Peeter Joot
peeter.joot@gmail.com

Reciprocity theorem

The class slides presented a derivation of the reciprocity theorem, a theorem that contained the integral of

$$\int \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{E}^{(b)} \times \mathbf{H}^{(a)} \right) \cdot d\mathbf{S} = \cdots \tag{1.1}$$

over a surface, where the RHS was a volume integral involving the fields and (electric and magnetic) current sources. The idea was to consider two different source loading configurations of the same system, and to show that the fields and sources in the two configurations can be related.

To derive the result in question, a simple way to start is to look at the divergence of the difference of cross products above. This will require the phasor form of the two cross product Maxwell's equations

$$\boldsymbol{\nabla} \times \mathbf{E} = -(\mathbf{M} + j\omega\mu_0 \mathbf{H}) \tag{1.2a}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J} + j\omega\epsilon_0 \mathbf{E}, \tag{1.2b}$$

so the divergence is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{E}^{(b)} \times \mathbf{H}^{(a)} \right) &= \mathbf{H}^{(b)} \cdot \left(\boldsymbol{\nabla} \times \mathbf{E}^{(a)} \right) - \mathbf{E}^{(a)} \cdot \left(\boldsymbol{\nabla} \times \mathbf{H}^{(b)} \right) \\
&\quad - \mathbf{H}^{(a)} \cdot \left(\boldsymbol{\nabla} \times \mathbf{E}^{(b)} \right) + \mathbf{E}^{(b)} \cdot \left(\boldsymbol{\nabla} \times \mathbf{H}^{(a)} \right) \\
&= -\mathbf{H}^{(b)} \cdot \left(\mathbf{M}^{(a)} + j\omega\mu_0 \mathbf{H}^{(a)} \right) - \mathbf{E}^{(a)} \cdot \left(\mathbf{J}^{(b)} + j\omega\epsilon_0 \mathbf{E}^{(b)} \right) \\
&\quad + \mathbf{H}^{(a)} \cdot \left(\mathbf{M}^{(b)} + j\omega\mu_0 \mathbf{H}^{(b)} \right) + \mathbf{E}^{(b)} \cdot \left(\mathbf{J}^{(a)} + j\omega\epsilon_0 \mathbf{E}^{(a)} \right).
\end{aligned} \tag{1.3}$$

The non-source terms cancel, leaving

$$\boxed{\boldsymbol{\nabla} \cdot \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{E}^{(b)} \times \mathbf{H}^{(a)} \right) = -\mathbf{H}^{(b)} \cdot \mathbf{M}^{(a)} - \mathbf{E}^{(a)} \cdot \mathbf{J}^{(b)} + \mathbf{H}^{(a)} \cdot \mathbf{M}^{(b)} + \mathbf{E}^{(b)} \cdot \mathbf{J}^{(a)}} \tag{1.4}$$

Should we be suprised to have a relation of this form? Probably not, given that the energy momentum relationship between the fields and currents of a single source has the form

$$\frac{\partial}{\partial t} \frac{\epsilon_0}{2} \left(\boldsymbol{\mathcal{E}}^2 + c^2 \boldsymbol{\mathcal{B}}^2 \right) + \boldsymbol{\nabla} \cdot \frac{1}{\mu_0} (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}) = -\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}}. \tag{1.5}$$

(this is without magnetic sources).

This initially suggests that the reciprocity theorem can be expressed more generally in terms of the energy-momentum tensor. However, there are some subtle differences since the time domain products lead to averages in terms of the real parts of conjugate pairs such as $\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}} \rightarrow \mathbf{E} \times \mathbf{B}^*$, and $\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{J}} \rightarrow \mathbf{E} \cdot \mathbf{J}^*$.

