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ECE1254H Modeling of Multiphysics Systems. Lecture 15: Nonlinear differential equations. Taught by Prof. Piero Triverio

1.1 Disclaimer

Peeter's lecture notes from class. These may be incoherent and rough.

1.2 Nonlinear differential equations

Assume that the relationships between the zeroth and first order derivatives has the form

$$F\left(x(t), \dot{x}(t)\right) = 0 \tag{1.1a}$$

$$x(0) = x_0 \tag{1.1b}$$

The backward Euler method where the derivative approximation is

$$\dot{x}(t_n) \approx \frac{x_n - x_{n-1}}{\Delta t}, \tag{1.2}$$

can be used to solve this numerically, reducing the problem to

$$F\left(x_n, \frac{x_n - x_{n-1}}{\Delta t}\right) = 0. \tag{1.3}$$

This can be solved with Newton's method. How do we find the initial guess for Newton's? Consider a possible system in fig. 1.1.

One strategy for starting each iteration of Newton's method is to base the initial guess for x_1 on the value x_0, and do so iteratively for each subsequent point. One can imagine that this may work up to some sample point x_n, but then break down (i.e. Newton's diverges when the previous value x_{n-1} is used to attempt to solve for x_n). At that point other possible strategies may work. One such strategy is to use an approximation of the derivative from the previous steps to attempt to get a better estimate of the next value. Another possibility is to reduce the time step, so the difference between successive points is reduced.

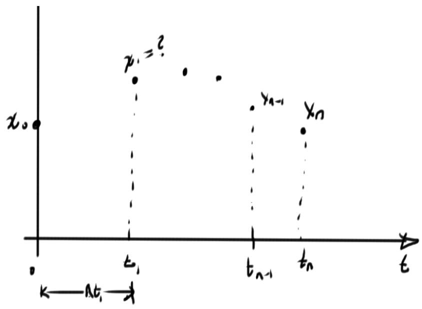

Figure 1.1: Possible solution points

1.3 Analysis, accuracy and stability ($\Delta t \to 0$)

Consider a differential equation

$$\dot{x}(t) = f(x(t), t) \tag{1.4a}$$

$$x(t_0) = x_0 \tag{1.4b}$$

A few methods of solution have been considered

(FE) $x_{n+1} - x_n = \Delta t f(x_n, t_n)$

(BE) $x_{n+1} - x_n = \Delta t f(x_{n+1}, t_{n+1})$

(TR) $x_{n+1} - x_n = \frac{\Delta t}{2} f(x_{n+1}, t_{n+1}) + \frac{\Delta t}{2} f(x_n, t_n)$

A common pattern can be observed, the generalization of which are called *linear multistep methods* (LMS) , which have the form

$$\sum_{j=-1}^{k-1} \alpha_j x_{n-j} = \Delta t \sum_{j=-1}^{k-1} \beta_j f(x_{n-j}, t_{n-j}) \tag{1.5}$$

The FE (explicit), BE (implicit), and TR methods are now special cases with

(FE) $\alpha_{-1} = 1, \alpha_0 = -1, \beta_{-1} = 0, \beta_0 = 1$

(BE) $\alpha_{-1} = 1, \alpha_0 = -1, \beta_{-1} = 1, \beta_0 = 0$

(TR) $\alpha_{-1} = 1, \alpha_0 = -1, \beta_{-1} = 1/2, \beta_0 = 1/2$

Here k is the number of timesteps used. The method is explicit if $\beta_{-1} = 0$.

Definition 1.1: Convergence

With

$x(t)$: exact solution

x_n : computed solution

e_n : where $e_n = x_n - x(t_n)$, is the global error

The LMS method is convergent if

$$\max_{n, \Delta t \to 0} |x_n - t(t_n)| \to 0$$

Convergence: zero-stability and consistency (small local errors made at each iteration), where zero-stability is “small sensitivity to changes in initial condition”.

Definition 1.2: Consistency

A local error R_{n+1} can be defined as

$$R_{n+1} = \sum_{j=-1}^{k-1} \alpha_j x(t_{n-j}) - \Delta t \sum_{j=-1}^{k-1} \beta_j f(x(t_{n-j}), t_{n-j}).$$

The method is consistent if

$$\lim_{\Delta t} \left(\max_n \left| \frac{1}{\Delta t} R_{n+1} \right| = 0 \right)$$

or $R_{n+1} \sim O(\Delta t^2)$