

$$u'^i_{\text{observer}} = \begin{bmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} u^i \tag{D.64}$$

$$p^0 = \gamma mc \tag{D.65}$$

D.3 frequency of light from a distant star (again very rough notes)

Suppose we have a star far away. What is the frequency of the light emitted

$$\hat{\omega} = \omega e^{-ac\tau} \tag{D.66}$$

FIXME: derive.

where ω is the emitted frequency.

FIXME: This implied an elapsed time before the star would no longer be visible?

E

3D GPS GEOMETRIES

For exercise 3.6 I'd initially gotten very carried away playing with algebra and geometry of circular intersection. I did end up with a numerical computation (handed in on paper) where I attempted to force a bit of SR into the mix in the end. I know that I really ought to have been tackling the problem without considering 3D geometries and only using SR concepts but had too much fun playing with things, and ran out of time. I paid for that with a really poor mark, and later reworked part (a) as a basic SR problem.

Here is the play I did the first time around.

Discussion of the non-toy model It is fairly easy to find interesting info about the mechanisms that real GPS works using. NASA has a nice How does GPS work page [14], and How stuff works has a nice How GPS Receivers Work article [12]. Reading these one finds that the GPS clocks are actually kept synchronized. The typical GPS receiver obviously has a clock, since we have countdown timers for time until arrival, is that clock accurate enough compared to the satellite atomic clocks to be used for the GPS location algorithm? What is done in fact is to use the local receiver time to seed the iterative algorithms, allowing the local time to be calculated eventually with an accuracy that actually approaches that of the satellite's atomic clocks. Some of the sources of error, like reflection of the signals, delaying them, and interference by atmosphere are also discussed in these articles. Also interesting is that there is a table lookup of the satellites position implemented in the GPS receivers. This table lookup is used to seed the iterative algorithms, and can be used to reduce calculation error.

Our basic GPS problem is to calculate the intersection of a number of "spherical" hypersurfaces. This is made more interesting by the fact that this is both a non-linear and an overspecified problem. Let us consider the geometric problem to get an idea of how to set up this problem. Suppose that we have a set of k satellites, located at position $\mathbf{p}_i, i \in [1, k]$, and we know that these are located with distance d_i from our position $\mathbf{x}$.

Our problem is then to find the simultaneous solution to the following set of equations

$$
\begin{aligned}
(\mathbf{x} - \mathbf{p}_1)^2 &= d_1^2 \\
(\mathbf{x} - \mathbf{p}_2)^2 &= d_2^2 \\
&\vdots \\
(\mathbf{x} - \mathbf{k}_2)^2 &= d_k^2.
\end{aligned}
\tag{E.1}
$$

Observe that even if we reduce this to a one dimensional problem in a single variable x, we still have a non-linear system

$$\begin{aligned}
0 &= (x - p_1)^2 - d_1^2 \\
0 &= (x - p_2)^2 - d_2^2 \\
&\vdots \\
0 &= (x - k_2)^2 - d_k^2.
\end{aligned} \tag{E.2}$$

We also need to be aware of the fact that each of the positions p_i, and the respective distances d_i will in reality both have associated errors, so there is not likely any specific single value of x that "solves" this problem, unless it is setup in a contrived and perfect fashion. This intrinsic error, and the k equations, one unknown nature of the problem (or three unknowns for spatial, or four for spatial and time position) suggests a least squares approach, but it will have to be one that also incorporates iteration.

We can setup our problem in matrix form, where we are looking for a solution to

$$F(x) = \begin{bmatrix} F_i(x) \end{bmatrix} = \begin{bmatrix} |\mathbf{x} - \mathbf{p}_1| - c|t - t_1| \\ |\mathbf{x} - \mathbf{p}_2| - c|t - t_2| \\ \vdots \\ |\mathbf{x} - \mathbf{p}_k| - c|t - t_k| \end{bmatrix} = 0. \tag{E.3}$$

We seek the spacetime event vector $x = (ct, \mathbf{x})$ for the spatial location and the exact local time at the location of the GPS receiver. Given any approximation of the solution, we can refine the solution using Newton's root finding method by taking partials, forming the Jacobian matrix for our function $F(x)$. That is

$$F(x_0 + \Delta x) \approx F(x_0) + \left.\frac{\partial F_i(x)}{\partial x^j}\right|_{x_0} \Delta x^j = F(x_0) + J(x_0)\Delta x = 0 \tag{E.4}$$

This leaves us with the our least squares problem, requiring the generalized inverse to the matrix equation

$$x_1 = x_0 - J^{\dagger}(x_0)F(x_0), \tag{E.5}$$

where

$$J^{\dagger} = (J^{\mathrm{T}}J)^{-1}J^{\mathrm{T}}. \tag{E.6}$$

This is a solution in the least squares sense that given $b = Jx$, the norm $|J\bar{x} - b|$ is minimized by $\bar{x} = J^\dagger b$.

This iterative method of solution, in the context of finding fitting circles and ellipses can be found discussed in detail in [3].

Goofing around with the geometry of it all For our toy model we have two satellites A and B both moving in the positive x-axis direction at velocity V_x at height h. As seen above, we do not require the velocity of the satellites to setup the problem, and could express the problem to solve as the numerical solution of the set of equations

$$F(x) = \begin{bmatrix} \sqrt{h^2 + (x - x'_A)^2} - c\left|t - t'_A\right| \\ \sqrt{h^2 + (x - x'_B)^2} - c\left|t - t'_B\right| \end{bmatrix} = 0. \tag{E.7}$$

If we assume that our GPS receiver's clock is synchronized sufficiently with satellites A and B, this single variable problem admits a closed form for one iteration of the least squares process. However, since we are asked for a result that includes a V_x/c term, we can augment our matrix equation by two additional rows, with a secondary set of data points introduced at an offset time interval. That is

$$F(x) = \begin{bmatrix} \sqrt{h^2 + (x - x'_A)^2} - \left|ct - ct'_A\right| \\ \sqrt{h^2 + (x - x'_B)^2} - \left|ct - ct'_B\right| \\ \sqrt{h^2 + (x - x'_A - (V_x/c)c\delta t)^2} - \left|ct - ct'_A - c\delta t\right| \\ \sqrt{h^2 + (x - x'_B - (V_x/c)c\delta t)^2} - \left|ct - ct'_B - c\delta t\right| \end{bmatrix} = 0. \tag{E.8}$$

With t, δt, x'_A, and x'_B given, and an initial seed value for the iterative procedure assumed to be the midpoint $x_0 = (x'_A + x'_B)/2$, we can calculate a first approximation to the receiver position $x_1 = x_0 + \Delta x$ using the Newton's procedure outlined above.

For this system our Jacobian elements are all differentials of the following form

$$\frac{\partial}{\partial x} \sqrt{h^2 + (x - p)^2} - \Delta = \frac{x - p}{\sqrt{h^2 + (x - p)^2}}, \tag{E.9}$$

so, our Jacobian is

$$J = \begin{bmatrix} \frac{x - x'_A}{\sqrt{h^2 + (x - x'_A)^2}} \\ \frac{x - x'_B}{\sqrt{h^2 + (x - x'_B)^2}} \\ \frac{x - x'_A - (V_x/c)c\delta t}{\sqrt{h^2 + (x - x'_A - (V_x/c)c\delta t)^2}} \\ \frac{x - x'_B - (V_x/c)c\delta t}{\sqrt{h^2 + (x - x'_B - (V_x/c)c\delta t)^2}} \end{bmatrix} \tag{E.10}$$

Our deviation from the midpoint to first order in V_x/c is

$$\Delta x = -\frac{J^{\mathrm{T}} F}{J^T J}\bigg|_{x=(x'_A + x'_B)/2} \tag{E.11}$$

To tidy this up, let

$$s = \frac{1}{2}(x'_B - x'_A) \tag{E.12}$$

$$D = (V_x/c)c\delta t \tag{E.13}$$

$$\begin{aligned} J^{\mathrm{T}} J\big|_{(x'_A + x'_B)/2} &= \frac{s^2}{h^2 + s^2} + \frac{(-s)^2}{h^2 + s^2} + \frac{(s-D)^2}{h^2 + (s-D)^2} + \frac{(-s-D)^2}{h^2 + (-s-D)^2} \\ &= \frac{2s^2}{h^2 + s^2} + \frac{(s-D)^2}{h^2 + (s-D)^2} + \frac{(s+D)^2}{h^2 + (s+D)^2} \end{aligned} \tag{E.14}$$

and

$$J^{\mathrm{T}}F\big|_{(x'_A+x'_B)/2} = \begin{bmatrix} \frac{s}{\sqrt{h^2+(s)^2}} & \frac{-s}{\sqrt{h^2+(-s)^2}} & \frac{s-D}{\sqrt{h^2+(s-D)^2}} & \frac{-s-D}{\sqrt{h^2+(-s-D)^2}} \end{bmatrix} \begin{bmatrix} \sqrt{h^2+(s)^2} - \left|ct - ct'_A\right| \\ \sqrt{h^2+(-s)^2} - \left|ct - ct'_B\right| \\ \sqrt{h^2+(s-D)^2} - \left|ct - ct'_A - c\delta t\right| \\ \sqrt{h^2+(-s-D)^2} - \left|ct - ct'_B - c\delta t\right| \end{bmatrix}$$
$$= \frac{s}{\sqrt{s^2+D^2}}\left(\left|ct - ct'_B\right| - \left|ct - ct'_A\right|\right) - 2D - \frac{(s-D)\left|ct - ct'_A - c\delta t\right|}{\sqrt{h^2+(s-D)^2}} + \frac{(s+D)\left|ct - ct'_B - c\delta t\right|}{\sqrt{h^2+(s+D)^2}} \tag{E.15}$$

The final beastly ugly result, utilizing the helper variables of eq. (E.13), and eq. (E.12), we have for the deviation from the midpoint (after one iteration of this least squares Newton's method):

$$\Delta x = \frac{-\frac{s}{\sqrt{s^2+D^2}}\left(\left|ct - ct'_B\right| - \left|ct - ct'_A\right|\right) + 2D + \frac{(s-D)\left|ct-ct'_A-c\delta t\right|}{\sqrt{h^2+(s-D)^2}} - \frac{(s+D)\left|ct-ct'_B-c\delta t\right|}{\sqrt{h^2+(s+D)^2}}}{\frac{2s^2}{h^2+s^2} + \frac{(s-D)^2}{h^2+(s-D)^2} + \frac{(s+D)^2}{h^2+(s+D)^2}} \tag{E.16}$$

In practice it does not make much sense to compute this. You will want to use a computer, and assuming the availability of a pre-canned SVD routine to compute the generalized inverse, the toy model would not be any easier to solve than the real thing.

