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# ECE1254H Modeling of Multiphysics Systems. Lecture 16: LMS systems and stability. Taught by Prof. Piero Triverio

## 1.1 Disclaimer

Peeter's lecture notes from class. These may be incoherent and rough.

## 1.2 Residual for LMS methods

*Mostly on slides:* 12_ODS.pdf

Residual is illustrated in fig. 1.1, assuming that the iterative method was accurate until $t_n$

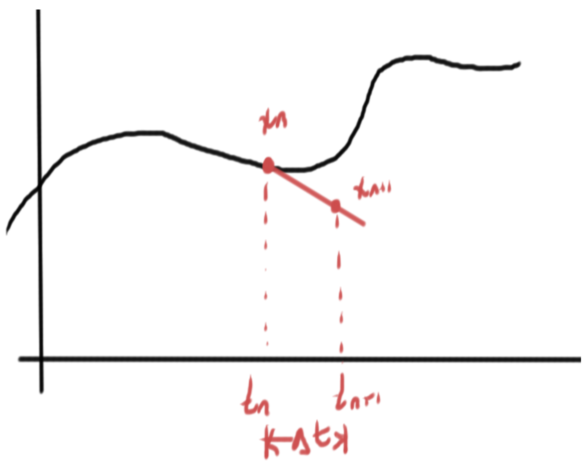

Figure 1.1: Residual illustrated

*Summary*

FE : $R_{n+1} \sim (\Delta t)^2$. This is of order $p = 1$.

BE : $R_{n+1} \sim (\Delta t)^2$. This is of order $p = 1$.

$$\text{TR}\ : R_{n+1} \sim (\Delta t)^3. \text{ This is of order } p = 2.$$
$$\text{BESTE}\ : R_{n+1} \sim (\Delta t)^4. \text{ This is of order } p = 3.$$

### 1.3 Global error estimate

Suppose $t \in [0, 1]s$, with $N = 1/\Delta t$ intervals. For a method with local error of order $R_{n+1} \sim (\Delta t)^2$ the global error is approximately $NR_{n+1} \sim \Delta t$.

### 1.4 Stability

Recall that a linear multistep method (LMS) was a system of the form

$$\sum_{j=-1}^{k-1} \alpha_j x_{n-j} = \Delta t \sum_{j=-1}^{k-1} \beta_j f(x_{n-j}, t_{n-j}) \tag{1.1}$$

Consider a one dimensional test problem

$$\dot{x}(t) = \lambda x(t) \tag{1.2}$$

where as in fig. 1.2, $\text{Re}(\lambda) < 0$ is assumed to ensure stability.

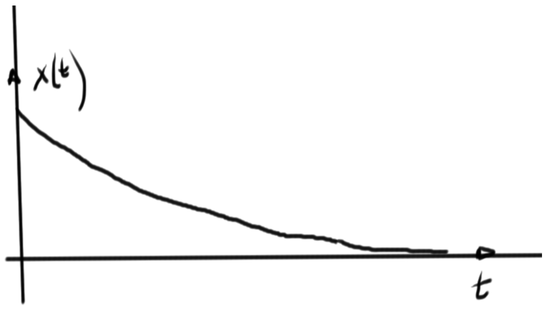


**Figure 1.2:** Stable system

Linear stability theory can be thought of as asking the question: "Is the solution of eq. (1.2) computed by my LMS method also stable?"

Application of eq. (1.1) to eq. (1.2) gives

$$\sum_{j=-1}^{k-1} \alpha_j x_{n-j} = \Delta t \sum_{j=-1}^{k-1} \beta_j \lambda x_{n-j}, \tag{1.3}$$

or

$$\sum_{j=-1}^{k-1} \left(\alpha_j - \Delta \beta_j \lambda\right) x_{n-j} = 0. \tag{1.4}$$

With

$$\gamma_j = \alpha_j - \Delta\beta_j\lambda, \tag{1.5}$$

this expands to

$$\gamma_{-1}x_{n+1} + \gamma_0 x_n + \gamma_1 x_{n-1} + \cdots + \gamma_{k-1}x_{n-k}. \tag{1.6}$$

This can be seen as a

- discrete time system
- FIR filter

The numerical solution $x_n$ will be stable if eq. (1.6) is stable.
A characteristic equation associated with eq. (1.6) can be defined as

$$\gamma_{-1}z^k + \gamma_0 z^{k-1} + \gamma_1 z^{k-2} + \cdots + \gamma_{k-1} = 0. \tag{1.7}$$

This is a polynomial with roots $z_n$ (poles). This is stable if the poles satisfy $|z_n| < 1$, as illustrated in fig. 1.3

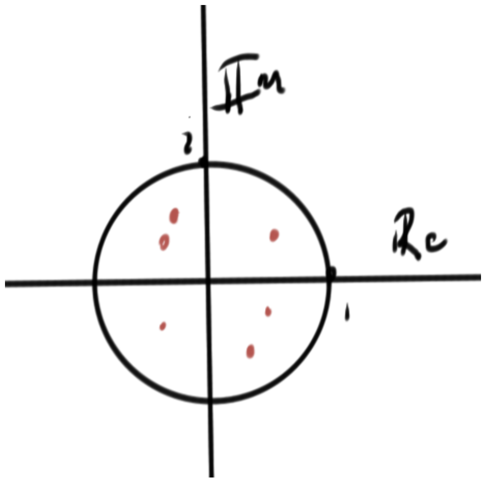


**Figure 1.3:** Stability

Observe that the $\gamma's$ are dependent on $\Delta t$.
FIXME: There's a lot of handwaving here that could use more strict justification. Check if the text covers this in more detail.

**Example 1.1: Forward Euler stability**

For $k = 1$ step.

$$x_{n+1} - x_n = \Delta t f(x_n, t_n), \tag{1.8}$$

the coefficients are $\alpha_{-1} = 1, \alpha_0 = -1, \beta_{-1} = 0, \beta_0 = 1$. For the simple function above

$$\gamma_{-1} = \alpha_{-1} - \Delta t \lambda \beta_{-1} = 1 \tag{1.9a}$$

$$\gamma_0 = \alpha_0 - \Delta t \lambda \beta_0 = -1 - \Delta t \lambda. \tag{1.9b}$$

The stability polynomial is

$$1z + (-1 - \Delta t \lambda) = 0, \tag{1.10}$$

or

$$\boxed{z = 1 + \delta t \lambda.} \tag{1.11}$$

This is the root, or pole.
For stability we must have

$$|1 + \Delta t \lambda| < 1, \tag{1.12}$$

or

$$\left| \lambda - \left( -\frac{1}{\Delta t} \right) \right| < \frac{1}{\Delta t}, \tag{1.13}$$

This inequality is illustrated roughly in fig. 1.4.

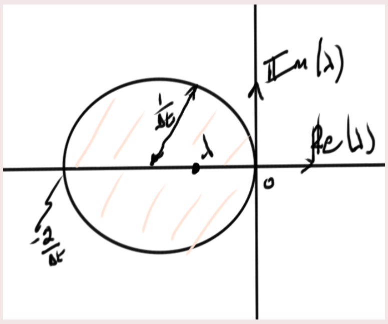


**Figure 1.4:** Stability region of FE

All poles of my system must be inside the stability region in order to get stable $\gamma$.