Peeter Joot
920798560
peeter.joot@utoronto.ca
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ECE1254H Modeling of Multiphysics Systems. Problem Set 1: Modified Nodal Analysis

Exercise 1.1 Heat conduction

a. In this problem we will examine the heat conducting bar basic example $\cdots$ Then, interpret the discretized equation as a KCL using the electrothermal analogy where temperature corresponds to node voltage, and heat flow to current. Draw the equivalent circuit you obtained.

b. Plot $T(x)$ in $x \in [0, 1]$.

c. In your numerical calculation, how did you choose Δx? Justify the choice of Δx.

d. Now use your simulator to numerically solve the above equation $\cdots$

e. Plot the new temperature profile.

f. Explain the temperature distributions that you obtained from a physical standpoint.

Answer for Exercise 1.1

Part a. To discretize the equation, let's divide the interval into N segments, as illustrated in fig. 1.1.

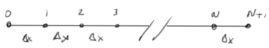

Figure 1.1: Discretization intervals

Let

$$
\begin{aligned}
x^i &= i \Delta x, \quad i \in \{0, 1, \cdots, N+1\} \\
H^i &= 50 \sin^2\left(2 \pi x^i\right), \quad i \in \{0, 1, \cdots, N+1\} \\
T^i &= T(x^i), \quad i \in \{0, 1, \cdots, N+1\} \\
Q^i &= \frac{T^{i+1} - T^i}{\Delta x}, \quad i \in \{0, \cdots, N\}
\end{aligned}
\tag{1.1}
$$

Here x^i are the discrete points at which the temperatures are evaluated with and $\Delta x = 1/N$. The values H^i, T^i are the heats and temperatures respectively, and Q^i is a temperature current (proportional to the heat flow) in the interval i flowing from node $i+1$ to i.

With $Q^i = \left(T^i - T^{i-1}\right)/\Delta x$ we have an equivalence to the circuit element $I = \Delta V/R$. This analogy makes sense physicsally since a smaller interval length has less resistance to heat flow.

The linearized Poisson equation at interior nodes $1 \le i \le N$ is

$$Q^i - Q^{i-1} - \frac{\kappa_a \Delta x}{\kappa_m}\left(T^i - T_0\right) + \frac{H^i \Delta x}{\kappa_m} = 0. \tag{1.2}$$

Identification of the $-\kappa_a \Delta x/\kappa_m \left(T^i - T_0\right)$ term as a current $I = \Delta V/R$ (with temperatures as voltages) means that we can identify $R \sim \kappa_m \Delta x/\kappa_a$. This allows us to create the equivalent circuit sketched in fig. 1.2.

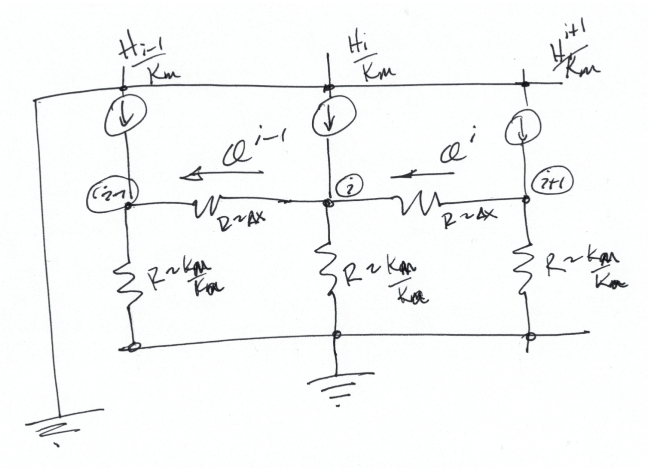

Figure 1.2: Equivalent circuit

At the endpoints things are a slightly different. At x^0, x^{N+1} respectively, we have

$$\begin{aligned} -q_0 + Q^0 - \frac{\kappa_a \Delta x}{\kappa_m}\left(T^0 - T_0\right) + \frac{H^0 \Delta x}{\kappa_m} &= 0 \\ -q_1 - Q^N - \frac{\kappa_a \Delta x}{\kappa_m}\left(T^{N+1} - T_0\right) + \frac{H^{N+1} \Delta x}{\kappa_m} &= 0. \end{aligned} \tag{1.3}$$

At these nodes is only one current term, but we also have to model the heat sinks. This equivalent circuit is sketched in fig. 1.3.

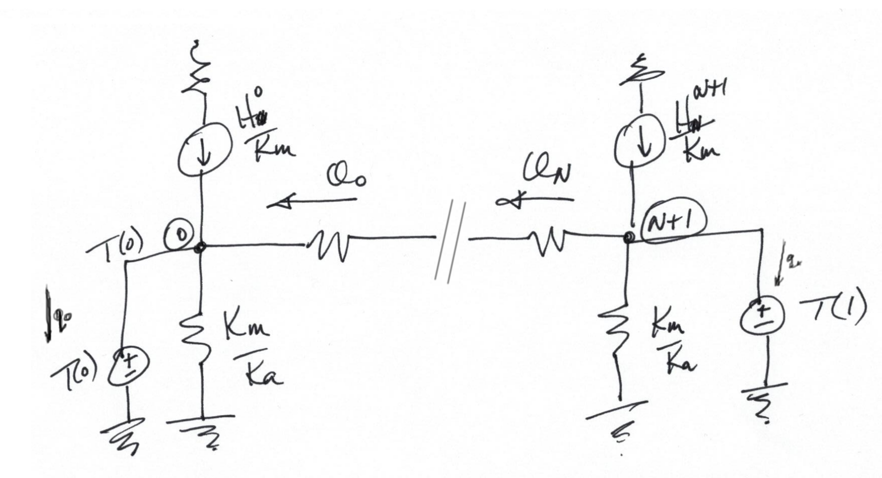

Figure 1.3: Endpoints in equivalent circuit

Notes

- I missed the Δx factors in the equivalent circuit figures.
- It looks like it would have been better to formulate these equations with $N - 1$ intervals, so that I could start off the node numbers at (1) instead of (0), since the ambiant temperature has been identified with ground. The code that generates the netlist offsets the internal node numbers by one to compensate.

Part b. TODO.

Part c. We'd like to keep the variation of the H_i term small. That diffrence between two adjacent intervals scales with the squared sine

$$\begin{aligned}\Delta\left(\sin^2(2\pi x)\right) &\approx 4\pi \sin\left(2\pi x\right)\cos\left(2\pi x\right)\Delta x \\ &= 2\pi \sin\left(4\pi x\right)\Delta x\end{aligned} \tag{1.4}$$

This is biggest when the sine is near unity. If we want the heat terms to differ by no more than a fraction f we can determine Δx from

$$\begin{aligned}f H^i &\approx H^i - H^{i-1} \\ &\approx 2\pi \Delta x H^i. \\ &= \frac{2\pi}{N} H^i.\end{aligned} \tag{1.5}$$

That is

$$N \approx \frac{2\pi}{f}. \tag{1.6}$$

For example, if we desire $f = 1/10$ we can pick $N \approx 63$.

Part d. TODO.

Part e. TODO.

Part f. TODO.

Bibliography