F

PLAYING WITH COMPLEX NOTATION FOR RELATIVISTIC APPLICATIONS IN A PLANE

F.1 MOTIVATION

In the electrodynamics midterm we had a question on circular motion. This screamed for use of complex numbers to describe the spatial parts of the spacetime trajectories.

Let us play with this a bit.

F.2 OUR INVARIANT

Suppose we describe our spacetime point as a paired time and complex number

$$X = (ct, z). \tag{F.1}$$

Our spacetime invariant interval in this form is thus

$$X^2 \equiv (ct)^2 - |z|^2. \tag{F.2}$$

Not much different than the usual coordinate representation of the spatial coordinates, except that we have a $|z|^2$ replacing the usual $\mathbf{x}^2$.

Taking the spacetime distance between X and another point, say $\tilde{X} = (c\tilde{t}, \tilde{z})$ motivates the inner product between two points in this representation

$$\begin{aligned}
(X - \tilde{X})^2 &= (ct - c\tilde{t})^2 - |z - \tilde{z}|^2 \\
&= (ct - c\tilde{t})^2 - (z - \tilde{z})(z^* - \tilde{z}^*) \\
&= (ct)^2 - 2(ct)(c\tilde{t}) + (c\tilde{t})^2 - |z|^2 - |\tilde{z}|^2 + (z\tilde{z}^* + z^*\tilde{z}) \\
&= X^2 + \tilde{X}^2 - 2\left((ct)(c\tilde{t}) - \frac{1}{2}(z\tilde{z}^* + z^*\tilde{z})\right)
\end{aligned} \tag{F.3}$$

It is clear that it makes sense to define

$$X \cdot \tilde{X} = (ct)(c\tilde{t}) - \mathrm{Re}(z\tilde{z}^*), \tag{F.4}$$

consistent with our original starting point

$$X^2 = X \cdot X. \tag{F.5}$$

Let us also introduce a complex inner product

$$\langle z, \tilde{z} \rangle \equiv \frac{1}{2}\left(z\tilde{z}^* + z^*\tilde{z}\right) = \mathrm{Re}(z\tilde{z}^*). \tag{F.6}$$

Our dot product can now be written

$$X \cdot \tilde{X} = (ct)(c\tilde{t}) - \langle z, \tilde{z} \rangle. \tag{F.7}$$

F.3 CHANGE OF BASIS

Our standard basis for our spatial components is $\{1, i\}$, but we are free to pick any other basis should we choose. In particular, if we rotate our basis counterclockwise by ϕ, our new basis, still orthonormal, is $\{e^{i\phi}, ie^{i\phi}\}$.

In any orthonormal basis the coordinates of a point with respect to that basis are real, so just as we can write

$$z = \langle 1, z \rangle + i\langle i, z \rangle, \tag{F.8}$$

we can extract the coordinates in the rotated frame, also simply by taking inner products

$$z = e^{i\phi}\langle e^{i\phi}, z \rangle + ie^{i\phi}\langle ie^{i\phi}, z \rangle. \tag{F.9}$$

The values $\langle e^{i\phi}, z \rangle$, and $\langle ie^{i\phi}, z \rangle$ are the (real) coordinates of the point z in this rotated basis.

This is enough that we can write the Lorentz boost immediately for a velocity $\vec{v} = c\beta e^{i\phi}$ at an arbitrary angle ϕ in the plane

$$\begin{bmatrix} ct' \\ \langle e^{i\phi}, z' \rangle \\ \langle ie^{i\phi}, z' \rangle \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ \langle e^{i\phi}, z \rangle \\ \langle ie^{i\phi}, z \rangle \end{bmatrix} \tag{F.10}$$

Let us translate this to ct, x, y coordinates as a check. For the spatial component parallel to the boost direction we have

$$\begin{aligned} \langle e^{i\phi}, x + iy \rangle &= \mathrm{Re}(e^{-i\phi}(x + iy)) \\ &= \mathrm{Re}((\cos\phi - i\sin\phi)(x + iy)) \\ &= x\cos\phi + y\sin\phi, \end{aligned} \tag{F.11}$$

and the perpendicular components are

$$\begin{aligned}\langle ie^{i\phi}, x+iy\rangle &= \mathrm{Re}(-ie^{-i\phi}(x+iy)) \\ &= \mathrm{Re}((-i\cos\phi - \sin\phi)(x+iy)) \\ &= -x\sin\phi + y\cos\phi.\end{aligned} \tag{F.12}$$

Grouping the two gives

$$\begin{bmatrix}\langle e^{i\phi}, x+iy\rangle \\ \langle ie^{i\phi}, x+iy\rangle\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi \\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = R_{-\phi}\begin{bmatrix}x \\ y\end{bmatrix} \tag{F.13}$$

The boost equation in terms of the cartesian coordinates is thus

$$\begin{bmatrix}1 & 0 \\ 0 & R_{-\phi}\end{bmatrix}\begin{bmatrix}ct' \\ x' \\ y'\end{bmatrix} = \begin{bmatrix}\gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & R_{-\phi}\end{bmatrix}\begin{bmatrix}ct \\ x \\ y\end{bmatrix}. \tag{F.14}$$

Writing

$$\begin{bmatrix}ct' \\ x' \\ y'\end{bmatrix} = \left\|\wedge^{\mu}{}_{\nu}\right\|\begin{bmatrix}ct \\ x \\ y\end{bmatrix}, \tag{F.15}$$

the boost matrix $\|\wedge^{\mu}{}_{\nu}\|$ is found to be (after a bit of work)

$$\begin{aligned}\left\|\wedge^{\mu}{}_{\nu}\right\| &= \begin{bmatrix}1 & 0 \\ 0 & R_{\phi}\end{bmatrix}\begin{bmatrix}\gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & R_{-\phi}\end{bmatrix} \\ &= \begin{bmatrix}\gamma & -\gamma\beta\cos\phi & -\gamma\beta\sin\phi \\ -\gamma\beta\cos\phi & \gamma\cos^2\phi + \sin^2\phi & (\gamma-1)\sin\phi\cos\phi \\ -\gamma\beta\sin\phi & (\gamma-1)\sin\phi\cos\phi & \gamma\sin^2\phi + \cos^2\phi\end{bmatrix}\end{aligned} \tag{F.16}$$

A final bit of regrouping gives

$$\left\|\wedge^{\mu}{}_{\nu}\right\| = \begin{bmatrix}\gamma & -\gamma\beta\cos\phi & -\gamma\beta\sin\phi \\ -\gamma\beta\cos\phi & 1+(\gamma-1)\cos^2\phi & (\gamma-1)\sin\phi\cos\phi \\ -\gamma\beta\sin\phi & (\gamma-1)\sin\phi\cos\phi & 1+(\gamma-1)\sin^2\phi\end{bmatrix}. \tag{F.17}$$

This is consistent with the result stated in [16], finishing the game for the day.

G

WAVEGUIDES: CONFINED EM WAVES

G.1 MOTIVATION

While this is not part of the course, the topic of waveguides is one of so many applications that it is worth a mention, and that will be done in this tutorial.

We will setup our system with a waveguide (conducting surface that confines the radiation) oriented in the $\hat{\mathbf{z}}$ direction. The shape can be arbitrary

PICTURE: cross section of wacky shape.

At the surface of a conductor At the surface of the conductor (I presume this means the interior surface where there is no charge or current enclosed) we have

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0
\end{aligned} \tag{G.1}$$

If we are talking about the exterior surface, do we need to make any other assumptions (perfect conductors, or constant potentials)?

