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PHY1520H Graduate Quantum Mechanics. Lecture 16: Addition of angular momenta. Taught by Prof. Arun Paramekanti

Disclaimer Peeter's lecture notes from class. These may be incoherent and rough.

These are notes for the UofT course PHY1520, Graduate Quantum Mechanics, taught by Prof. Paramekanti, covering ch. 3 [1] content.

1.1 Addition of angular momenta (cont.)

- For orbital angular momentum

  $$\begin{aligned}\hat{\mathbf{L}}_1 &= \hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1 \\ \hat{\mathbf{L}}_1 &= \hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1,\end{aligned} \tag{1.1}$$

  We can show that it is true that

  $$\left[L_{1i} + L_{2i}, L_{1j} + L_{2j} \right] = i\hbar \epsilon_{ijk} \left(L_{1k} + L_{2k} \right), \tag{1.2}$$

  because the angular momentum of the independent particles commute. Given this is it fair to consider that the sum

  $$\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2 \tag{1.3}$$

  is also angular momentum.

- Given $\lvert l_1, m_1 \rangle$ and $\lvert l_2, m_2 \rangle$, if a measurement is made of $\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2$, what do we get?

  Specifically, what do we get for

  $$\left(\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2 \right)^2, \tag{1.4}$$

  and for

  $$\left(\hat{L}_{1z} + \hat{L}_{2z} \right). \tag{1.5}$$

For the latter, we get

$$\left(\hat{L}_{1z} + \hat{L}_{2z}\right) |l_1, m_1; l_2, m_2\rangle = (\hbar m_1 + \hbar m_2) |l_1, m_1; l_2, m_2\rangle \tag{1.6}$$

Given

$$\hat{L}_{1z} + \hat{L}_{2z} = \hat{L}_z^{\text{tot}}, \tag{1.7}$$

we find

$$\begin{aligned}
\left[\hat{L}_z^{\text{tot}}, \hat{\mathbf{L}}_1^2\right] &= 0 \\
\left[\hat{L}_z^{\text{tot}}, \hat{\mathbf{L}}_2^2\right] &= 0 \\
\left[\hat{L}_z^{\text{tot}}, \hat{\mathbf{L}}_{1z}\right] &= 0 \\
\left[\hat{L}_z^{\text{tot}}, \hat{\mathbf{L}}_{1z}\right] &= 0.
\end{aligned} \tag{1.8}$$

We also find

$$\begin{aligned}
\left[(\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2)^2, \mathbf{L}_1^2\right] &= \left[\hat{\mathbf{L}}_1^2 + \hat{\mathbf{L}}_2^2 + 2\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2, \mathbf{L}_1^2\right] \\
&= 0,
\end{aligned} \tag{1.9}$$

but for

$$\begin{aligned}
\left[(\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2)^2, \hat{L}_{1z}\right] &= \left[\hat{\mathbf{L}}_1^2 + \hat{\mathbf{L}}_2^2 + 2\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2, \hat{L}_{1z}\right] \\
&= 2\left[\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2, \hat{L}_{1z}\right] \\
&\neq 0.
\end{aligned} \tag{1.10}$$

Classically if we have measured $\hat{\mathbf{L}}_1$ and $\hat{\mathbf{L}}_2$ then we know the total angular momentum as sketched in fig. 1.1.

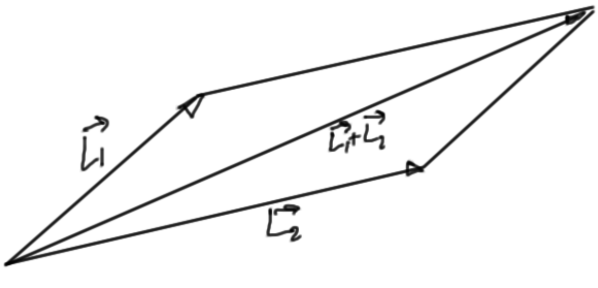

Figure 1.1: Classical addition of angular momenta.

In QM where we don't know all the components of the angular momentum simultaneously, things get fuzzier. For example, if the $\hat{L}_{1z}$ and $\hat{L}_{2z}$ components have been measured, we have the angular momentum defined within a conical region as sketched in fig. 1.2.

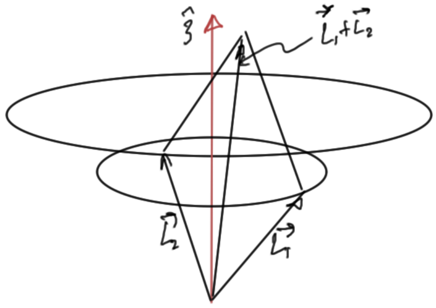

Figure 1.2: Addition of angular momenta given measured $\hat{L}_z$.