1.1 far field integral form

Employing the divergence theorem over a sphere the identity above takes the form

$$\int_S \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{E}^{(b)} \times \mathbf{H}^{(a)} \right) \cdot \hat{\mathbf{r}} dS = \int_V \left(-\mathbf{H}^{(b)} \cdot \mathbf{M}^{(a)} - \mathbf{E}^{(a)} \cdot \mathbf{J}^{(b)} + \mathbf{H}^{(a)} \cdot \mathbf{M}^{(b)} + \mathbf{E}^{(b)} \cdot \mathbf{J}^{(a)} \right) dV \tag{1.6}$$

In the far field, the cross products are strictly radial. That surface integral can be written as

$$\begin{aligned} \int_S \left(\mathbf{E}^{(a)} \times \mathbf{H}^{(b)} - \mathbf{E}^{(b)} \times \mathbf{H}^{(a)} \right) \cdot \hat{\mathbf{r}} dS &= \frac{1}{\mu_0} \int_S \left(\mathbf{E}^{(a)} \times \left(\hat{\mathbf{r}} \times \mathbf{E}^{(b)} \right) - \mathbf{E}^{(b)} \times \left(\hat{\mathbf{r}} \times \mathbf{E}^{(a)} \right) \right) \cdot \hat{\mathbf{r}} dS \\ &= \frac{1}{\mu_0} \int_S \left(\mathbf{E}^{(a)} \cdot \mathbf{E}^{(b)} - \mathbf{E}^{(b)} \cdot \mathbf{E}^{(a)} \right) dS \\ &= 0 \end{aligned} \tag{1.7}$$

The above expansions used eq. (1.11) to expand the terms of the form

$$(\mathbf{A} \times (\hat{\mathbf{r}} \times \mathbf{C})) \cdot \hat{\mathbf{r}} = \mathbf{A} \cdot \mathbf{C} - (\mathbf{A} \cdot \hat{\mathbf{r}}) (\mathbf{C} \cdot \hat{\mathbf{r}}), \tag{1.8}$$

in which only the first dot product survives due to the transverse nature of the fields.

So in the far field we have a direct relation between the fields and sources of two source configurations of the same system of the form

$$\boxed{\int_V \left(\mathbf{H}^{(a)} \cdot \mathbf{M}^{(b)} + \mathbf{E}^{(b)} \cdot \mathbf{J}^{(a)} \right) dV = \int_V \left(\mathbf{H}^{(b)} \cdot \mathbf{M}^{(a)} + \mathbf{E}^{(a)} \cdot \mathbf{J}^{(b)} \right) dV} \tag{1.9}$$

1.2 Application to antenna

This is the underlying reason that we are able to pose the problem of what an antenna can recieve, in terms of what the antenna can transmit.

Prof. Eleftheriades explained the the send-recieve equivalence using the concepts of a two-port network ([2], [3]).

An alternate, and very intuitive, explaination can be found in appendix A.1 [1], that directly related the current density sources and scalar current to the voltages in those regions using an integral representation of the reciprocity theorem.

1.3 Identities

Lemma 1.1: Divergence of a cross product

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B}\,(\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A}\,(\boldsymbol{\nabla} \times \mathbf{B})\,.$$

Proof.

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) &= \partial_a \epsilon_{abc} A_b B_c \\ &= \epsilon_{abc}(\partial_a A_b) B_c - \epsilon_{bac} A_b (\partial_a B_c) \\ &= \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}). \end{aligned} \tag{1.10}$$

Lemma 1.2: Triple cross product dotted

$$(\mathbf{A} \times (\mathbf{B} \times \mathbf{C})) \cdot \mathbf{D} = (\mathbf{A} \cdot \mathbf{C})\,(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{B})\,(\mathbf{C} \cdot \mathbf{D})$$

Proof.

$$\begin{aligned} (\mathbf{A} \times (\mathbf{B} \times \mathbf{C})) \cdot \mathbf{D} &= \epsilon_{abc} A_b \epsilon_{rsc} B_r C_s D_a \\ &= \delta^{rs}_{[ab]} A_b B_r C_s D_a \\ &= A_s B_r C_s D_r - A_r B_r C_s D_s \\ &= (\mathbf{A} \cdot \mathbf{C})\,(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{B})\,(\mathbf{C} \cdot \mathbf{D})\,. \end{aligned} \tag{1.11}$$

Bibliography

[1] Jiuping Chen, Douglas W Oldenburg, and Edad Haber. Reciprocity in electromagnetics: application to modelling marine magnetometric resistivity data. *Physics of the earth and Planetary Interiors*, 150(1):45–61, 2005. URL http://www.mathcs.emory.edu/~haber/pubs/recip.pdf.

[2] J.D. Irwin. *Basic Engineering Circuit Analysis*. MacMillian, 1993. 1.2

[3] Adel S Sedra and Kenneth Carless Smith. *Microelectronic circuits*. Saunders College Publishing, 3rd edition, 1991. 1.2