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ECE1254H Modeling of Multiphysics Systems. Lecture 2: Assembling system equations automatically. Taught by Prof. Piero Triverio

1.1 Disclaimer

Peeter's lecture notes from class. These may be incoherent and rough.

1.2 Assembling system equations automatically. Node/branch method

Consider the sample circuit of fig. 1.1.

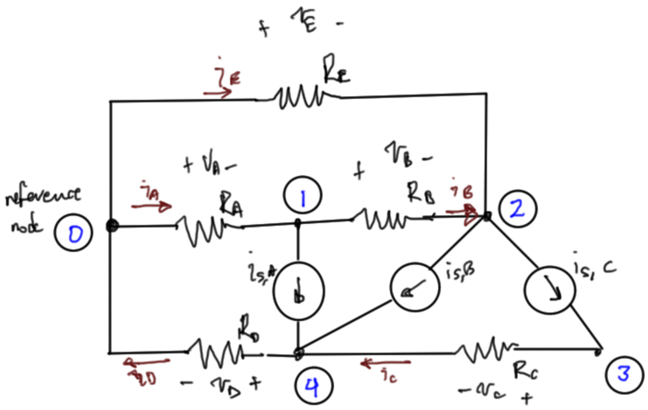

Figure 1.1: Sample resistive circuit

Step 1. Choose unknowns: For this problem, let's take

- node voltages: V_1, V_2, V_3, V_4
- branch currents: i_A, i_B, i_C, i_D, i_E

We do not need to introduce additional labels for the source current sources. We always introduce a reference node and call that zero.

For a circuit with N nodes, and B resistors, there will be $N-1$ unknown node voltages and B unknown branch currents, for a total number of $N-1+B$ unknowns.

Step 2. Conservation equations: KCL

- 0: $i_A + i_E - i_D = 0$
- 1: $-i_A + i_B + i_{S,A} = 0$
- 2: $-i_B + i_{S,B} - i_E + i_{S,C} = 0$
- 3: $i_C - i_{S,C} = 0$
- 4: $-i_{S,A} - i_{S,B} + i_D - i_C = 0$

Grouping unknown currents, this is

- 0: $i_A + i_E - i_D = 0$
- 1: $-i_A + i_B = -i_{S,A}$
- 2: $-i_B - i_E = -i_{S,B} - i_{S,C}$
- 3: $i_C = i_{S,C}$
- 4: $i_D - i_C = i_{S,A} + i_{S,B}$

Note that one of these equations is redundant (sum 1-4). In a circuit with N nodes, we can write at most $N-1$ independent KCLs.

In matrix form

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_A \\ i_B \\ i_C \\ i_D \\ i_E \end{bmatrix} = \begin{bmatrix} -i_{S,A} \\ -i_{S,B} - i_{S,C} \\ i_{S,C} \\ i_{S,A} + i_{S,B} \end{bmatrix} \tag{1.1}$$

We call this first matrix of ones and minus ones the incidence matrix A. This matrix has B columns and $N-1$ rows. We call the known current matrix $\bar{I}_S$, and the branch currents $\bar{I}_B$. That is

$$A\bar{I}_B = \bar{I}_S. \tag{1.2}$$

Observe that we have both a plus and minus one in all columns except for those columns impacted by our neglect of the reference node current conservation equation.

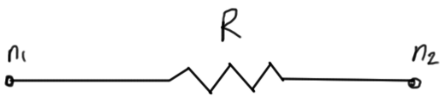

Figure 1.2: Resistor node convention

Algorithm for filling A In the input file, to describe a resistor of fig. 1.2, you'll have a line of the form

R name n_1 n_2 value

The algorithm to process resistor lines is

$A \leftarrow 0$
$ic \leftarrow 0$
for all resistor lines **do**
  $ic \leftarrow ic + 1$, adding one column to A
  if $n_1 != 0$ **then**
    $A(n_1, ic) \leftarrow +1$
  end if
  if $n_2 != 0$ **then**
    $A(n_2, ic) \leftarrow -1$
  end if
end for

Algorithm for filling $\bar{I}_S$ Current sources, as in fig. 1.3, a line will have the specification (FIXME: confirm... didn't write this down).

I name n_1 n_2 value

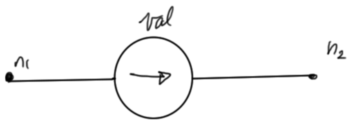

Figure 1.3: Current source conventions

$\bar{I}_S = \text{zeros}(N-1, 1)$
for all current lines **do**
  $\bar{I}_S(n_1) \leftarrow \bar{I}_S(n_1) - \text{value}$
  $\bar{I}_S(n_2) \leftarrow \bar{I}_S(n_1) + \text{value}$
end for

Step 3. Constitutive equations:

$$\begin{bmatrix} i_A \\ i_B \\ i_C \\ i_D \\ i_E \end{bmatrix} = \begin{bmatrix} 1/R_A & & & & \\ & 1/R_B & & & \\ & & 1/R_C & & \\ & & & 1/R_D & \\ & & & & 1/R_E \end{bmatrix} \begin{bmatrix} v_A \\ v_B \\ v_C \\ v_D \\ v_E \end{bmatrix} \tag{1.3}$$

Or

$$\bar{I}_B = \alpha \overline{V}_B, \tag{1.4}$$

where $\overline{V}_B$ are the branch voltages, not unknowns of interest directly. We can write

$$\begin{bmatrix} v_A \\ v_B \\ v_C \\ v_D \\ v_E \end{bmatrix} = \begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & & 1 & -1 \\ & & & 1 \\ & -1 & & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \tag{1.5}$$

Observe that α is the transpose of A, allowing us to write

$$\overline{V}_B = A^{\mathrm{T}} \overline{V}_N. \tag{1.6}$$

Solving

- KCLs: $A\bar{I}_B = \bar{I}_S$
- constitutive: $\bar{I}_B = \alpha \overline{V}_B \implies \bar{I}_B = \alpha A^{\mathrm{T}} \overline{V}_N$
- branch and node voltages: $\overline{V}_B = A^{\mathrm{T}} \overline{V}_N$

In block matrix form, this is

$$\begin{bmatrix} A & 0 \\ I & -\alpha A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \bar{I}_B \\ \overline{V}_N \end{bmatrix} = \begin{bmatrix} \bar{I}_S \\ 0 \end{bmatrix}. \tag{1.7}$$

Is it square? We see that it is after observing that we have

- $N - 1$ rows in A, and B columns.
- B rows in I.
- $N - 1$ columns.