Suppose we know $\hat{L}_z^{\text{tot}}$ precisely, but have imprecise information about $\left(\hat{\mathbf{L}}^{\text{tot}}\right)^2$. Can we determine bounds for this? Let $\left|\psi\right\rangle = \left|l_1, m_2; l_2, m_2\right\rangle$, so

$$\begin{aligned}\left\langle\psi\right|\left(\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2\right)^2\left|\psi\right\rangle &= \left\langle\psi\right|\hat{\mathbf{L}}_1^2\left|\psi\right\rangle + \left\langle\psi\right|\hat{\mathbf{L}}_2^2\left|\psi\right\rangle + 2\left\langle\psi\right|\hat{\mathbf{L}}_1\cdot\hat{\mathbf{L}}_2\left|\psi\right\rangle \\ &= l_1\left(l_1+1\right)\hbar^2 + l_2\left(l_2+1\right)\hbar^2 + 2\left\langle\psi\right|\hat{\mathbf{L}}_1\cdot\hat{\mathbf{L}}_2\left|\psi\right\rangle.\end{aligned} \tag{1.11}$$

Using the Cauchy-Schwartz inequality

$$\left|\left\langle\phi|\psi\right\rangle\right|^2 \le \left|\left\langle\phi|\phi\right\rangle\right|\left|\left\langle\psi|\psi\right\rangle\right|, \tag{1.12}$$

which is the equivalent of the classical relationship

$$\left(\mathbf{A}\cdot\mathbf{B}\right)^2 \le \mathbf{A}^2\mathbf{B}^2. \tag{1.13}$$

Applying this to the last term, we have

$$\begin{aligned}\left(\left\langle\psi\right|\hat{\mathbf{L}}_1\cdot\hat{\mathbf{L}}_2\left|\psi\right\rangle\right)^2 &\le \left\langle\psi\right|\hat{\mathbf{L}}_1\cdot\hat{\mathbf{L}}_1\left|\psi\right\rangle\left\langle\psi\right|\hat{\mathbf{L}}_2\cdot\hat{\mathbf{L}}_2\left|\psi\right\rangle \\ &= \hbar^4 l_1\left(l_1+1\right)l_2\left(l_2+2\right).\end{aligned} \tag{1.14}$$

Thus for the max we have

$$\left\langle\psi\right|\left(\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2\right)^2\left|\psi\right\rangle \le \hbar^2 l_1\left(l_1+1\right) + \hbar^2 l_2\left(l_2+1\right) + 2\hbar^2\sqrt{l_1\left(l_1+1\right)l_2\left(l_2+2\right)} \tag{1.15}$$

and for the min

$$\left\langle\psi\right|\left(\hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2\right)^2\left|\psi\right\rangle \ge \hbar^2 l_1\left(l_1+1\right) + \hbar^2 l_2\left(l_2+1\right) - 2\hbar^2\sqrt{l_1\left(l_1+1\right)l_2\left(l_2+2\right)}. \tag{1.16}$$

To try to pretty up these estimate, starting with the max, note that if we replace a portion of the RHS with something bigger, we are left with a strict less than relationship.
That is

$$\begin{aligned}l_1\left(l_1+1\right) &< \left(l_1 + \frac{1}{2}\right)^2 \\ l_2\left(l_2+1\right) &< \left(l_2 + \frac{1}{2}\right)^2\end{aligned} \tag{1.17}$$

That is

$$\begin{aligned}\langle\psi|\left(\hat{\mathbf{L}}_1+\hat{\mathbf{L}}_2\right)^2|\psi\rangle &< \hbar^2\left(l_1(l_1+1)+l_2(l_2+1)+2\left(l_1+\frac{1}{2}\right)\left(l_2+\frac{1}{2}\right)\right)\\ &= \hbar^2\left(l_1^2+l_2^2+l_1+l_2+2l_1l_2+l_1+l_2+\frac{1}{2}\right)\\ &= \hbar^2\left(\left(l_1+l_2+\frac{1}{2}\right)\left(l_1+l_2+\frac{3}{2}\right)-\frac{1}{4}\right)\end{aligned} \tag{1.18}$$

or

$$l_{\text{tot}}(l_{\text{tot}}+1) < \left(l_1+l_2+\frac{1}{2}\right)\left(l_1+l_2+\frac{3}{2}\right), \tag{1.19}$$