Wave equations For electric and magnetic fields in vacuum, we can show easily that these, like the potentials, separately satisfy the wave equation

Taking curls of the Maxwell curl equations above we have

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2},
\end{aligned} \tag{G.2}$$

but we have for vector $\mathbf{M}$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{M}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{M}) - \Delta \mathbf{M}, \tag{G.3}$$

which gives us a pair of wave equations

$$\begin{aligned} \Box \mathbf{E} &= 0 \\ \Box \mathbf{B} &= 0. \end{aligned} \tag{G.4}$$

We still have the original constraints of Maxwell's equations to deal with, but we are free now to pick the complex exponentials as fundamental solutions, as our starting point

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{ik^a x_a} = \mathbf{E}_0 e^{i(k^0 x_0 - \mathbf{k}\cdot\mathbf{x})} \\ \mathbf{B} &= \mathbf{B}_0 e^{ik^a x_a} = \mathbf{B}_0 e^{i(k^0 x_0 - \mathbf{k}\cdot\mathbf{x})}, \end{aligned} \tag{G.5}$$

With $k_0 = \omega/c$ and $x_0 = ct$ this is

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \\ \mathbf{B} &= \mathbf{B}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}. \end{aligned} \tag{G.6}$$

For the vacuum case, with monochromatic light, we treated the amplitudes as constants. Let us see what happens if we relax this assumption, and allow for spatial dependence (but no time dependence) of $\mathbf{E}_0$ and $\mathbf{B}_0$. For the LHS of the electric field curl equation we have

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \times \mathbf{E}_0 e^{ik_a x^a} \\ &= (\boldsymbol{\nabla} \times \mathbf{E}_0 - \mathbf{E}_0 \times \boldsymbol{\nabla}) e^{ik_a x^a} \\ &= (\boldsymbol{\nabla} \times \mathbf{E}_0 - \mathbf{E}_0 \times \mathbf{e}^\alpha i k_a \partial_\alpha x^a) e^{ik_a x^a} \\ &= (\boldsymbol{\nabla} \times \mathbf{E}_0 + \mathbf{E}_0 \times \mathbf{e}^\alpha i k^a {\delta_\alpha}^a) e^{ik_a x^a} \\ &= (\boldsymbol{\nabla} \times \mathbf{E}_0 + i \mathbf{E}_0 \times \mathbf{k}) e^{ik_a x^a}. \end{aligned} \tag{G.7}$$

Similarly for the divergence we have

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \cdot \mathbf{E}_0 e^{ik_a x^a} \\ &= (\boldsymbol{\nabla} \cdot \mathbf{E}_0 + \mathbf{E}_0 \cdot \boldsymbol{\nabla}) e^{ik_a x^a} \\ &= (\boldsymbol{\nabla} \cdot \mathbf{E}_0 + \mathbf{E}_0 \cdot \mathbf{e}^\alpha i k_a \partial_\alpha x^a) e^{ik_a x^a} \\ &= (\boldsymbol{\nabla} \cdot \mathbf{E}_0 - \mathbf{E}_0 \cdot \mathbf{e}^\alpha i k^a {\delta_\alpha}^a) e^{ik_a x^a} \\ &= (\boldsymbol{\nabla} \cdot \mathbf{E}_0 - i \mathbf{k} \cdot \mathbf{E}_0) e^{ik_a x^a}. \end{aligned} \tag{G.8}$$

This provides constraints on the amplitudes

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{E}_0 - i\mathbf{k} \times \mathbf{E}_0 &= -i\frac{\omega}{c}\mathbf{B}_0 \\
\boldsymbol{\nabla} \times \mathbf{B}_0 - i\mathbf{k} \times \mathbf{B}_0 &= i\frac{\omega}{c}\mathbf{E}_0 \\
\boldsymbol{\nabla} \cdot \mathbf{E}_0 - i\mathbf{k} \cdot \mathbf{E}_0 &= 0 \\
\boldsymbol{\nabla} \cdot \mathbf{B}_0 - i\mathbf{k} \cdot \mathbf{B}_0 &= 0
\end{aligned}
\tag{G.9}
$$

Applying the wave equation operator to our phasor we get

$$
\begin{aligned}
0 &= \left(\frac{1}{c^2}\partial_{tt} - \boldsymbol{\nabla}^2\right)\mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \\
&= \left(-\frac{\omega^2}{c^2} - \boldsymbol{\nabla}^2 + \mathbf{k}^2\right)\mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}
\end{aligned}
\tag{G.10}
$$

So the momentum space equivalents of the wave equations are

$$
\begin{aligned}
\left(\boldsymbol{\nabla}^2 + \frac{\omega^2}{c^2} - \mathbf{k}^2\right)\mathbf{E}_0 &= 0 \\
\left(\boldsymbol{\nabla}^2 + \frac{\omega^2}{c^2} - \mathbf{k}^2\right)\mathbf{B}_0 &= 0.
\end{aligned}
\tag{G.11}
$$

Observe that if $c^2\mathbf{k}^2 = \omega^2$, then these amplitudes are harmonic functions (solutions to the Laplacian equation). However, it does not appear that we require such a light like relation for the four vector $k^a = (\omega/c, \mathbf{k})$.

G.2 BACK TO THE TUTORIAL NOTES

In class we went straight to an assumed solution of the form

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_0(x, y)e^{i(\omega t - kz)} \\
\mathbf{B} &= \mathbf{B}_0(x, y)e^{i(\omega t - kz)},
\end{aligned}
\tag{G.12}
$$

where $\mathbf{k} = k\hat{\mathbf{z}}$. Our Laplacian was also written as the sum of components in the propagation and perpendicular directions

$$
\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial {x_\perp}^2} + \frac{\partial^2}{\partial z^2}.
\tag{G.13}
$$

With no z dependence in the amplitudes we have

$$\begin{aligned}\left(\frac{\partial^2}{\partial {x_\perp}^2} + \frac{\omega^2}{c^2} - \mathbf{k}^2\right)\mathbf{E}_0 &= 0\\ \left(\frac{\partial^2}{\partial {x_\perp}^2} + \frac{\omega^2}{c^2} - \mathbf{k}^2\right)\mathbf{B}_0 &= 0.\end{aligned} \tag{G.14}$$

G.3 SEPARATION INTO COMPONENTS

It was left as an exercise to separate out our Maxwell equations, so that our field components $\mathbf{E}_0 = \mathbf{E}_\perp + \mathbf{E}_z$ and $\mathbf{B}_0 = \mathbf{B}_\perp + \mathbf{B}_z$ in the propagation direction, and components in the perpendicular direction are separated

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{E}_0 &= (\boldsymbol{\nabla}_\perp + \hat{\mathbf{z}}\partial_z) \times \mathbf{E}_0\\ &= \boldsymbol{\nabla}_\perp \times \mathbf{E}_0\\ &= \boldsymbol{\nabla}_\perp \times (\mathbf{E}_\perp + \mathbf{E}_z)\\ &= \boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp + \boldsymbol{\nabla}_\perp \times \mathbf{E}_z\\ &= (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y) \times (\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y) + \boldsymbol{\nabla}_\perp \times \mathbf{E}_z\\ &= \hat{\mathbf{z}}(\partial_x E_y - \partial_z E_z) + \boldsymbol{\nabla}_\perp \times \mathbf{E}_z.\end{aligned} \tag{G.15}$$

We can do something similar for $\mathbf{B}_0$. This allows for a split of eq. (G.9) into $\hat{\mathbf{z}}$ and perpendicular components

$$\begin{aligned}\boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp &= -i\frac{\omega}{c}\mathbf{B}_z\\ \boldsymbol{\nabla}_\perp \times \mathbf{B}_\perp &= i\frac{\omega}{c}\mathbf{E}_z\\ \boldsymbol{\nabla}_\perp \times \mathbf{E}_z - i\mathbf{k} \times \mathbf{E}_\perp &= -i\frac{\omega}{c}\mathbf{B}_\perp\\ \boldsymbol{\nabla}_\perp \times \mathbf{B}_z - i\mathbf{k} \times \mathbf{B}_\perp &= i\frac{\omega}{c}\mathbf{E}_\perp\\ \boldsymbol{\nabla}_\perp \cdot \mathbf{E}_\perp &= ikE_z - \partial_z E_z\\ \boldsymbol{\nabla}_\perp \cdot \mathbf{B}_\perp &= ikB_z - \partial_z B_z.\end{aligned} \tag{G.16}$$

So we see that once we have a solution for $\mathbf{E}_z$ and $\mathbf{B}_z$ (by solving the wave equation above for those components), the components for the fields in terms of those components can be found. Alternately, if one solves for the perpendicular components of the fields, these propagation components are available immediately with only differentiation.

In the case where the perpendicular components are taken as given

$$
\begin{aligned}
\mathbf{B}_z &= i\frac{c}{\omega}\boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp \\
\mathbf{E}_z &= -i\frac{c}{\omega}\boldsymbol{\nabla}_\perp \times \mathbf{B}_\perp,
\end{aligned}
\tag{G.17}
$$

we can express the remaining ones strictly in terms of the perpendicular fields

$$
\begin{aligned}
\frac{\omega}{c}\mathbf{B}_\perp &= \frac{c}{\omega}\boldsymbol{\nabla}_\perp \times (\boldsymbol{\nabla}_\perp \times \mathbf{B}_\perp) + \mathbf{k} \times \mathbf{E}_\perp \\
\frac{\omega}{c}\mathbf{E}_\perp &= \frac{c}{\omega}\boldsymbol{\nabla}_\perp \times (\boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp) - \mathbf{k} \times \mathbf{B}_\perp \\
\boldsymbol{\nabla}_\perp \cdot \mathbf{E}_\perp &= -i\frac{c}{\omega}(ik - \partial_z)\hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_\perp \times \mathbf{B}_\perp) \\
\boldsymbol{\nabla}_\perp \cdot \mathbf{B}_\perp &= i\frac{c}{\omega}(ik - \partial_z)\hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp).
\end{aligned}
\tag{G.18}
$$

Is it at all helpful to expand the double cross products?

$$
\begin{aligned}
\frac{\omega^2}{c^2}\mathbf{B}_\perp &= \boldsymbol{\nabla}_\perp(\boldsymbol{\nabla}_\perp \cdot \mathbf{B}_\perp) - {\boldsymbol{\nabla}_\perp}^2\mathbf{B}_\perp + \frac{\omega}{c}\mathbf{k} \times \mathbf{E}_\perp \\
&= i\frac{c}{\omega}(ik - \partial_z)\boldsymbol{\nabla}_\perp\hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp) - {\boldsymbol{\nabla}_\perp}^2\mathbf{B}_\perp + \frac{\omega}{c}\mathbf{k} \times \mathbf{E}_\perp
\end{aligned}
\tag{G.19}
$$

This gives us

$$
\begin{aligned}
\left({\boldsymbol{\nabla}_\perp}^2 + \frac{\omega^2}{c^2}\right)\mathbf{B}_\perp &= -\frac{c}{\omega}(k + i\partial_z)\boldsymbol{\nabla}_\perp\hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp) + \frac{\omega}{c}\mathbf{k} \times \mathbf{E}_\perp \\
\left({\boldsymbol{\nabla}_\perp}^2 + \frac{\omega^2}{c^2}\right)\mathbf{E}_\perp &= -\frac{c}{\omega}(k + i\partial_z)\boldsymbol{\nabla}_\perp\hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_\perp \times \mathbf{B}_\perp) - \frac{\omega}{c}\mathbf{k} \times \mathbf{B}_\perp,
\end{aligned}
\tag{G.20}
$$

but that does not seem particularly useful for completely solving the system? It appears fairly messy to try to solve for $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ given the propagation direction fields. I wonder if there is a simplification available that I am missing?

