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# PHY1520H Graduate Quantum Mechanics. Lecture 12: Symmetry (cont.). Taught by Prof. Arun Paramekanti

*Disclaimer* Peeter's lecture notes from class. These may be incoherent and rough.

These are notes for the UofT course PHY1520, Graduate Quantum Mechanics, taught by Prof. Paramekanti, covering ch. 1 [1] content.

*Parity (review)*

$$\hat{\Pi}\hat{x}\hat{\Pi} = -\hat{x} \tag{1.1}$$

$$\hat{\Pi}\hat{p}\hat{\Pi} = -\hat{p} \tag{1.2}$$

These are polar vectors, in contrast to an axial vector such as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

$$\hat{\Pi}^2 = 1 \tag{1.3}$$

$$\Psi(x) \rightarrow \Psi(-x) \tag{1.4}$$

If $\left[\hat{\Pi}, \hat{H}\right] = 0$ then all the eigenstates are either

- even: $\hat{\Pi}$ eigenvalue is $+1$.
- odd: $\hat{\Pi}$ eigenvalue is $-1$.

We are done with discrete symmetry operators for now.

*Translations* Define a (continuous) translation operator

$$\hat{T}_\epsilon \left|x\right\rangle = \left|x + \epsilon\right\rangle \tag{1.5}$$

The action of this operator is sketched in fig. 1.1.
This is a unitary operator

$$\hat{T}_{-\epsilon} = \hat{T}_\epsilon^\dagger = \hat{T}_\epsilon^{-1} \tag{1.6}$$

In a position basis, the action of this operator is

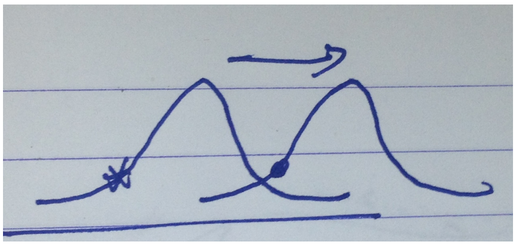


Figure 1.1: Translation operation.

$$\begin{aligned}\langle x|\,\hat{T}_\epsilon\,|\psi\rangle &= \langle x-\epsilon|\psi\rangle \\ &= \psi(x-\epsilon)\end{aligned} \tag{1.7}$$

$$\Psi(x-\epsilon) \approx \Psi(x) - \epsilon\frac{\partial \Psi}{\partial x} \tag{1.8}$$

$$\langle x|\,\hat{T}_\epsilon\,|\Psi\rangle = \langle x|\Psi\rangle - \frac{\epsilon}{\hbar}\langle x|\,i\hat{p}\,|\Psi\rangle \tag{1.9}$$

$$\hat{T}_\epsilon \approx \left(1 - i\frac{\epsilon}{\hbar}\hat{p}\right) \tag{1.10}$$

A non-infinitesimal translation can be composed of many small translations, as sketched in fig. 1.2.

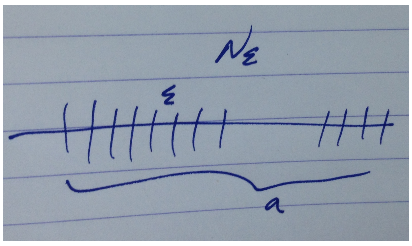



Figure 1.2: Composition of small translations

For $\epsilon \to 0, N \to \infty, N\epsilon = a$, the total translation operator is

$$\begin{aligned}\hat{T}_a &= \hat{T}_\epsilon^N \\ &= \lim_{\epsilon\to 0, N\to\infty, N\epsilon = a}\left(1 - \frac{\epsilon}{\hbar}\hat{p}\right)^N \\ &= e^{-ia\hat{p}/\hbar}\end{aligned} \tag{1.11}$$

The momentum $\hat{p}$ is called a "Generator" of translations. If a Hamiltonian $H$ is translationally invariant, then

$$\left[\hat{T}_a, H\right] = 0, \qquad \forall a. \tag{1.12}$$

This means that momentum will be a good quantum number

$$\left[\hat{p}, H\right] = 0. \tag{1.13}$$

*Rotations* Rotations form a non-Abelian group , since the order of rotations $\hat{R}_1\hat{R}_2 \neq \hat{R}_2\hat{R}_1$. Given a rotation acting on a ket

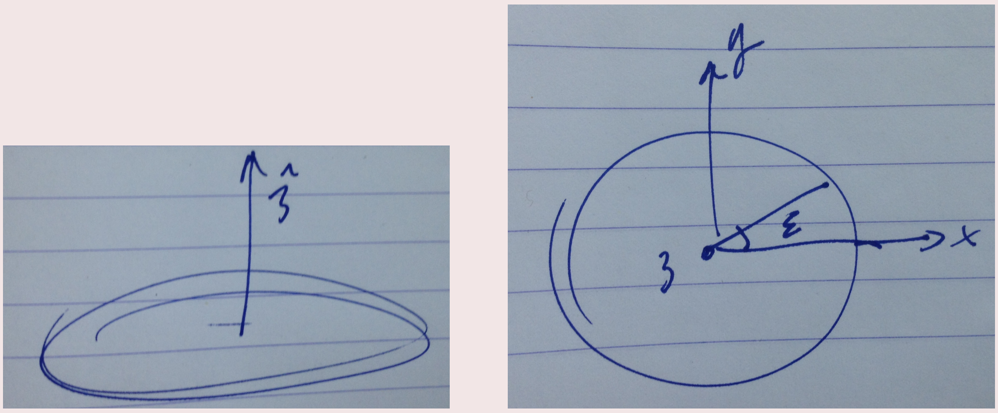
$$\hat{R} \left|\mathbf{r}\right\rangle = \left|R\mathbf{r}\right\rangle , \tag{1.14}$$

