Peeter Joot
peeter.joot@gmail.com

PHY1520H Graduate Quantum Mechanics. Lecture 4: Quantum Harmonic oscillator and coherent states. Taught by Prof. Arun Paramekanti

Disclaimer Peeter's lecture notes from class. These may be incoherent and rough. This lecture reviewed a lot of quantum harmonic oscillator theory, and wouldn't make sense without having seen raising and lowering operators (ladder operators), number operators, and the like.

These are notes for the UofT course PHY1520, Graduate Quantum Mechanics, taught by Prof. Paramekanti, covering ch. 2 [1] content.

Classical Harmonic Oscillator Recall the classical Harmonic oscillator equations in their Hamiltonian form

$$\frac{dx}{dt} = \frac{p}{m} \tag{1.1a}$$

$$\frac{dp}{dt} = -kx. \tag{1.1b}$$

With

$$\begin{aligned} x(t = 0) &= x_0 \\ p(t = 0) &= p_0 \\ k &= m\omega^2, \end{aligned} \tag{1.2}$$

the solutions are ellipses in phase space

$$x(t) = x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) \tag{1.3a}$$

$$p(t) = p_0 \cos(\omega t) - m\omega x_0 \sin(\omega t). \tag{1.3b}$$

After a suitable scaling of the variables, these elliptical orbits can be transformed into circular trajectories.

Quantum Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 \tag{1.4}$$

Set

$$\hat{X} = \sqrt{\frac{m\omega}{\hbar}}\hat{x} \tag{1.5a}$$

$$\hat{P} = \sqrt{\frac{1}{m\omega\hbar}}\hat{p} \tag{1.5b}$$

The commutators after this change of variables goes from

$$[\hat{x}, \hat{p}] = i\hbar, \tag{1.6}$$

to

$$\left[\hat{X}, \hat{P}\right] = i. \tag{1.7}$$

The Hamiltonian takes the form

$$\begin{aligned} \hat{H} &= \frac{\hbar\omega}{2}\left(\hat{X}^2 + \hat{P}^2\right) \\ &= \hbar\omega\left(\left(\frac{\hat{X} - i\hat{P}}{\sqrt{2}}\right)\left(\frac{\hat{X} + i\hat{P}}{\sqrt{2}}\right) + \frac{1}{2}\right). \end{aligned} \tag{1.8}$$

Define ladder operators (raising and lowering operators respectively)

$$\hat{a}^\dagger = \frac{\hat{X} - i\hat{P}}{\sqrt{2}} \tag{1.9a}$$

$$\hat{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}} \tag{1.9b}$$

so

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{1.10}$$

We can show

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1, \tag{1.11}$$

and

$$N\left|n\right\rangle \equiv \hat{a}^\dagger a = n\left|n\right\rangle, \tag{1.12}$$

where $n \geq 0$ is an integer. Recall that

$$\hat{a}\,|0\rangle = 0, \tag{1.13}$$

and

$$\langle X|\, X + iP\,|0\rangle = 0. \tag{1.14}$$

With

$$\langle x|0\rangle = \Psi_0(x), \tag{1.15}$$

we can show

$$\frac{1}{\sqrt{2}}\left(X + \frac{\partial}{\partial X}\right)\Psi_0(X) = 0. \tag{1.16}$$

Also recall that

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{1.17a}$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{1.17b}$$

Coherent states Coherent states for the quantum harmonic oscillator are the eigenkets for the creation and annihilation operators

$$\hat{a}\,|z\rangle = z\,|z\rangle \tag{1.18a}$$

$$\hat{a}^\dagger\,|\tilde{z}\rangle = \tilde{z}\,|\tilde{z}\rangle\,, \tag{1.18b}$$

where

$$|z\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle\,, \tag{1.19}$$

and z is allowed to be a complex number.
Looking for such a state, we compute

$$\begin{aligned}\hat{a}\,|z\rangle &= \sum_{n=1}^{\infty} c_n \hat{a}\,|n\rangle \\ &= \sum_{n=1}^{\infty} c_n \sqrt{n}\,|n-1\rangle\end{aligned} \tag{1.20}$$

compare this to

$$\begin{aligned}z\,|z\rangle &= z\sum_{n=0}^{\infty} c_n\,|n\rangle \\ &= \sum_{n=1}^{\infty} c_n \sqrt{n}\,|n-1\rangle \\ &= \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle\,,\end{aligned} \tag{1.21}$$

so

$$c_{n+1}\sqrt{n+1} = zc_n \tag{1.22}$$

This gives

$$c_{n+1} = \frac{zc_n}{\sqrt{n+1}} \tag{1.23}$$

$$\begin{aligned} c_1 &= c_0 z \\ c_2 &= \frac{zc_1}{\sqrt{2}} = \frac{z^2 c_0}{\sqrt{2}} \\ &\vdots \end{aligned} \tag{1.24}$$

or

$$c_n = \frac{z^n}{\sqrt{n!}}. \tag{1.25}$$

So the desired state is

$$|z\rangle = c_0 \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle . \tag{1.26}$$