G.4 SOLVING THE MOMENTUM SPACE WAVE EQUATIONS

Back to the class notes. We proceeded to solve for $\mathbf{E}_z$ and $\mathbf{B}_z$ from the wave equations by separation of variables. We wish to solve equations of the form

$$
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - \mathbf{k}^2\right)\phi(x, y) = 0
\tag{G.21}
$$

Write $\phi(x, y) = X(x)Y(y)$, so that we have

$$\frac{X''}{X} + \frac{Y''}{Y} = \mathbf{k}^2 - \frac{\omega^2}{c^2} \tag{G.22}$$

One solution is sinusoidal

$$\begin{aligned} \frac{X''}{X} &= -k_1^2 \\ \frac{Y''}{Y} &= -k_2^2 \\ -k_1^2 - k_2^2 &= \mathbf{k}^2 - \frac{\omega^2}{c^2}. \end{aligned} \tag{G.23}$$

The example in the tutorial now switched to a rectangular waveguide, still oriented with the propagation direction down the z-axis, but with lengths a and b along the x and y axis respectively.

Writing $k_1 = 2\pi m/a$, and $k_2 = 2\pi n/b$, we have

$$\phi(x, y) = \sum_{mn} a_{mn} \exp\left(\frac{2\pi i m}{a} x\right) \exp\left(\frac{2\pi i n}{b} y\right) \tag{G.24}$$

We were also provided with some definitions

Definition G.1

TE (Transverse Electric)
$\mathbf{E}_3 = 0.$

Definition G.2

TM (Transverse Magnetic)
$\mathbf{B}_3 = 0.$

Definition G.3

TEM (Transverse Electromagnetic)
$\mathbf{E}_3 = \mathbf{B}_3 = 0.$

claim: TEM do not existing in a hollow waveguide.

Why: I had in my notes

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} = 0 &\implies \frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} = 0 \\ \boldsymbol{\nabla} \cdot \mathbf{E} = 0 &\implies \frac{\partial E_1}{\partial x^1} + \frac{\partial E_2}{\partial x^2} = 0 \end{aligned} \tag{G.25}$$

and then

$$\begin{aligned} \boldsymbol{\nabla}^2 \phi &= 0 \\ \phi &= \text{const} \end{aligned} \tag{G.26}$$

In retrospect I fail to see how these are connected? What happened to the $\partial_t \mathbf{B}$ term in the curl equation above?

It was argued that we have $\mathbf{E}_{\|} = \mathbf{B}_{\perp} = 0$ on the boundary.

So for the TE case, where $\mathbf{E}_3 = 0$, we have from the separation of variables argument

$$\hat{\mathbf{z}} \cdot \mathbf{B}_0(x, y) = \sum_{mn} a_{mn} \cos\left(\frac{2\pi i m}{a} x\right) \cos\left(\frac{2\pi i n}{b} y\right). \tag{G.27}$$

No sines because

$$B_1 \sim \frac{\partial B_3}{\partial x_a} \rightarrow \cos(k_1 x^1). \tag{G.28}$$

The quantity

$$a_{mn} \cos\left(\frac{2\pi i m}{a} x\right) \cos\left(\frac{2\pi i n}{b} y\right). \tag{G.29}$$

is called the TE_{mn} mode. Note that since $B = \text{const}$ an ampere loop requires $\mathbf{B} = 0$ since there is no current.

Writing

$$\begin{aligned} k &= \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} \\ \omega_{mn} &= 2\pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \end{aligned} \tag{G.30}$$

Since $\omega < \omega_{mn}$ we have k purely imaginary, and the term

$$e^{-ikz} = e^{-|k|z} \tag{G.31}$$

represents the die off.
ω_{10} is the smallest.
Note that the convention is that the m in TE_{mn} is the bigger of the two indices, so $\omega > \omega_{10}$.
The phase velocity

$$V_\phi = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} \geq c \tag{G.32}$$

However, energy is transmitted with the group velocity, the ratio of the Poynting vector and energy density

$$\frac{\langle \mathbf{S} \rangle}{\langle U \rangle} = V_g = \frac{\partial \omega}{\partial k} = 1/\frac{\partial k}{\partial \omega} \tag{G.33}$$

(This can be shown).
Since

$$\left(\frac{\partial k}{\partial \omega}\right)^{-1} = \left(\frac{\partial}{\partial \omega} \sqrt{(\omega/c)^2 - (\omega_{mn}/c)^2}\right)^{-1} = c\sqrt{1 - (\omega_{mn}/\omega)^2} \leq c \tag{G.34}$$

We see that the energy is transmitted at less than the speed of light as expected.

G.5 FINAL REMARKS

I had started converting my handwritten scrawl for this tutorial into an attempt at working through these ideas with enough detail that they self contained, but gave up part way. This appears to me to be too big of a sub-discipline to give it justice in one hours class. As is, it is enough to at least get an concept of some of the ideas involved. I think were I to learn this for real, I had need a good text as a reference (or the time to attempt to blunder through the ideas in much much more detail).

H

THREE DIMENSIONAL DIVERGENCE THEOREM WITH GENERALLY PARAMETRIZED VOLUME ELEMENT

With the divergence of the energy momentum tensor converted from a volume to a surface integral given by

$$\int_V d^3\mathbf{x} \partial_\beta T^{\beta\alpha} = \oint_{\partial V} d^2\sigma^\beta T^{\beta\alpha}, \tag{H.1}$$

I got to wondering what a closed form algebraic expression for this curious (and foreign seeming) quantity $d^2\sigma^\beta$ was. It obviously must be related to the normal to the surface. It seemed to me that a natural way to answer this question was to consider this divergence integral over an arbitrarily parametrized volume. This turns out to be overkill, but a useful seeming digression.

H.1 A GENERALLY PARAMETRIZED PARALLELEPIPED VOLUME ELEMENT

Suppose we parametrize a volume by specifying that all the points in that volume are covered by the position vector from the origin, given by

$$\mathbf{x} = \mathbf{x}(a_1, a_2, a_3). \tag{H.2}$$

At any point in the volume of interest, we can create a level curve, holding two of the parameters a_α constant, and varying the remaining one. In particular, we can construct three direction vectors along these level curves, one for each parameter not held constant

$$\begin{aligned} d\mathbf{x}_1 &= da_1 \frac{\partial \mathbf{x}}{\partial a_1} \\ d\mathbf{x}_2 &= da_2 \frac{\partial \mathbf{x}}{\partial a_2} \\ d\mathbf{x}_3 &= da_3 \frac{\partial \mathbf{x}}{\partial a_3} \end{aligned} \tag{H.3}$$

The span of these vectors, provided they are non-degenerate, forms a parallelepiped, the volume of which is

$$d^3\mathbf{x} = d\mathbf{x}_3 \cdot (d\mathbf{x}_1 \times d\mathbf{x}_2). \tag{H.4}$$

This volume element can be expanded in a number of ways

$$
\begin{aligned}
d^3\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial a_1} \cdot \left(\frac{\partial \mathbf{x}}{\partial a_2} \times \frac{\partial \mathbf{x}}{\partial a_3} \right) \\
&= \frac{\partial x^\alpha}{\partial a_1} \frac{\partial x^\beta}{\partial a_2} \frac{\partial x^\gamma}{\partial a_3} \epsilon_{\alpha\beta\gamma} da_1 da_2 da_3 \\
&= \frac{\partial x^1}{\partial a_\alpha} \frac{\partial x^2}{\partial a_\beta} \frac{\partial x^3}{\partial a_\gamma} \epsilon_{\alpha\beta\gamma} da_1 da_2 da_3 \\
&= \frac{\partial x^1}{\partial a_{[1}} \frac{\partial x^2}{\partial a_2} \frac{\partial x^3}{\partial a_{3]}} da_1 da_2 da_2 \\
&= \left| \frac{\partial (x^1, x^2, x^3)}{\partial (a_1, a_2, a_3)} \right| da_1 da_2 da_3
\end{aligned}
\tag{H.5}
$$

where the Jacobian determinant is given by

$$
\left| \frac{\partial (x^1, x^2, x^3)}{\partial (a_1, a_2, a_3)} \right| = \begin{vmatrix} \frac{\partial x^1}{\partial a_1} & \frac{\partial x^2}{\partial a_1} & \frac{\partial x^3}{\partial a_1} \\ \frac{\partial x^1}{\partial a_2} & \frac{\partial x^2}{\partial a_2} & \frac{\partial x^3}{\partial a_2} \\ \frac{\partial x^1}{\partial a_3} & \frac{\partial x^2}{\partial a_3} & \frac{\partial x^3}{\partial a_3} \end{vmatrix}.
\tag{H.6}
$$

Provided we are interested in a volume for which the sign of this Jacobian determinant does not change sign, our task is to evaluate and reduce the integral

$$
\int \left| \frac{\partial (x^1, x^2, x^3)}{\partial (a_1, a_2, a_3)} \right| da_1 da_2 da_3 \frac{\partial T^{\beta\alpha}}{\partial x^\beta}
\tag{H.7}
$$

to a set (and sum of) two dimensional integrals.

