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ECE1254H Modeling of Multiphysics Systems. Lecture 11: Nonlinear equations. Taught by Prof. Piero Triverio

1.1 Disclaimer

Peeter's lecture notes from class. These may be incoherent and rough.

1.2 Solution of N nonlinear equations in N unknowns

We'd now like to move from solutions of nonlinear functions in one variable:

$$f(x^*) = 0, \tag{1.1}$$

to multivariable systems of the form

$$\begin{aligned} f_1(x_1, x_2, \cdots, x_N) &= 0 \\ &\vdots \\ f_N(x_1, x_2, \cdots, x_N) &= 0 \end{aligned}, \tag{1.2}$$

where our unknowns are

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}. \tag{1.3}$$

Form the vector F

$$F(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2, \cdots, x_N) \\ \vdots \\ f_N(x_1, x_2, \cdots, x_N) \end{bmatrix}, \tag{1.4}$$

so that the equation to solve is

$$\boxed{F(\mathbf{x}) = 0.} \tag{1.5}$$

The Taylor expansion of F around point $\mathbf{x}_0$ is

$$F(\mathbf{x}) = F(\mathbf{x}_0) + \underbrace{\boxed{J_F(\mathbf{x}_0)}}_{\text{Jacobian}}(\mathbf{x} - \mathbf{x}_0)\,, \tag{1.6}$$

where the Jacobian is

$$J_F(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ & \ddots & \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} \tag{1.7}$$

1.3 Multivariable Newton's iteration

Given $\mathbf{x}^k$, expand $F(\mathbf{x})$ around $\mathbf{x}^k$

$$F(\mathbf{x}) \approx F(\mathbf{x}^k) + J_F(\mathbf{x}^k)\left(\mathbf{x} - \mathbf{x}^k\right) \tag{1.8}$$

With the approximation

$$0 = F(\mathbf{x}^k) + J_F(\mathbf{x}^k)\left(\mathbf{x}^{k+1} - \mathbf{x}^k\right), \tag{1.9}$$

then multiplying by the inverse Jacobian , and rearranging, we have

$$\boxed{\mathbf{x}^{k+1} = \mathbf{x}^k - J_F^{-1}(\mathbf{x}^k)F(\mathbf{x}^k).} \tag{1.10}$$

Our algorithm is

Guess $\mathbf{x}^0$, $k = 0$.

repeat

  Compute F and J_F at $\mathbf{x}^k$

  Solve linear system $J_F(\mathbf{x}^k)\Delta\mathbf{x}^k = -F(\mathbf{x}^k)$

  $\mathbf{x}^{k+1} = \mathbf{x}^k + \Delta\mathbf{x}^k$

  $k = k + 1$

until converged

As with one variable, our convergence is after we have all of the convergence conditions satisfied

$$\begin{aligned} \left\|\Delta\mathbf{x}^k\right\| &< \epsilon_1 \\ \left\|F(\mathbf{x}^{k+1})\right\| &< \epsilon_2 \\ \frac{\|\Delta\mathbf{x}^k\|}{\|\mathbf{x}^{k+1}\|} &< \epsilon_3 \end{aligned} \tag{1.11}$$

Typical termination is some multiple of eps, where eps is the machine precision. This may be something like:

$$4 \times N \times \text{eps}, \tag{1.12}$$

where N is the "size of the problem". Sometimes we may be able to find meaningful values for the problem. For example, for a voltage problem, we may not be interested in precisions greater than a millivolt.

1.4 Automatic assembly of equations for nolinear system

Nonlinear circuits We will start off considering a non-linear resistor, designated within a circuit as sketched in fig. 1.1.