Also recall that

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}} |0\rangle , \tag{1.27}$$

which gives

$$\begin{aligned} |z\rangle &= c_0 \sum_{n=0}^{\infty} \frac{\left(z\hat{a}^\dagger\right)^n}{n!} |0\rangle \\ &= c_0 e^{z\hat{a}^\dagger} |0\rangle . \end{aligned} \tag{1.28}$$

The normalization is

$$c_0 = e^{-|z|^2/2}. \tag{1.29}$$

While we have $\langle n_1 | n_2 \rangle = \delta_{n_1,n_2}$, these $|z\rangle$ states are not orthonormal. Figuring out that this overlap

$$\langle z_1 | z_2 \rangle \neq 0, \tag{1.30}$$

will be left for homework.

Dynamics We don't know much about these coherent states. For example does a coherent state at time zero evolve to a coherent state?

$$\left|z\right\rangle \xrightarrow{?} \left|z(t)\right\rangle \tag{1.31}$$

It turns out that these questions are best tackled in the Heisenberg picture, considering

$$e^{-i\hat{H}t/\hbar} \left|z\right\rangle . \tag{1.32}$$

For example, what is the average of the position operator

$$\left\langle z\right| e^{i\hat{H}t/\hbar} \hat{x} e^{-i\hat{H}t/\hbar} \left|z\right\rangle = \sum_{n,n'=0}^{\infty} \left\langle n\right| c_n^* e^{iE_n t/\hbar} \left(a + a^\dagger\right) \sqrt{\frac{\hbar}{m\omega}} c_{n'} e^{iE_{n'} t/\hbar} \left|n\right\rangle . \tag{1.33}$$

This is very messy to attempt. Instead if we know how the operator evolves we can calculate

$$\left\langle z\right| \hat{x}_{\mathrm{H}}(t) \left|z\right\rangle , \tag{1.34}$$

that is

$$\left\langle \hat{x}\right\rangle (t) = \left\langle z\right| \hat{x}_{\mathrm{H}}(t) \left|z\right\rangle , \tag{1.35}$$

and for momentum

$$\left\langle \hat{p}\right\rangle (t) = \left\langle z\right| \hat{p}_{\mathrm{H}}(t) \left|z\right\rangle . \tag{1.36}$$

The question to ask is what are the expansions of

$$\hat{a}_{\mathrm{H}}(t) = e^{i\hat{H}t/\hbar} \hat{a} e^{-i\hat{H}t/\hbar}. \tag{1.37a}$$

$$\hat{a}_{\mathrm{H}}^\dagger(t) = e^{i\hat{H}t/\hbar} \hat{a}^\dagger e^{-i\hat{H}t/\hbar}. \tag{1.37b}$$

The question to ask is how do these operators ask on the basis states

$$\begin{aligned}
\hat{a}_{\mathrm{H}}(t) \left|n\right\rangle &= e^{i\hat{H}t/\hbar} \hat{a} e^{-i\hat{H}t/\hbar} \left|n\right\rangle \\
&= e^{i\hat{H}t/\hbar} \hat{a} e^{-it\omega(n+1/2)} \left|n\right\rangle \\
&= e^{-it\omega(n+1/2)} e^{i\hat{H}t/\hbar} \sqrt{n} \left|n-1\right\rangle \\
&= \sqrt{n} e^{-it\omega(n+1/2)} e^{it\omega(n-1/2)} \left|n-1\right\rangle \\
&= \sqrt{n} e^{-i\omega t} \left|n-1\right\rangle \\
&= e^{-i\omega t} \left|n\right\rangle .
\end{aligned} \tag{1.38}$$

So we have found

$$\begin{aligned}
\hat{a}_{\mathrm{H}}(t) &= a e^{-i\omega t} \\
\hat{a}_{\mathrm{H}}^\dagger(t) &= a^\dagger e^{i\omega t}
\end{aligned} \tag{1.39}$$

Position and momentum operator time evolution

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1