H.2 ON THE GEOMETRY OF THE SURFACES

Suppose that we integrate over the ranges $[a_{1-}, a_{1+}]$, $[a_{2-}, a_{2+}]$, $[a_{3-}, a_{3+}]$. Observe that the outwards normals along the $a_1 = a_1+$ face is $d\mathbf{n}_{1+} = da_2 da_3 \partial \mathbf{x}/\partial a_2 \times \partial \mathbf{x}/\partial a_3$. This is

$$
d\mathbf{n}_{1+} = da_2 da_3 \frac{\partial \mathbf{x}}{\partial a_2} \times \frac{\partial \mathbf{x}}{\partial a_3} = da_2 da_3 \frac{\partial x^\mu}{\partial a_2} \frac{\partial x^\nu}{\partial a_3} \epsilon_{\mu\nu\gamma} \mathbf{e}_\gamma
\tag{H.8}
$$

Similarly our normal on the $a_2 = a_{2+}$ face is

$$
d\mathbf{n}_{2+} = da_3 da_1 \frac{\partial \mathbf{x}}{\partial a_3} \times \frac{\partial \mathbf{x}}{\partial a_1} = da_3 da_1 \frac{\partial x^\mu}{\partial a_3} \frac{\partial x^\nu}{\partial a_1} \epsilon_{\mu\nu\gamma} \mathbf{e}_\gamma,
\tag{H.9}
$$

and on the $a_3 = a_{3+}$ face the outward normal is

$$d\mathbf{n}_{3+} = da_1 da_2 \frac{\partial \mathbf{x}}{\partial a_1} \times \frac{\partial \mathbf{x}}{\partial a_2} = da_1 da_2 \frac{\partial x^\mu}{\partial a_1} \frac{\partial x^\nu}{\partial a_2} \epsilon_{\mu\nu\gamma} \mathbf{e}_\gamma. \tag{H.10}$$

Along the $a_{\alpha-}$ faces these are just negated. We can summarize these as

$$d\mathbf{n}_{\sigma\pm} = \pm \frac{1}{2!} da_\alpha da_\beta \frac{\partial \mathbf{x}}{\partial a_\alpha} \times \frac{\partial \mathbf{x}}{\partial a_\beta} \epsilon_{\alpha\beta\sigma} = \pm \frac{1}{2!} da_\alpha da_\beta \frac{\partial x^\mu}{\partial a_\alpha} \frac{\partial x^\nu}{\partial a_\beta} \epsilon_{\alpha\beta\sigma} \epsilon_{\mu\nu\gamma} \mathbf{e}_\gamma \tag{H.11}$$

H.3 EXPANSION OF THE JACOBIAN DETERMINANT

Suppose, to start with, our divergence volume integral eq. (H.7) has just the following term

$$\int d^3\mathbf{x} \partial_3 M. \tag{H.12}$$

The specifics of how the scalar $M = T^{3\alpha}$ is indexed will not matter yet, so let us suppress it. The Jacobian determinant can be expanded along the $\frac{\partial x^3}{\partial a_\alpha}$ column for

$$
\begin{aligned}
\int d^3\mathbf{x}\partial_3 M &= \int da_1 da_2 da_3 \left|\frac{\partial(x^1, x^2, x^3)}{\partial(a_1, a_2, a_3)}\right| \frac{\partial M}{\partial x^3} \\
&= \int da_1 da_2 da_3 \left(\frac{\partial x^1}{\partial a_{[1}} \frac{\partial x^2}{\partial a_2} \frac{\partial x^3}{\partial a_{3]}}\right) \frac{\partial M}{\partial x^3} \\
&= \int da_1 da_2 da_3 \left(\frac{\partial x^1}{\partial a_{[1}} \frac{\partial x^2}{\partial a_{2]}} \frac{\partial x^3}{\partial a_3} + \frac{\partial x^1}{\partial a_{[2}} \frac{\partial x^2}{\partial a_{3]}} \frac{\partial x^3}{\partial a_1} + \frac{\partial x^1}{\partial a_{[3}} \frac{\partial x^2}{\partial a_{1]}} \frac{\partial x^3}{\partial a_2}\right) \frac{\partial M}{\partial x^3} \\
&= \int da_1 da_2 da_3 \left(\left|\frac{\partial(x^1, x^2)}{\partial(a_1, a_2)}\right| \frac{\partial x^3}{\partial a_3} + \left|\frac{\partial(x^1, x^2)}{\partial(a_2, a_3)}\right| \frac{\partial x^3}{\partial a_1} + \left|\frac{\partial(x^1, x^2)}{\partial(a_3, a_1)}\right| \frac{\partial x^3}{\partial a_2}\right) \frac{\partial M}{\partial x^3} \\
&= \int da_1 da_2 \left|\frac{\partial(x^1, x^2)}{\partial(a_1, a_2)}\right| \int da_3 \frac{\partial x^3}{\partial a_3} \frac{\partial M}{\partial x^3} \\
&\quad + \int da_2 da_3 \left|\frac{\partial(x^1, x^2)}{\partial(a_2, a_3)}\right| \int da_1 \frac{\partial x^3}{\partial a_1} \frac{\partial M}{\partial x^3} \\
&\quad + \int da_3 da_1 \left|\frac{\partial(x^1, x^2)}{\partial(a_3, a_1)}\right| \int da_2 \frac{\partial x^3}{\partial a_2} \frac{\partial M}{\partial x^3} \\
&= \int da_1 da_2 \left|\frac{\partial(x^1, x^2)}{\partial(a_1, a_2)}\right| \int da_3 \frac{\partial M}{\partial a_3} \\
&\quad + \int da_2 da_3 \left|\frac{\partial(x^1, x^2)}{\partial(a_2, a_3)}\right| \int da_1 \frac{\partial M}{\partial a_1} \\
&\quad + \int da_3 da_1 \left|\frac{\partial(x^1, x^2)}{\partial(a_3, a_1)}\right| \int da_2 \frac{\partial M}{\partial a_2} \\
&= \int da_1 da_2 \left|\frac{\partial(x^1, x^2)}{\partial(a_1, a_2)}\right| \big(M(a_{3+}) - M(a_{3+})\big) \\
&\quad + \int da_2 da_3 \left|\frac{\partial(x^1, x^2)}{\partial(a_2, a_3)}\right| \big(M(a_{1+}) - M(a_{1+})\big) \\
&\quad + \int da_3 da_1 \left|\frac{\partial(x^1, x^2)}{\partial(a_3, a_1)}\right| \big(M(a_{2+}) - M(a_{2+})\big)
\end{aligned}
\tag{H.13}
$$

Performing the same task (really just performing cyclic permutation of indices) we can now construct the whole divergence integral

$$\begin{aligned}
\int d^3\mathbf{x}\partial_\beta T^{\beta\alpha} = \int da_1 da_2 \left|\frac{\partial(x^1,x^2)}{\partial(a_1,a_2)}\right|\left(T^{3\alpha}(a_{3+}) - T^{3\alpha}(a_{3+})\right) \\
+ \int da_2 da_3 \left|\frac{\partial(x^1,x^2)}{\partial(a_2,a_3)}\right|\left(T^{3\alpha}(a_{1+}) - T^{3\alpha}(a_{1+})\right) \\
+ \int da_3 da_1 \left|\frac{\partial(x^1,x^2)}{\partial(a_3,a_1)}\right|\left(T^{3\alpha}(a_{2+}) - T^{3\alpha}(a_{2+})\right) \\
+ \int da_1 da_2 \left|\frac{\partial(x^2,x^3)}{\partial(a_1,a_2)}\right|\left(T^{1\alpha}(a_{3+}) - T^{1\alpha}(a_{3+})\right) \\
+ \int da_2 da_3 \left|\frac{\partial(x^2,x^3)}{\partial(a_2,a_3)}\right|\left(T^{1\alpha}(a_{1+}) - T^{1\alpha}(a_{1+})\right) \\
+ \int da_3 da_1 \left|\frac{\partial(x^2,x^3)}{\partial(a_3,a_1)}\right|\left(T^{1\alpha}(a_{2+}) - T^{1\alpha}(a_{2+})\right) \\
+ \int da_1 da_2 \left|\frac{\partial(x^3,x^1)}{\partial(a_1,a_2)}\right|\left(T^{2\alpha}(a_{3+}) - T^{2\alpha}(a_{3+})\right) \\
+ \int da_2 da_3 \left|\frac{\partial(x^3,x^1)}{\partial(a_2,a_3)}\right|\left(T^{2\alpha}(a_{1+}) - T^{2\alpha}(a_{1+})\right) \\
+ \int da_3 da_1 \left|\frac{\partial(x^3,x^1)}{\partial(a_3,a_1)}\right|\left(T^{2\alpha}(a_{2+}) - T^{2\alpha}(a_{2+})\right).
\end{aligned} \tag{H.14}$$

