

Plane wave solution directly from Maxwell's equations

Here's a problem that I thought was fun, an exercise for the reader to show that the plane wave solution to Maxwell's equations can be found with ease directly from Maxwell's equations. This is in contrast to the what seems like the usual method of first showing that Maxwell's equations imply wave equations for the fields, and then solving those wave equations.

Exercise 1.1 \hat{x} oriented plane wave electric field ([1] ex. 4.1)

A uniform plane wave having only an x component of the electric field is traveling in the $+z$ direction in an unbounded lossless, source-free region. Using Maxwell's equations write expressions for the electric and corresponding magnetic field intensities.

Answer for Exercise 1.1

The phasor form of Maxwell's equations for a source free region are

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} \quad (1.1a)$$

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} \quad (1.1b)$$

$$\nabla \cdot \mathbf{D} = 0 \quad (1.1c)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (1.1d)$$

Since $\mathbf{E} = \hat{x}E(z)$, the magnetic field follows from eq. (1.1a)

$$\begin{aligned} -j\omega\mathbf{B} &= \nabla \times \mathbf{E} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E & 0 & 0 \end{vmatrix} \\ &= \hat{y}\partial_z E(z) - \hat{z}\partial_y E(z), \end{aligned} \quad (1.2)$$

or

$$\mathbf{B} = -\frac{1}{j\omega}\partial_z E. \quad (1.3)$$

This is constrained by eq. (1.1b)

$$\begin{aligned}
j\omega\epsilon\hat{\mathbf{x}}E &= \frac{1}{\mu}\nabla \times \mathbf{B} \\
&= -\frac{1}{\mu j\omega} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ 0 & \partial_z E & 0 \end{vmatrix} \\
&= -\frac{1}{\mu j\omega} (-\hat{\mathbf{x}}\partial_{zz}E + \hat{\mathbf{z}}\partial_x\partial_zE)
\end{aligned} \tag{1.4}$$

Since $\partial_x\partial_zE = \partial_z(\partial_xE) = \partial_z\frac{1}{\epsilon}\nabla \cdot \mathbf{D} = \partial_z0$, this means

$$\partial_{zz}E = -\omega^2\epsilon\mu E = -k^2E. \tag{1.5}$$

This is the usual starting place that we use to show that the plane wave has an exponential form

$$\mathbf{E}(z) = \hat{\mathbf{x}} \left(E_+e^{-jkz} + E_-e^{jkz} \right). \tag{1.6}$$

The magnetic field from eq. (1.3) is

$$\begin{aligned}
\mathbf{B} &= \frac{j}{\omega} \left(-jkE_+e^{-jkz} + jkE_-e^{jkz} \right) \\
&= \frac{1}{c} \left(E_+e^{-jkz} - E_-e^{jkz} \right),
\end{aligned} \tag{1.7}$$

or

$$\begin{aligned}
\mathbf{H} &= \frac{1}{\mu c} \left(E_+e^{-jkz} - E_-e^{jkz} \right) \\
&= \frac{1}{\eta} \left(E_+e^{-jkz} - E_-e^{jkz} \right).
\end{aligned} \tag{1.8}$$

A solution requires zero divergence for the magnetic field, but that can be seen to be the case by inspection.

Bibliography

- [1] Constantine A Balanis. *Advanced engineering electromagnetics*. Wiley New York, 1989. 1.1