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Hermite polynomial normalization constant

Exercise 1.1 Hermite polynomial normalization constant ([1] *pr. 2.21*)

Derive the normalization constant c_n for the Harmonic oscillator solution

$$u_n(x) = c_n H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right) e^{-m\omega x^2/2\hbar}, \tag{1.1}$$

by deriving the orthogonality relationship using generating functions

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{1.2}$$

Start by working out the integral

$$I = \int_{-\infty}^{\infty} g(x,t)g(x,s)e^{-x^2}dx, \tag{1.3}$$

consider the integral twice with each side definition of the generating function.

Answer for Exercise 1.1

First using the exponential definition of the generating function

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x,t)g(x,s)e^{-x^2}dx &= \int_{-\infty}^{\infty} e^{-t^2+2tx}e^{-s^2+2sx}e^{-x^2}dx \\
&= e^{-t^2-s^2}\int_{-\infty}^{\infty} e^{-(x^2-2tx-2sx)}dx \\
&= e^{-t^2-s^2+(s+t)^2}\int_{-\infty}^{\infty} e^{-(x-t-s)^2}dx \\
&= e^{2st}\int_{-\infty}^{\infty} e^{-u^2}du \\
&= \sqrt{\pi}e^{2st}.
\end{aligned} \tag{1.4}$$

With the Hermite polynomial definition of the generating function, this integral is

$$\begin{aligned}\int_{-\infty}^{\infty} g(x,t)g(x,s)e^{-x^2}dx &= \int_{-\infty}^{\infty}\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} H_m(x)\frac{s^m}{m!}e^{-x^2}dx \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{s^m}{m!}\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx.\end{aligned} \tag{1.5}$$

Let

$$\alpha_{nm} = \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx, \tag{1.6}$$

and equate the two expansions of this integral

$$\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2st)^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{s^m}{m!}\alpha_{nm}, \tag{1.7}$$

or, after equating powers of t^n

$$\sqrt{\pi}(2s)^n = \sum_{m=0}^{\infty}\frac{s^m}{m!}\alpha_{nm}. \tag{1.8}$$

This requires α_{nm} to be zero for $n \neq m$, so

$$\sqrt{\pi}2^n = \frac{1}{n!}\alpha_{nn}, \tag{1.9}$$

and

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = \delta_{nm}\sqrt{\pi}2^n n!\,. \tag{1.10}$$

The SHO normalization is fixed by

$$\begin{aligned}\int_{-\infty}^{\infty} u_n^2(x)dx &= c_n^2\int_{-\infty}^{\infty} H_n^2(x/x_0)e^{-(x/x_0)^2}dx \\ &= c_n^2 x_0\sqrt{\pi}2^n n!\,,\end{aligned} \tag{1.11}$$

or

$$\begin{aligned}c_n &= \frac{1}{\sqrt{\sqrt{\pi}2^n n!\sqrt{\frac{\hbar}{m\omega}}}} \\ &= \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} 2^{-n/2}\frac{1}{\sqrt{n!}}\end{aligned} \tag{1.12}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1