Regrouping we have

$$\begin{aligned}
\int d^3\mathbf{x}\partial_\beta T^{\beta\alpha} = \int da_1 da_2 \left(\left|\frac{\partial(x^1,x^2)}{\partial(a_1,a_2)}\right| T^{3\alpha}\Big|_{\Delta a_3} + \left|\frac{\partial(x^2,x^3)}{\partial(a_1,a_2)}\right| T^{1\alpha}\Big|_{\Delta a_3} + \left|\frac{\partial(x^3,x^1)}{\partial(a_1,a_2)}\right| T^{2\alpha}\Big|_{\Delta a_3}\right) \\
+ \int da_2 da_3 \left(\left|\frac{\partial(x^1,x^2)}{\partial(a_2,a_3)}\right| T^{3\alpha}\Big|_{\Delta a_3} + \left|\frac{\partial(x^2,x^3)}{\partial(a_2,a_3)}\right| T^{1\alpha}\Big|_{\Delta a_3} + \left|\frac{\partial(x^3,x^1)}{\partial(a_2,a_3)}\right| T^{2\alpha}\Big|_{\Delta a_3}\right) \\
+ \int da_3 da_1 \left(\left|\frac{\partial(x^1,x^2)}{\partial(a_3,a_1)}\right| T^{3\alpha}\Big|_{\Delta a_3} + \left|\frac{\partial(x^2,x^3)}{\partial(a_3,a_1)}\right| T^{1\alpha}\Big|_{\Delta a_3} + \left|\frac{\partial(x^3,x^1)}{\partial(a_3,a_1)}\right| T^{2\alpha}\Big|_{\Delta a_3}\right).
\end{aligned} \tag{H.15}$$

Observe that we can factor these sums utilizing the normals for the parallelepiped volume element

$$\begin{aligned}\int d^3\mathbf{x}\partial_\beta T^{\beta\alpha} &= \int da_1 da_2 \left|\frac{\partial(x^\mu, x^\nu)}{\partial(a_1, a_2)}\right| \epsilon_{\mu\nu\gamma}\mathbf{e}_\gamma \cdot \mathbf{e}_\beta T^{\beta\alpha}\Big|_{\Delta a_3} \\ &+ \int da_2 da_3 \left|\frac{\partial(x^\mu, x^\nu)}{\partial(a_2, a_3)}\right| \epsilon_{\mu\nu\gamma}\mathbf{e}_\gamma \cdot \mathbf{e}_\beta T^{\beta\alpha}\Big|_{\Delta a_1} \\ &+ \int da_3 da_1 \left|\frac{\partial(x^\mu, x^\nu)}{\partial(a_3, a_1)}\right| \epsilon_{\mu\nu\gamma}\mathbf{e}_\gamma \cdot \mathbf{e}_\beta T^{\beta\alpha}\Big|_{\Delta a_2}\end{aligned} \tag{H.16}$$

Let us look at the first of these integrals in more detail. We integrate the values of the $\mathbf{e}_\beta T^{\beta\alpha}$ evaluated on the points of the surface for which $a_3 = a_{3+}$. To perform this integral we dot against the outward normal area element

$$da_1 da_2 \partial x^\mu / \partial a_1 \partial x^\nu / \partial a_2 \epsilon_{\mu\nu\gamma}\mathbf{e}_\gamma. \tag{H.17}$$

We do the same, but subtract the integral where $\mathbf{e}_\beta T^{\beta\alpha}$ is evaluated on the surface $a_3 = a_{3-}$, where we dot with the area element that has the inwards normal direction on that surface. This is then done for each of the surfaces of the parallelepiped that we are integrating over.

In terms of the outwards (area scaled) normals $d\mathbf{n}_3, d\mathbf{n}_1, d\mathbf{n}_2$ on the a_{3+}, a_{1+} and a_{2+} surfaces respectively we can write

$$\int d^3\mathbf{x}\partial_\beta T^{\beta\alpha} = \int d\mathbf{n}_3 \cdot \mathbf{e}_\beta T^\beta \alpha\Big|_{\Delta a_3} + \int d\mathbf{n}_1 \cdot \mathbf{e}_\beta T^\beta \alpha\Big|_{\Delta a_1} + \int d\mathbf{n}_2 \cdot \mathbf{e}_\beta T^\beta \alpha\Big|_{\Delta a_2}. \tag{H.18}$$

This can be written more concisely in index form with

$$d^2\sigma^\beta = \epsilon_{\mu\nu\beta}\left(\frac{\partial x^\mu}{\partial a_2}\frac{\partial x^\nu}{\partial a_3} da_2 da_3 + \frac{\partial x^\mu}{\partial a_3}\frac{\partial x^\nu}{\partial a_1} da_3 da_1 + \frac{\partial x^\mu}{\partial a_1}\frac{\partial x^\nu}{\partial a_2} da_1 da_2\right), \tag{H.19}$$

so that the divergence integral is just

$$\int d^3\mathbf{x} = \int_{\text{over level surfaces } a_{1+},\, a_{2+},\, a_{3+}} d^2\sigma^\beta T^{\beta\alpha} - \int_{\text{over level surfaces } a_{1-},\, a_{2-},\, a_{3-}} d^2\sigma^\beta T^{\beta\alpha} \tag{H.20}$$

In each case, for the $a_{\alpha-}$ surfaces, our negated inwards normal form can be redefined so that we integrate over only the outwards normal directions, and we can use the oriented integral notation

$$\int d^3\mathbf{x} = \oint d^2\sigma^\beta T^{\beta\alpha}, \tag{H.21}$$

To encode (or imply) whether we require a positive or negative sign on the area element tensor of eq. (H.19) for the surface in question.

H.4 A LOOK BACK, AND LOOKING FORWARD

Now, having performed this long winded calculation, the meaning of $d^2\sigma^\beta$ becomes clear. What is also clear is how this could have been arrived at directly utilizing the divergence theorem in its normal vector form. We had only to re-write our equation as a vector equation in terms of the gradient

$$\int_V d^3\mathbf{x}\frac{\partial T^{\beta\alpha}}{\partial x^\alpha} = \int_V d^3\mathbf{x}\boldsymbol{\nabla}\cdot(\mathbf{e}_\beta T^{\beta\alpha}) = \int_{\partial V} dA\mathbf{n}\cdot\mathbf{e}_\beta T^{\beta\alpha} \tag{H.22}$$

From this we see directly that $d^2\sigma^\beta = dA\mathbf{n}\cdot\mathbf{e}_\beta$.

Despite there being an easier way to find the form of $d^2\sigma^\beta$, I still consider this a worthwhile exercise. It hints how one could generalize the arguments to the higher dimensional cases. The main task would be to construct the normals to the hypersurfaces bounding the hypervolume, and how to do this algebraically utilizing determinants may not be too hard (since we want a Jacobian determinant as the hypervolume element in the “volume” integral). We also got more than the normal physics text book proof of the divergence theorem for Cartesian coordinates, and did it here for a general parametrization. This was not a complete argument since we did not consider a general surface, broken down into a triangular mesh. We really want volume elements with triangular sides instead of parallelograms.

I

EM FIELDS FROM MAGNETIC DIPOLE CURRENT

I.1 REVIEW

Recall for the electric dipole we started with a system like

$$
\begin{aligned}
z_+ &= 0 \\
z_- &= \mathbf{e}_3(z_0 + a\sin(\omega t))
\end{aligned}
\tag{I.1}
$$

(we did it with the opposite polarity)

$$
\begin{aligned}
\mathbf{E} &= \frac{qa\omega^2}{c^2}\sin\omega t_o \sin\theta \frac{1}{|\mathbf{x}|}(-\hat{\boldsymbol{\theta}}) = \frac{1}{c^2|\mathbf{x}|}(\ddot{\mathbf{d}}(t_r)\times\hat{\mathbf{r}})\times\hat{\mathbf{r}} \\
\mathbf{B} &= -\frac{qa\omega^2}{c^2}\sin\omega t_o \sin\theta \frac{1}{|\mathbf{x}|}(-\hat{\boldsymbol{\phi}}) = \hat{\mathbf{r}}\times\mathbf{E}.
\end{aligned}
\tag{I.2}
$$

This was after the multipole expansion ($\lambda \gg l$).

Physical analogy: a high and low frequency wave interacting. The low frequency wave becomes the envelope, and does not really "see" the dynamics of the high frequency wave.

We also figured out the Poynting vector was

$$
\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \hat{\mathbf{r}}\frac{\sin^2\theta\left|\ddot{\mathbf{d}}(t_r)\right|^2}{4\pi c^3|\mathbf{x}|^2},
\tag{I.3}
$$

and our Power was

$$
\text{Power}(R) = \oint_{S_R{}^2} d^2\boldsymbol{\sigma}\cdot\langle\mathbf{S}\rangle = \frac{q^2a^2\omega^4}{3c^3}.
\tag{I.4}
$$

I.2 MAGNETIC DIPOLE

PICTURE: positively oriented current I circulating around the normal $\mathbf{m}$ at radius b in the x-y plane. We have

(from third year)

$$|\mathbf{m}| = I\pi b^2. \tag{I.5}$$

With the magnetic moment directed upwards along the z-axis

$$\mathbf{m} = I\pi b^2 \mathbf{e}_3, \tag{I.6}$$

where we have a frequency dependence in the current

$$I = I_o \sin(\omega t). \tag{I.7}$$

With no static charge distribution we have zero scalar potential

$$\rho = 0 \implies A^0 = 0. \tag{I.8}$$

Our first moments approximation of the vector potential was

$$A^\alpha(\mathbf{x}, t) \approx \frac{1}{c|\mathbf{x}|} \int d^3\mathbf{x}' \, j^\alpha(\mathbf{x}', t) + O(\text{higher moments}). \tag{I.9}$$