which, gives

$$l_{\text{tot}} < l_1+l_2+\frac{1}{2}. \tag{1.20}$$

Finally, given a quantization requirement, that is

$$\boxed{l_{\text{tot}} \le l_1+l_2.} \tag{1.21}$$

Similarly, for the min, we find

$$\begin{aligned}\langle\psi|\left(\hat{\mathbf{L}}_1+\hat{\mathbf{L}}_2\right)^2|\psi\rangle &> \hbar^2\left(l_1(l_1+1)+l_2(l_2+1)-2\left(l_1+\frac{1}{2}\right)\left(l_2+\frac{1}{2}\right)\right)\\ &= \hbar^2\left(l_1^2+l_2^2-2l_1l_2-\frac{1}{2}\right)\\ &= \hbar^2\left(\left(l_1-l_2-\frac{1}{2}\right)\left(l_1-l_2+\frac{1}{2}\right)-\frac{1}{4}\right).\end{aligned} \tag{1.22}$$

The total angular momentum quantum number must then satisfy

$$l_{\text{tot}}(l_{\text{tot}}+1) > \left(l_1-l_2-\frac{1}{2}\right)\left(l_1-l_2+\frac{1}{2}\right)-\frac{1}{4} \tag{1.23}$$

Is it true that

$$l_{\text{tot}}(l_{\text{tot}}+1) > \left(l_1-l_2-\frac{1}{2}\right)\left(l_1-l_2+\frac{1}{2}\right)? \tag{1.24}$$

This is true when $l_{\text{tot}} > l_1 - l_2 - \frac{1}{2}$, assuming that $l_1 > l_2$. Suppose $l_{\text{tot}} = l_1 - l_2 - \frac{1}{2}$, then

$$\begin{aligned}l_{\text{tot}}(l_{\text{tot}}+1) &= \left(l_1-l_2-\frac{1}{2}\right)\left(l_1-l_2+\frac{1}{2}\right)\\ &= (l_1-l_2)^2-\frac{1}{4}.\end{aligned} \tag{1.25}$$

So, is it true that

$$(l_1 - l_2)^2 - \frac{1}{4} \geq l_1^2 + l_1 + l_2^2 + l_2 - 2\sqrt{l_1(l_1+1)l_2(l_2+1)}? \tag{1.26}$$

If that is the case we have

$$-2l_1 l_2 - \frac{1}{4} \geq l_1 + l_2 - 2\sqrt{l_1(l_1+1)l_2(l_2+1)}, \tag{1.27}$$

$$\begin{aligned} 2\sqrt{l_2(l_1+1)l_1(l_2+1)} &\geq l_1 + l_2 + 2l_1 l_2 + \frac{1}{4} \\ &= l_1(l_2+1) + l_2(l_1+1) + \frac{1}{4}. \end{aligned} \tag{1.28}$$

This has the structure

$$2\sqrt{xy} \geq x + y + \frac{1}{4}, \tag{1.29}$$

or

$$4xy \geq (x+y)^2 + \frac{1}{16} + \frac{1}{2}(x+y), \tag{1.30}$$

or

$$0 \geq (x-y)^2 + \frac{1}{16} + \frac{1}{2}(x+y), \tag{1.31}$$

But since $x + y \geq 0$ this inequality is not satisfied when $l_{\text{tot}} = l_1 - l_2 - \frac{1}{2}$. We can conclude

$$l_1 - l_2 - \frac{1}{2} < l_{\text{tot}} < l_1 + l_2 + \frac{1}{2}. \tag{1.32}$$

Is it true that

$$l_1 - l_2 \geq l_{\text{tot}} \geq l_1 + l_2? \tag{1.33}$$

Note that we have two separate Hilbert spaces $l_1 \otimes l_2$ of dimension $2l_1 + 1$ and $2l_2 + 1$ respectively. The total number of states is

$$\begin{aligned} \sum_{l_{\text{tot}} = l_1 - l_2}^{l_1 + l_2} (2l_{\text{tot}} + 1) &= 2 \sum_{n = l_1 - l_2}^{l_1 + l_2} n + \cancel{l_1} + l_2 - (\cancel{l_1} - l_2) + 1 \\ &= 2\frac{1}{2}\left(l_1 + l_2 + (l_1 - l_2)\right)\left(l_1 + l_2 - (l_1 - l_2) + 1\right) + 2l_2 + 1 \\ &= 2l_1\left(2l_2 + 1\right) + 2l_2 + 1 \\ &= (2l_1 + 1)(2l_2 + 1). \end{aligned} \tag{1.34}$$

So the end result is that given $|l_1, m_1\rangle, |l_2, m_2\rangle$, with $l_1 \geq l_2$, where, in steps of 1,

$$\boxed{l_1 - l_2 \leq l_{\text{tot}} \leq l_1 + l_2.} \tag{1.35}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1