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Matrix form for discrete time Fourier transform

1.1 Transform pair

In [2] a verification of the discrete Fourier transform pairs was performed. A much different looking discrete Fourier transform pair is given in [1] §A.4. This transform pair samples the points at what are called the Nykvist time instants given by

$$t_k = \frac{Tk}{2N+1}, \qquad k \in [-N, \cdots N] \tag{1.1}$$

Note that the endpoints of these sampling points are not $\pm T/2$, but are instead at

$$\pm \frac{T}{2} \frac{1}{1 + 1/N}, \tag{1.2}$$

which are slightly within the interior of the $[-T/2, T/2]$ range of interest. The reason for this slightly odd seeming selection of sampling times becomes clear if one calculate the inversion relations.

Given a periodic ($\omega_0 T = 2\pi$) bandwidth limited signal evaluated only at the Nykvist times t_k,

$$\boxed{x(t_k) = \sum_{n=-N}^{N} X_n e^{j n \omega_0 t_k},} \tag{1.3}$$

assume that an inversion relation can be found. To find X_n evaluate the sum

$$\begin{aligned}\sum_{k=-N}^{N} x(t_k) e^{-j m \omega_0 t_k} &= \sum_{k=-N}^{N} \left(\sum_{n=-N}^{N} X_n e^{j n \omega_0 t_k} \right) e^{-j m \omega_0 t_k} \\ &= \sum_{n=-N}^{N} X_n \sum_{k=-N}^{N} e^{j(n-m)\omega_0 t_k}\end{aligned} \tag{1.4}$$

This interior sum has the value $2N+1$ when $n = m$. For $n \ne m$, and $a = e^{j(n-m)\frac{2\pi}{2N+1}}$, this is

$$
\begin{aligned}
\sum_{k=-N}^{N} e^{j(n-m)\omega_0 t_k} &= \sum_{k=-N}^{N} e^{j(n-m)\omega_0 \frac{Tk}{2N+1}} \\
&= \sum_{k=-N}^{N} a^k \\
&= a^{-N} \sum_{k=-N}^{N} a^{k+N} \\
&= a^{-N} \sum_{r=0}^{2N} a^r \\
&= a^{-N} \frac{a^{2N+1} - 1}{a - 1}.
\end{aligned}
\tag{1.5}
$$

Since $a^{2N+1} = e^{2\pi j(n-m)} = 1$, this sum is zero when $n \neq m$. This means that

$$
\sum_{k=-N}^{N} e^{j(n-m)\omega_0 t_k} = (2N+1)\delta_{n,m}, \tag{1.6}
$$

which provides the desired Fourier inversion relation

$$
\boxed{X_m = \frac{1}{2N+1} \sum_{k=-N}^{N} x(t_k) e^{-jm\omega_0 t_k}.} \tag{1.7}
$$

1.2 Matrix form

The discrete time Fourier transform has been seen to have the form

$$
x_k = \sum_{n=-N}^{N} X_n e^{2\pi jnk/(2N+1)} \tag{1.8a}
$$

$$
X_n = \frac{1}{2N+1} \sum_{k=-N}^{N} x_k e^{-2\pi jnk/(2N+1)}. \tag{1.8b}
$$

A matrix representation of this form is desired. Let

$$
\mathbf{x} = \begin{bmatrix} x_{-N} \\ \vdots \\ x_0 \\ \vdots \\ x_N \end{bmatrix} \tag{1.9a}
$$

$$\mathbf{X} = \begin{bmatrix} X_{-N} \\ \vdots \\ X_0 \\ \vdots \\ X_N \end{bmatrix} \tag{1.9b}$$

Equation (1.8a) written out in full is

$$\begin{aligned} x_k = X_{-N}e^{-2\pi jNk/(2N+1)} \\ + X_{1-N}e^{-2\pi j(N-1)k/(2N+1)} \\ + \cdots \\ + X_0 \\ + \cdots \\ + X_{N-1}e^{2\pi j(N-1)k/(2N+1)} \\ + X_N e^{2\pi jNk/(2N+1)} \end{aligned} \tag{1.10}$$