Now we use our new trick introducing a $1 = 1$ to rewrite the current

$$\left(\frac{\partial x'^\alpha}{\partial x'^\beta}\right) j^\beta = {\delta^\alpha}_\beta j^\beta = j^\alpha, \tag{I.10}$$

or equivalently

$$\boldsymbol{\nabla} x^\alpha = \mathbf{e}_\alpha. \tag{I.11}$$

Carrying out the trickery we have

$$
\begin{aligned}
A^\alpha &= \frac{1}{c|\mathbf{x}|}\int d^3\mathbf{x}' (\boldsymbol{\nabla}' x'^\alpha) \cdot \mathbf{J}(\mathbf{x}', t_r) \\
&= \frac{1}{c|\mathbf{x}|}\int d^3\mathbf{x}' (\partial_{\beta'} x'^\alpha) j^\beta(\mathbf{x}', t_r) \\
&= \frac{1}{c|\mathbf{x}|}\int d^3\mathbf{x}' (\partial_{\beta'} (x'^\alpha j^\beta(\mathbf{x}', t_r)) - x'^\alpha (\overbrace{\boxed{\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{x}', t_r)}}^{= -\partial_0 \rho = 0})) \\
&= \frac{1}{c|\mathbf{x}|}\int d^3\mathbf{x}' \boldsymbol{\nabla}' \cdot (x'^\alpha \mathbf{J}) \\
&= \oint_{S_R{}^2} d^2\boldsymbol{\sigma} \cdot (x'^\alpha \mathbf{J}) \\
&= 0.
\end{aligned}
\tag{I.12}
$$

We see that the first order approximation is insufficient to calculate the vector potential for the magnetic dipole system, and that we have

$$
A^\alpha = 0 + \text{higher moments} \tag{I.13}
$$

Looking back to what we would done in class, we would also dropped this term of the vector potential, using the same arguments. What we had left was

$$
\mathbf{A}(\mathbf{x}, t) = \frac{1}{c|\mathbf{x}|}\dot{\mathbf{d}}\left(t - \frac{|\mathbf{x}|}{c}\right) = \frac{1}{c|\mathbf{x}|}\int d^3\mathbf{x}' x'^\alpha \frac{\partial}{\partial t}\rho\left(\mathbf{x}', t - \frac{|\mathbf{x}|}{c}\right), \tag{I.14}
$$

but that additional term is also zero in this magnetic dipole system since we have no static charge distribution.

There are two options to resolve this

1. calculate $\mathbf{A}$ using higher order moments $\lambda \gg b$. Go to next order in b/λ.

   This is complicated!

2. Use EM dualities (the slick way!)

Recall that Maxwell's equations are

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J}
\end{aligned} \tag{I.15}$$

If $j^i = 0$, then taking $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{E}$ we get the same equations.
Introduce dual charges ρ_m and $\mathbf{J}_m$

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho_e \\
\nabla \cdot \mathbf{B} &= 4\pi\rho_m \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + 4\pi\mathbf{J}_m \\
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J}_e
\end{aligned} \tag{I.16}$$

Duality $\mathbf{E} \to \mathbf{B}$ provided $\rho_e \to \rho_m$ and $\mathbf{J}_e \to \mathbf{J}_m$, or

$$\begin{aligned}
F^{ij} &\to \tilde{F}^{ij} = \epsilon^{ijkl} F_{kl} \\
j^k &\to \tilde{j}^k
\end{aligned} \tag{I.17}$$

With radiation : the duality transformation takes the electric dipole moment to the magnetic dipole moment $\mathbf{d} \to \mathbf{m}$.

$$\begin{aligned}
\mathbf{B} &= -\frac{1}{c^2|\mathbf{x}|}(\ddot{\mathbf{m}} \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} \\
\mathbf{E} &= \hat{\mathbf{r}} \times \mathbf{B}
\end{aligned} \tag{I.18}$$

with

$$\text{Power} \sim \left\langle \left| \ddot{\mathbf{m}}^2 \right| \right\rangle \tag{I.19}$$

$$\left\langle \left| \ddot{\mathbf{m}}^2 \right| \right\rangle = \frac{1}{2}(I_o \pi b^2 \omega^2)^2 \tag{I.20}$$

where

$$I_o = \dot{q} = \omega q \tag{I.21}$$

So the power of the magnetic dipole is

$$P_m(R) = \frac{b^4 q^2 \pi^2 \omega^6}{3c^5} \tag{I.22}$$

Taking ratios of the magnetic and electric power we find

$$\begin{aligned}
\frac{P_m}{E_m} &= \frac{b^4 q^2 \pi^2 \omega^6}{b^2 q^2 \omega^4 c^2} \\
&\sim \frac{b^2 \omega^2}{c^2} \\
&= \left(\frac{b\omega}{c}\right)^2 \\
&= \left(\frac{b}{\lambda}\right)^2
\end{aligned} \tag{I.23}$$

This difference in power shows the second order moment dependence, in the $\lambda \gg b$ approximations.

FIXME: go back and review the “third year” content and see where the magnetic dipole moment came from. That is the key to this argument, since we need to see how this ends up equivalent to a pair of charges in the electric field case.

J

YUKAWA POTENTIAL NOTE

In the last part of the tutorial, the bonus question from the tutorial was covered. This was to determine the Yukawa potential from the differential equation that we found in the earlier part of the problem.

I took a couple notes about this on paper, but do not intend to write them up. Everything proceeded exactly as I would have expected them to for solving the problem (I barely finished the midterm as is, so I did not have a chance to try it). Take Fourier transforms and then evaluate the inverse Fourier integral. This is exactly what we can do for the Coulomb potential, but actually easier since we do not have to introduce anything to offset the poles (and we recover the Coulomb potential in the $M \rightarrow 0$ case).

There was one notable point in this Yukawa potential derivation, which was not obvious to me immediately

$$\tilde{\rho}(\mathbf{k}) = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \rho(\mathbf{x}) = 1. \tag{J.1}$$

However, the Fourier transform equal to unity followed straight from the definition of the potential, which was a delta function

$$\rho(x) = \int ds \delta^4(x - x(\tau)). \tag{J.2}$$

K

PROOF OF THE D'ALEMBERTIAN GREEN'S FUNCTION

Our Prof is excellent at motivating any results that he pulls out of magic hats. He is said that he is included a derivation using Fourier transforms and tricky contour integration arguments in the class notes for anybody who is interested (and for those who also know how to do contour integration). For those who do not know contour integration yet (some people are taking it concurrently), one can actually prove this by simply applying the wave equation operator to this function. This treats the delta function as a normal function that one can take the derivatives of, something that can be well defined in the context of generalized functions. Chugging ahead with this approach we have

$$\square G(\mathbf{x}, t) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\frac{\delta\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|} = \frac{\delta''\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi c^2|\mathbf{x}|} - \Delta\frac{\delta\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|}. \tag{K.1}$$

This starts things off and now things get a bit hairy. It is helpful to consider a chain rule expansion of the Laplacian

$$\begin{aligned}\Delta(uv) &= \partial_{\alpha\alpha}(uv) \\ &= \partial_\alpha(v\partial_\alpha u + u\partial_\alpha v) \\ &= (\partial_\alpha v)(\partial_\alpha u) + v\partial_{\alpha\alpha} u + (\partial_\alpha u)(\partial_\alpha v) + u\partial_{\alpha\alpha} v).\end{aligned} \tag{K.2}$$

In vector form this is

$$\Delta(uv) = u\Delta v + 2(\boldsymbol{\nabla} u)\cdot(\boldsymbol{\nabla} v) + v\Delta u. \tag{K.3}$$

Applying this to the Laplacian portion of eq. (K.1) we have

$$\begin{aligned}\Delta\frac{\delta\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|} &= \delta\left(t - \frac{|\mathbf{x}|}{c}\right)\Delta\frac{1}{4\pi|\mathbf{x}|} \\ &\quad + \left(\boldsymbol{\nabla}\frac{1}{2\pi|\mathbf{x}|}\right)\cdot\left(\boldsymbol{\nabla}\delta\left(t - \frac{|\mathbf{x}|}{c}\right)\right) + \frac{1}{4\pi|\mathbf{x}|}\Delta\delta\left(t - \frac{|\mathbf{x}|}{c}\right).\end{aligned} \tag{K.4}$$

Here we make the identification

$$\Delta\frac{1}{4\pi|\mathbf{x}|} = -\delta^3(\mathbf{x}). \tag{K.5}$$

This could be considered a given from our knowledge of electrostatics, but it is not too much work to just do so.

K.1 AN ASIDE. PROVING THE LAPLACIAN GREEN'S FUNCTION

If $-1/4\pi|\mathbf{x}|$ is a Green's function for the Laplacian, then the Laplacian of the convolution of this with a test function should recover that test function

$$\Delta \int d^3\mathbf{x}' \left(-\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) f(\mathbf{x}') = f(\mathbf{x}). \tag{K.6}$$

We can directly evaluate the LHS of this equation, following the approach in [13]. First note that the Laplacian can be pulled into the integral and operates only on the presumed Green's function. For that operation we have

$$\Delta \left(-\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \right) = -\frac{1}{4\pi} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \left| \mathbf{x} - \mathbf{x}' \right|. \tag{K.7}$$

It will be helpful to compute the gradient of various powers of $|\mathbf{x}|$

$$\begin{aligned} \boldsymbol{\nabla}|\mathbf{x}|^a &= e_\alpha \partial_\alpha (x^\beta x^\beta)^{a/2} \\ &= e_\alpha \left(\frac{a}{2} \right) 2 x^\beta {\delta_\beta}^\alpha |\mathbf{x}|^{a-2}. \end{aligned} \tag{K.8}$$

In particular we have, when $\mathbf{x} \neq 0$, this gives us

$$\begin{aligned} \boldsymbol{\nabla}|\mathbf{x}| &= \frac{\mathbf{x}}{|\mathbf{x}|} \\ \boldsymbol{\nabla} \frac{1}{|\mathbf{x}|} &= -\frac{\mathbf{x}}{|\mathbf{x}|^3} \\ \boldsymbol{\nabla} \frac{1}{|\mathbf{x}|^3} &= -3 \frac{\mathbf{x}}{|\mathbf{x}|^5}. \end{aligned} \tag{K.9}$$