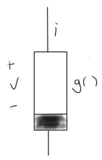

Figure 1.1: Non-linear resistor

Example: diode, with $i = g(v)$, such as

$$i = I_0 \left(e^{v/\eta V_T} - 1 \right). \tag{1.13}$$

Consider the example circuit of fig. 1.2. KCL's at each of the nodes are

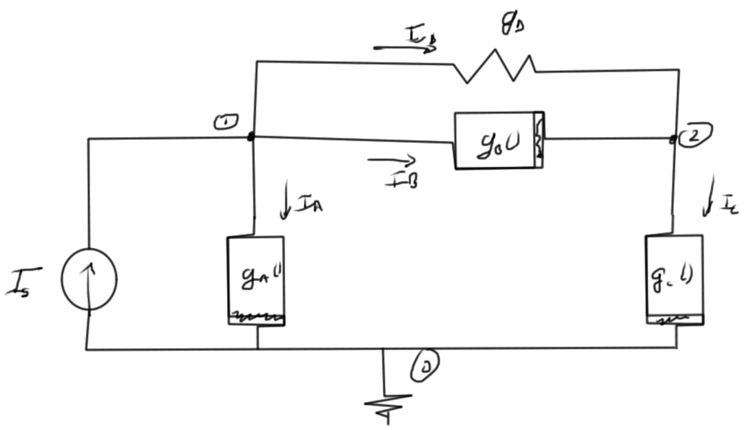

Figure 1.2: Example circuit

1. $I_A + I_B + I_D - I_s = 0$
2. $-I_B + I_C - I_D = 0$

Introducing the consistuative equations this is

1. $g_A(V_1) + g_B(V_1 - V_2) + g_D(V_1 - V_2) - I_s = 0$
2. $-g_B(V_1 - V_2) + g_C(V_2) - g_D(V_1 - V_2) = 0$

In matrix form this is

$$\begin{bmatrix} g_D & -g_D \\ -g_D & g_D \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} g_A(V_1) & +g_B(V_1 - V_2) & & -I_s \\ & -g_B(V_1 - V_2) & +g_C(V_2) & \end{bmatrix} = 0. \tag{1.14}$$

We can write the entire system as

$$\boxed{F(\mathbf{x}) = G\mathbf{x} + F'(\mathbf{x}) = 0.} \tag{1.15}$$

The first term, a product of a nodal matrix G represents the linear subnetwork, and is filled with the stamps we are already familiar with.

The second term encodes the relationships of the nonlinear subnetwork. This non-linear component has been marked with a prime to distinguish it from the complete network function that includes both linear and non-linear elements.

Observe the similarity with the stamp analysis that we did previously. With $g_A()$ connected on one end to ground we have it only once in the resulting vector, whereas the nonlinear elements connected to two non-zero nodes in the network occur once with each sign.

Stamp for nonlinear resistor For the non-linear circuit element of fig. 1.3.

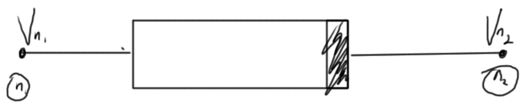

Figure 1.3: Non-linear resistor circuit element

$$F'(\mathbf{x}) = \begin{matrix} n_1 \rightarrow \\ n_2 \rightarrow \end{matrix} \begin{bmatrix} +g(V_{n_1} - V_{n_2}) \\ -g(V_{n_1} - V_{n_2}) \end{bmatrix} \tag{1.16}$$

Stamp for Jacobian

$$J_F(\mathbf{x}^k) = G + J_{F'}(\mathbf{x}^k). \tag{1.17}$$

Here the stamp for the Jacobian, an $N \times N$ matrix, is

$$
J_{F'}(\mathbf{x}^k) = \begin{array}{c} \\ 1 \\ \vdots \\ n_1 \\ n_2 \\ \vdots \\ N \end{array}
\begin{array}{c} \begin{array}{cccccc} V_1 & \cdots & V_{n_1} & V_{n_2} & \cdots & V_N \end{array} \\
\left[\begin{array}{cccccc}
 & & & & & \\
 & & & & & \\
 & & \frac{\partial g(V_{n_1}-V_{n_2})}{\partial V_{n_1}} & \frac{\partial g(V_{n_1}-V_{n_2})}{\partial V_{n_2}} & & \\
 & & -\frac{\partial g(V_{n_1}-V_{n_2})}{\partial V_{n_1}} & -\frac{\partial g(V_{n_1}-V_{n_2})}{\partial V_{n_2}} & & \\
 & & & & & \\
 & & & & &
\end{array} \right] \end{array}. \tag{1.18}
$$