With $\alpha = e^{2\pi j/(2N+1)}$ the matrix form is

$$\mathbf{x} = \begin{bmatrix} \alpha^{NN} & \alpha^{(N-1)N} & \cdots & 1 & \cdots & \alpha^{-(N-1)N} & \alpha^{-NN} \\ \alpha^{N(N-1)} & \alpha^{(N-1)(N-1)} & \cdots & 1 & \cdots & \alpha^{-(N-1)(N-1)} & \alpha^{-N(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^{-N(N-1)} & \alpha^{-(N-1)(N-1)} & \cdots & 1 & \cdots & \alpha^{N-1(N-1)} & \alpha^{N(N-1)} \\ \alpha^{-NN} & \alpha^{-NN} & \cdots & 1 & \cdots & \alpha^{(N-1)N} & \alpha^{NN} \end{bmatrix} \mathbf{X} \tag{1.11}$$

Similarily, from eq. (1.8b), the inverse relation expands out to

$$\begin{aligned} (2N+1)X_n = x_{-N}e^{2\pi jnN/(2N+1)} \\ + x_{1-N}e^{2\pi jn(N-1)/(2N+1)} \\ \cdots \\ + x_0 \\ \cdots \\ + x_{N-1}e^{-2\pi jn(N-1)/(2N+1)} \\ + x_N e^{-2\pi jnN/(2N+1)}, \end{aligned} \tag{1.12}$$

with a matrix form of

$$(2N+1)\mathbf{X} = \begin{bmatrix} \alpha^{-NN} & \alpha^{-N(N-1)} & \cdots & 1 & \cdots & \alpha^{N(N-1)} & \alpha^{NN} \\ \alpha^{-(N-1)N} & \alpha^{-(N-1)(N-1)} & \cdots & 1 & \cdots & \alpha^{(N-1)(N-1)} & \alpha^{(N-1)N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^{(N-1)N} & \alpha^{(N-1)(N-1)} & \cdots & 1 & \cdots & \alpha^{-(N-1)(N-1)} & \alpha^{-(N-1)N} \\ \alpha^{NN} & \alpha^{N(N-1)} & \cdots & 1 & \cdots & \alpha^{-N(N-1)} & \alpha^{-NN} \end{bmatrix} \tag{1.13}$$

Letting

$$\mathbf{F} = \begin{bmatrix} \alpha^{NN} & \alpha^{(N-1)N} & \cdots & 1 & \cdots & \alpha^{-(N-1)N} & \alpha^{-NN} \\ \alpha^{N(N-1)} & \alpha^{(N-1)(N-1)} & \cdots & 1 & \cdots & \alpha^{-(N-1)(N-1)} & \alpha^{-N(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^{-N(N-1)} & \alpha^{-(N-1)(N-1)} & \cdots & 1 & \cdots & \alpha^{N-1(N-1)} & \alpha^{N(N-1)} \\ \alpha^{-NN} & \alpha^{-NN} & \cdots & 1 & \cdots & \alpha^{(N-1)N} & \alpha^{NN} \end{bmatrix}, \tag{1.14}$$

the discrete transform pair has the following compactly matrix representation

$$\mathbf{x} = \mathbf{F}\mathbf{X} \tag{1.15a}$$

$$\mathbf{X} = \frac{1}{2N+1}\bar{\mathbf{F}}\mathbf{x}, \tag{1.15b}$$

where $\bar{\mathbf{F}}$ is the complex conjugate of $\mathbf{F}$.

Bibliography

[1] Franco Giannini and Giorgio Leuzzi. *Nonlinear Microwave Circuit Design*. Wiley Online Library, 2004. 1.1

[2] Peeter Joot. *Condensed matter physics.*, chapter Discrete Fourier transform. 2013. URL http://peeterjoot.com/archives/math2013/phy487.pdf. [Online; accessed 02-December-2014]. 1.1