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ECE1254H Modeling of Multiphysics Systems. Lecture 6: Singular value decomposition, and conditioning number. Taught by Prof. Piero Triverio

1.1 Disclaimer

Peeter's lecture notes from class. These may be incoherent and rough.

1.2 Matrix norm

We've defined the matrix norm of M, for the system $\bar{y} = M\bar{x}$ as

$$\left\lVert M \right\rVert = \max_{\left\lVert \bar{x} \right\rVert = 1} \left\lVert M\bar{x} \right\rVert . \tag{1.1}$$

We will typically use the L_2 norm, so that the matrix norm is

$$\left\lVert M \right\rVert_2 = \max_{\left\lVert \bar{x} \right\rVert_2 = 1} \left\lVert M\bar{x} \right\rVert_2 . \tag{1.2}$$

It can be shown that

$$\left\lVert M \right\rVert_2 = \max_i \sigma_i(M), \tag{1.3}$$

where $\sigma_i(M)$ are the (SVD) singular values.

Definition 1.1: Singular value decomposition (SVD)

Given $M \in \mathbb{R}^{m \times n}$, we can find a representation of M

$$M = U \Sigma V^\mathrm{T}, \tag{1.4}$$

where U and V are orthogonal matrices such that $U^\mathrm{T} U = 1$, and $V^\mathrm{T} V = 1$, and

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix} \tag{1.5}$$

The values $\sigma_i, i \le \min(n, m)$ are called the singular values of M. The singular values are subject to the ordering

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge 0 \tag{1.6}$$

If r is the rank of M, then the σ_r above is the minimum non-zero singular value (but the zeros are also called singular values).

Observe that the condition $U^{\mathrm{T}} U = 1$ is a statement of orthonormality. In terms of column vectors $\bar{u}_i$, such a product written out explicitly is

$$\begin{bmatrix} \bar{u}_1^{\mathrm{T}} \\ \bar{u}_2^{\mathrm{T}} \\ \vdots \\ \bar{u}_m^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_m \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}. \tag{1.7}$$

This is both normality $\bar{u}_i^{\mathrm{T}} \bar{u}_i = 1$, and orthonormality $\bar{u}_i^{\mathrm{T}} \bar{u}_j = 1, i \ne j$.

Example 1.1: 2 x 2 case

(for column vectors $\bar{u}_i, \bar{v}_j$).

$$M = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \bar{v}_1^{\mathrm{T}} \\ \bar{v}_2^{\mathrm{T}} \end{bmatrix} \tag{1.8}$$

Consider $\bar{y} = M\bar{x}$, and take an $\bar{x}$ with $\|\bar{x}\|_2 = 1$

Note: I've chosen not to sketch what was drawn on the board. See instead the animated gif of the same in [**?**].

1.3 Conditioning number

Given a perturbation of $M\bar{x} = \bar{b}$ to

$$(M + \delta M)(\bar{x} + \delta\bar{x}) = \bar{b}, \tag{1.9}$$

or

$$\cancel{M\bar{x} - b} + \delta M\bar{x} + M\delta\bar{x} + \delta M\delta\bar{x} = 0. \tag{1.10}$$

This gives

$$M\delta\bar{x} = -\delta M\bar{x} - \delta M\delta\bar{x}, \tag{1.11}$$

or

$$\delta\bar{x} = -M^{-1}\delta M\left(\bar{x} + \delta\bar{x}\right). \tag{1.12}$$

Taking norms

$$\begin{aligned} \left\|\delta\bar{x}\right\|_2 &= \left\|M^{-1}\delta M\left(\bar{x} + \delta\bar{x}\right)\right\|_2 \\ &\le \left\|M^{-1}\right\|_2 \left\|\delta M\right\|_2 \left\|\bar{x} + \delta\bar{x}\right\|_2, \end{aligned} \tag{1.13}$$

or

$$\overbrace{\frac{\left\|\delta\bar{x}\right\|_2}{\left\|\bar{x} + \delta\bar{x}\right\|_2}}^{\text{relative error of solution}} \le \underbrace{\left\|M\right\|_2 \left\|M^{-1}\right\|_2}_{\text{conditioning number of } M} \overbrace{\frac{\left\|\delta M\right\|_2}{\left\|M\right\|_2}}^{\text{relative perturbation of } M}. \tag{1.14}$$

The conditioning number can be shown to be

$$\mathrm{cond}(M) = \frac{\sigma_{\mathrm{max}}}{\sigma_{\mathrm{min}}} \ge 1 \tag{1.15}$$

FIXME: justify.

1.3.1 *sensitivity to conditioning number*

Double precision relative rounding errors can be of the order $10^{-16} \sim 2^{-54}$, which allows us to gauge the relative error of the solution

relative error of solution	$\le$	$\mathrm{cond}(M)$	$\frac{\|\delta M\|}{\|M\|}$
10^{-15}	$\le$	10	$\sim 10^{-16}$
10^{-2}	$\le$	10^{14}	10^{-16}