Peeter Joot
peeterjoot@protonmail.com

More on (SHO) coherent states

[1] pr. 2.19(c) Show that $|f(n)|^2$ for a coherent state written as

$$|z\rangle = \sum_{n=0}^{\infty} f(n) |n\rangle \tag{1.1}$$

has the form of a Poisson distribution, and find the most probable value of n, and thus the most probable energy.

A: The Poisson distribution has the form

$$P(n) = \frac{\mu^n e^{-\mu}}{n!}. \tag{1.2}$$

Here μ is the mean of the distribution

$$\begin{aligned}
\langle n \rangle &= \sum_{n=0}^{\infty} n P(n) \\
&= \sum_{n=1}^{\infty} n \frac{\mu^n e^{-\mu}}{n!} \\
&= \mu e^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^{n-1}}{(n-1)!} \\
&= \mu e^{-\mu} e^{\mu} \\
&= \mu.
\end{aligned} \tag{1.3}$$

We found that the coherent state had the form

$$|z\rangle = c_0 \sum_{n=0} \frac{z^n}{\sqrt{n!}} |n\rangle , \tag{1.4}$$

so the probability coefficents for $|n\rangle$ are

$$\begin{aligned}
P(n) &= c_0^2 \frac{|z^n|^2}{n!} \\
&= e^{-|z|^2} \frac{|z^n|^2}{n!}.
\end{aligned} \tag{1.5}$$

This has the structure of the Poisson distrubution with mean $\mu = |z|^2$. The most probable value of n is that for which $|f(n)|^2$ is the largest. This is, in general, hard to compute, since we have a maximization problem in the integer domain that falls outside the normal toolbox. If we assume that n is large, so that Stirlings approximation can be used to approximate the factorial, and also seek a non-integer value that maximizes the distribution, the most probable value will be the closest integer to that, and this can be computed. Let

$$\begin{aligned} g(n) &= |f(n)|^2 \\ &= \frac{e^{-\mu}\mu^n}{n!} \\ &= \frac{e^{-\mu}\mu^n}{e^{\ln n!}} \\ &\approx e^{-\mu - n\ln n + n}\mu^n. \\ &= e^{-\mu - n\ln n + n + n\ln\mu} \end{aligned} \tag{1.6}$$

This is maximized when

$$\begin{aligned} 0 &= \frac{dg}{dn} \\ &= \left(-\ln n - 1 + 1 + \ln\mu\right) g(n), \end{aligned} \tag{1.7}$$

which is maximized at $n = \mu$. One of the integers $n = \lfloor\mu\rfloor$ or $n = \lceil\mu\rceil$ that brackets this value $\mu = |z|^2$ is the most probable. So, if an energy measurement is made of a coherent state $|z\rangle$, the most probable value will be one of

$$E = \hbar\left(\lfloor|z|^2\rfloor + \frac{1}{2}\right), \tag{1.8}$$

or

$$E = \hbar\left(\lceil|z|^2\rceil + \frac{1}{2}\right), \tag{1.9}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1