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Numeric LU factorization example

To get a better feel for LU factorization before attempting a numeric implementation, let's look at a numeric example in detail. This will strip some of the abstraction away.

Let's compute the LU factorization for

$$M = \begin{bmatrix} 5 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix}. \tag{1.1}$$

This matrix was picked to avoid having to think of selecting the right pivot row. Our first two operations give us

$$\begin{matrix} \\ (r_2 \rightarrow r_2 - \frac{2}{5} r_1) \\ (r_3 \rightarrow r_3 - \frac{3}{5} r_1) \end{matrix} \begin{bmatrix} 5 & 1 & 1 \\ 0 & 13/5 & 18/5 \\ 0 & 2/5 & 7/5 \end{bmatrix}. \tag{1.2}$$

The row operations (left multiplication) that produce this matrix are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2/5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/5 & 1 & 0 \\ -3/5 & 0 & 1 \end{bmatrix}. \tag{1.3}$$

These operations happen to be commutative and also both invert simply. The inverse operations are

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3/5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 1 & 0 \\ 3/5 & 0 & 1 \end{bmatrix}. \tag{1.4}$$

In matrix form the elementary matrix operations that take us to the first stage of the Gaussian reduction are

$$\begin{bmatrix} 1 & 0 & 0 \\ -2/5 & 1 & 0 \\ -3/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 13/5 & 18/5 \\ 0 & 2/5 & 7/5 \end{bmatrix}. \tag{1.5}$$

Inverted that is

$$\begin{bmatrix} 5 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 1 & 0 \\ 3/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 1 \\ 0 & 13/5 & 18/5 \\ 0 & 2/5 & 7/5 \end{bmatrix}. \tag{1.6}$$

This is the first stage of the LU decomposition, although the U matrix is not yet in upper triangular form. Again, with our pivot row in the desired position already, the last row operation to perform is

$$r_3 \rightarrow r_3 - \frac{2/5}{5/13} r_2 = r_3 - \frac{2}{13} r_2. \tag{1.7}$$

The final stage of this Gaussian reduction is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/13 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 1 \\ 0 & 13/5 & 18/5 \\ 0 & 2/5 & 7/5 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 13/5 & 18/5 \\ 0 & 0 & 11/13 \end{bmatrix} = U, \tag{1.8}$$

and our desired lower triangular matrix factor is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 1 & 0 \\ 3/5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/13 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/5 & 1 & 0 \\ 3/5 & 2/13 & 1 \end{bmatrix} = L. \tag{1.9}$$

A bit of matlab code easily verifies that the above manual computation recovers $M = LU$

```
l = [ 1 0 0 ; 2/5 1 0 ; 3/5 2/13 1 ] ;
u = [ 5 1 1 ; 0 13/5 18/5 ; 0 0 11/13 ] ;
l * u
```