For the Laplacian of $1/|\mathbf{x}|$, at the points $\mathbf{e} \neq 0$ where this is well defined we have

$$\begin{aligned}
\Delta \frac{1}{|\mathbf{x}|} &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \frac{1}{|\mathbf{x}|} \\
&= -\partial_\alpha \frac{x^\alpha}{|\mathbf{x}|^3} \\
&= -\frac{3}{|\mathbf{x}|^3} - x^\alpha \partial_\alpha \frac{1}{|\mathbf{x}|^3} \\
&= -\frac{3}{|\mathbf{x}|^3} - \mathbf{x} \cdot \boldsymbol{\nabla} \frac{1}{|\mathbf{x}|^3} \\
&= -\frac{3}{|\mathbf{x}|^3} + 3 \frac{\mathbf{x}^2}{|\mathbf{x}|^5}
\end{aligned} \tag{K.10}$$

So we have a zero. This means that the Laplacian operation

$$\Delta \int d^3\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} f(\mathbf{x}') = \lim_{\epsilon = |\mathbf{x}-\mathbf{x}'| \rightarrow 0} f(\mathbf{x}) \int d^3\mathbf{x}' \Delta \frac{1}{|\mathbf{x} - \mathbf{x}'|}, \tag{K.11}$$

can only have a value in a neighborhood of point $\mathbf{x}$. Writing $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ we have

$$\Delta \int d^3\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} f(\mathbf{x}') = \lim_{\epsilon = |\mathbf{x}-\mathbf{x}'| \rightarrow 0} f(\mathbf{x}) \int d^3\mathbf{x}' \boldsymbol{\nabla} \cdot - \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}. \tag{K.12}$$

Observing that $\boldsymbol{\nabla} \cdot f(\mathbf{x} - \mathbf{x}') = -\boldsymbol{\nabla}' f(\mathbf{x} - \mathbf{x}')$ we can put this in a form that allows for use of Stokes theorem so that we can convert this to a surface integral

$$\begin{aligned}
\Delta \int d^3\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} f(\mathbf{x}') &= \lim_{\epsilon = |\mathbf{x}-\mathbf{x}'| \rightarrow 0} f(\mathbf{x}) \int d^3\mathbf{x}' \boldsymbol{\nabla}' \cdot \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \\
&= \lim_{\epsilon = |\mathbf{x}-\mathbf{x}'| \rightarrow 0} f(\mathbf{x}) \int d^2\mathbf{x}' \mathbf{n} \cdot \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \\
&= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \epsilon^2 \sin\theta d\theta d\phi \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x} - \mathbf{x}'|} \cdot \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \\
&= -\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \epsilon^2 \sin\theta d\theta d\phi \frac{\epsilon^2}{\epsilon^4}
\end{aligned} \tag{K.13}$$

where we use $(\mathbf{x}' - \mathbf{x})/|\mathbf{x}' - \mathbf{x}|$ as the outwards normal for a sphere centered at $\mathbf{x}$ of radius ϵ. This integral is just -4π, so we have

$$\Delta \int d^3\mathbf{x}' \frac{1}{-4\pi|\mathbf{x} - \mathbf{x}'|} f(\mathbf{x}') = f(\mathbf{x}). \tag{K.14}$$

The convolution of $f(\mathbf{x})$ with $-\Delta/4\pi|\mathbf{x}|$ produces $f(\mathbf{x})$, allowing an identification of this function with a delta function, since the two have the same operational effect

$$\int d^3\mathbf{x}'\delta(\mathbf{x}-\mathbf{x}')f(\mathbf{x}') = f(\mathbf{x}). \tag{K.15}$$

K.2 RETURNING TO THE D'ALEMBERTIAN GREEN'S FUNCTION

We need two additional computations to finish the job. The first is the gradient of the delta function

$$\begin{aligned}
\boldsymbol{\nabla}\delta\left(t-\frac{|\mathbf{x}|}{c}\right) &= ? \\
\Delta\delta\left(t-\frac{|\mathbf{x}|}{c}\right) &= ?
\end{aligned} \tag{K.16}$$

Consider $\boldsymbol{\nabla} f(g(\mathbf{x}))$. This is

$$\begin{aligned}
\boldsymbol{\nabla} f(g(\mathbf{x})) &= e_\alpha \frac{\partial f(g(\mathbf{x}))}{\partial x^\alpha} \\
&= e_\alpha \frac{\partial f}{\partial g}\frac{\partial g}{\partial x^\alpha},
\end{aligned} \tag{K.17}$$

so we have

$$\boldsymbol{\nabla} f(g(\mathbf{x})) = \frac{\partial f}{\partial g}\boldsymbol{\nabla} g. \tag{K.18}$$

The Laplacian is similar

$$\begin{aligned}
\Delta f(g) &= \boldsymbol{\nabla}\cdot\left(\frac{\partial f}{\partial g}\boldsymbol{\nabla} g\right) \\
&= \partial_\alpha\left(\frac{\partial f}{\partial g}\partial_\alpha g\right) \\
&= \left(\partial_\alpha\frac{\partial f}{\partial g}\right)\partial_\alpha g + \frac{\partial f}{\partial g}\partial_{\alpha\alpha} g \\
&= \frac{\partial^2 f}{\partial g^2}\left(\partial_\alpha g\right)\left(\partial_\alpha g\right) + \frac{\partial f}{\partial g}\Delta g,
\end{aligned} \tag{K.19}$$

so we have

$$\Delta f(g) = \frac{\partial^2 f}{\partial g^2}(\boldsymbol{\nabla} g)^2 + \frac{\partial f}{\partial g}\Delta g \tag{K.20}$$

With $g(\mathbf{x}) = |\mathbf{x}|$, we will need the Laplacian of this vector magnitude

$$\begin{aligned}
\Delta|\mathbf{x}| &= \partial_\alpha \frac{x_\alpha}{|\mathbf{x}|} \\
&= \frac{3}{|\mathbf{x}|} + x_\alpha \partial_\alpha (x^\beta x^\beta)^{-1/2} \\
&= \frac{3}{|\mathbf{x}|} - \frac{x_\alpha x_\alpha}{|\mathbf{x}|^3} \\
&= \frac{2}{|\mathbf{x}|}
\end{aligned} \tag{K.21}$$

So that we have

$$\begin{aligned}
\boldsymbol{\nabla}\delta\left(t - \frac{|\mathbf{x}|}{c}\right) &= -\frac{1}{c}\delta'\left(t - \frac{|\mathbf{x}|}{c}\right)\frac{\mathbf{x}}{|\mathbf{x}|} \\
\Delta\delta\left(t - \frac{|\mathbf{x}|}{c}\right) &= \frac{1}{c^2}\delta''\left(t - \frac{|\mathbf{x}|}{c}\right) - \frac{1}{c}\delta'\left(t - \frac{|\mathbf{x}|}{c}\right)\frac{2}{|\mathbf{x}|}
\end{aligned} \tag{K.22}$$

Now we have all the bits and pieces of eq. (K.4) ready to assemble

$$\begin{aligned}
\Delta \frac{\delta\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|} &= -\delta\left(t - \frac{|\mathbf{x}|}{c}\right)\delta^3(\mathbf{x}) \\
&\quad + \frac{1}{2\pi}\left(-\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) \cdot -\frac{1}{c}\delta'\left(t - \frac{|\mathbf{x}|}{c}\right)\frac{\mathbf{x}}{|\mathbf{x}|} \\
&\quad + \frac{1}{4\pi|\mathbf{x}|}\left(\frac{1}{c^2}\delta''\left(t - \frac{|\mathbf{x}|}{c}\right) - \frac{1}{c}\delta'\left(t - \frac{|\mathbf{x}|}{c}\right)\frac{2}{|\mathbf{x}|}\right) \\
&= -\delta\left(t - \frac{|\mathbf{x}|}{c}\right)\delta^3(\mathbf{x}) + \frac{1}{4\pi|\mathbf{x}|c^2}\delta''\left(t - \frac{|\mathbf{x}|}{c}\right)
\end{aligned} \tag{K.23}$$

Since we also have

$$\frac{1}{c^2}\partial_{tt}\frac{\delta\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|} = \frac{\delta''\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|c^2} \tag{K.24}$$

The δ'' terms cancel out in the d'Alembertian, leaving just

$$\Box \frac{\delta\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|} = \delta\left(t - \frac{|\mathbf{x}|}{c}\right)\delta^3(\mathbf{x}) \tag{K.25}$$

Noting that the spatial delta function is non-zero only when $\mathbf{x} = 0$, which means $\delta(t - |\mathbf{x}|/c) = \delta(t)$ in this product, and we finally have

$$\Box \frac{\delta\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|} = \delta(t)\delta^3(\mathbf{x}) \tag{K.26}$$

We write

$$G(\mathbf{x}, t) = \frac{\delta\left(t - \frac{|\mathbf{x}|}{c}\right)}{4\pi|\mathbf{x}|}, \tag{K.27}$$

L

MATHEMATICA NOTEBOOKS

These Mathematica notebooks, some just trivial ones used to generate figures, others more elaborate, and perhaps some even polished, can be found in

https://raw.github.com/peeterjoot/mathematica/master/.

The free Wolfram CDF player, is capable of read-only viewing these notebooks to some extent.

- Apr 21, 2011 phy450/dipolePlot.nb

  plot of dipole moment

- Apr 21, 2011 phy450/ps5IntegralTakeII.nb

  Integrate x sin(a - |x|)/|x|

Part III

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Part IV

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