observe that the action of the rotation operator on a wave function is inverted

$$\begin{aligned} \left\langle\mathbf{r}\right| \hat{R} \left|\Psi\right\rangle &= \left\langle R^{-1}\mathbf{r}\right| \left|\Psi\right\rangle \\ &= \Psi(R^{-1}\mathbf{r}). \end{aligned} \tag{1.15}$$

**Example 1.1: Z axis normal rotation**

Consider an infinitesimal rotation about the z-axis as sketched in fig. 1.3.




(a) (b)

**Figure 1.3:** Rotation about z-axis.

$$
\begin{aligned}
x' &= x - \epsilon y \\
y' &= y + \epsilon y \\
z' &= z
\end{aligned}
\tag{1.16}
$$

The rotated wave function is

$$
\begin{aligned}
\tilde{\Psi}(x, y, z) &= \Psi(x + \epsilon y, y - \epsilon x, z) \\
&= \Psi(x, y, z) + \epsilon y \underbrace{\frac{\partial \Psi}{\partial x}}_{i\hat{p}_x/\hbar} - \epsilon x \underbrace{\frac{\partial \Psi}{\partial y}}_{i\hat{p}_y/\hbar} .
\end{aligned}
\tag{1.17}
$$

The state must then transform as

$$
\left|\tilde{\Psi}\right\rangle = \left(1 + i\frac{\epsilon}{\hbar}\hat{y}\hat{p}_x - i\frac{\epsilon}{\hbar}\hat{x}\hat{p}_y\right)\left|\Psi\right\rangle . \tag{1.18}
$$

Observe that the combination $\hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ is the $\hat{L}_z$ component of angular momentum $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$, so the infinitesimal rotation can be written

$$\boxed{\hat{R}_z(\epsilon) \left|\Psi\right\rangle = \left(1 - i\frac{\epsilon}{\hbar}\hat{L}_z\right) \left|\Psi\right\rangle .} \tag{1.19}$$

For a finite rotation $\epsilon \rightarrow 0, N \rightarrow \infty, \phi = \epsilon N$, the total rotation is

$$\hat{R}_z(\phi) = \left(1 - \frac{i\epsilon}{\hbar}\hat{L}_z\right)^N , \tag{1.20}$$

or

$$\boxed{\hat{R}_z(\phi) = e^{-i\frac{\phi}{\hbar}\hat{L}_z}.} \tag{1.21}$$

Note that $\left[\hat{L}_x, \hat{L}_y\right] \neq 0$.

By construction using Euler angles or any other method, a general rotation will include contributions from components of all the angular momentum operator, and will have the structure

$$\boxed{\hat{R}_{\hat{\mathbf{n}}}(\phi) = e^{-i\frac{\phi}{\hbar}\left(\hat{\mathbf{L}}\cdot\hat{\mathbf{n}}\right)}.} \tag{1.22}$$

*Rotationally invariant $\hat{H}$.* Given a rotationally invariant Hamiltonian

$$\left[\hat{R}_{\hat{\mathbf{n}}}(\phi), \hat{H}\right] = 0 \qquad \forall \hat{\mathbf{n}}, \phi, \tag{1.23}$$

then every

$$\left[\mathbf{L}\cdot\hat{\mathbf{n}}, \hat{H}\right] = 0, \tag{1.24}$$

or

$$\left[L_i, \hat{H}\right] = 0, \tag{1.25}$$

Non-Abelian implies degeneracies in the spectrum.

*Time-reversal* Imagine that we have something moving along a curve at time $t = 0$, and ending up at the final position at time $t = t_f$, as sketched in fig. 1.4.

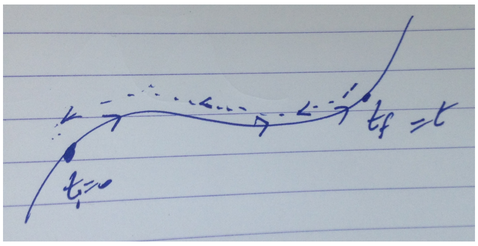



**Figure 1.4:** Time reversal trajectory.

Now imagine that we flip the direction of motion (i.e. flipping the velocity) and run time backwards so the final-time state becomes the initial state.

If the time reversal operator is designated $\hat{\Theta}$, with operation

$$\hat{\Theta} |\Psi\rangle = |\tilde{\Psi}\rangle , \tag{1.26}$$

so that

$$\hat{\Theta}^{-1} e^{-i\hat{H}t/\hbar} \hat{\Theta} |\Psi(t)\rangle = |\Psi(0)\rangle , \tag{1.27}$$

or

$$\hat{\Theta}^{-1} e^{-i\hat{H}t/\hbar} \hat{\Theta} |\Psi(0)\rangle = |\Psi(-t)\rangle . \tag{1.28}$$

# Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1