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Momentum space representation of Schrödinger equation

Exercise 1.1 Momentum space representation of Schrödinger equation ([1] *pr. 2.15*)

Using

$$\left\langle x' \middle| p' \right\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ip'x'/\hbar}, \tag{1.1}$$

show that

$$\left\langle p' \right| x \left| \alpha \right\rangle = i\hbar \frac{\partial}{\partial p'} \left\langle p' \middle| \alpha \right\rangle. \tag{1.2}$$

Use this to find the momentum space representation of the Schrödinger equation for the one dimensional SHO and the energy eigenfunctions in their momentum representation.

Answer for Exercise 1.1

To expand the matrix element, introduce both momentum and position space identity operators

$$\begin{aligned}
\left\langle p' \right| x \left| \alpha \right\rangle &= \int dx' dp'' \left\langle p' \middle| x' \right\rangle \left\langle x' \right| x \left| p'' \right\rangle \left\langle p'' \middle| \alpha \right\rangle \\
&= \int dx' dp'' \left\langle p' \middle| x' \right\rangle x' \left\langle x' \middle| p'' \right\rangle \left\langle p'' \middle| \alpha \right\rangle \\
&= \frac{1}{2\pi\hbar} \int dx' dp'' e^{-ip'x'/\hbar} x' e^{ip''x'/\hbar} \left\langle p'' \middle| \alpha \right\rangle \\
&= \frac{1}{2\pi\hbar} \int dx' dp'' x' e^{i(p''-p')x'/\hbar} \left\langle p'' \middle| \alpha \right\rangle \\
&= \frac{1}{2\pi\hbar} \int dx' dp'' \frac{d}{dp''} \left(\frac{-i\hbar}{e}^{i(p''-p')x'/\hbar} \right) \left\langle p'' \middle| \alpha \right\rangle \\
&= i\hbar \int dp'' \left(\frac{1}{2\pi\hbar} \int dx' e^{i(p''-p')x'/\hbar} \right) \frac{d}{dp''} \left\langle p'' \middle| \alpha \right\rangle \\
&= i\hbar \int dp'' \delta(p''-p') \frac{d}{dp''} \left\langle p'' \middle| \alpha \right\rangle \\
&= i\hbar \frac{d}{dp'} \left\langle p' \middle| \alpha \right\rangle. \qquad \square
\end{aligned} \tag{1.3}$$

Schrödinger's equation for a time dependent state $|\alpha\rangle = U(t)|\alpha, 0\rangle$ is

$$i\hbar \frac{\partial}{\partial t}|\alpha\rangle = H|\alpha\rangle, \tag{1.4}$$

with the momentum representation

$$i\hbar \frac{\partial}{\partial t}\langle p'|\alpha\rangle = \langle p'|H|\alpha\rangle. \tag{1.5}$$

Expansion of the Hamiltonian matrix element for a strictly spatial dependent potential $V(x)$ gives

$$\begin{aligned}\langle p'|H|\alpha\rangle &= \langle p'|\left(\frac{p^2}{2m} + V(x)\right)|\alpha\rangle \\ &= \frac{(p')^2}{2m} + \langle p'|V(x)|\alpha\rangle.\end{aligned} \tag{1.6}$$

Assuming a Taylor representation of the potential $V(x) = \sum c_k x^k$, we want to calculate

$$\langle p'|V(x)|\alpha\rangle = \sum c_k \langle p'|x^k|\alpha\rangle. \tag{1.7}$$

With $|\alpha\rangle = |p''\rangle$ eq. (1.2) provides the $k = 1$ term

$$\begin{aligned}\langle p'|x|p''\rangle &= i\hbar \frac{d}{dp'}\langle p'|p''\rangle \\ &= i\hbar \frac{d}{dp'}\delta(p' - p''),\end{aligned} \tag{1.8}$$

where it is implied here that the derivative is operating on not just the delta function, but on all else that follows.

Using this the higher powers of $\langle p'|x^k|\alpha\rangle$ can be found easily. For example for $k = 2$ we have

$$\begin{aligned}\langle p'|x^2|\alpha\rangle &= \int dp'' \langle p'|x|p''\rangle \langle p''|x|\alpha\rangle \\ &= \int dp'' i\hbar \frac{d}{dp'}\delta(p' - p'') i\hbar \frac{d}{dp''}\langle p''|\alpha\rangle \\ &= (i\hbar)^2 \frac{d^2}{d(p')^2}\langle p'|\alpha\rangle.\end{aligned} \tag{1.9}$$

This means that the potential matrix element is

$$\begin{aligned}\langle p'|V(x)|\alpha\rangle &= \sum c_k \left(i\hbar \frac{d}{dp'}\right)^k \langle p'|\alpha\rangle \\ &= V\left(i\hbar \frac{d}{dp'}\right).\end{aligned} \tag{1.10}$$

Writing $\Psi_\alpha(p') = \langle p'|\alpha\rangle$, the momentum space representation of Schrödinger's equation for a position dependent potential is

$$\boxed{i\hbar\frac{\partial}{\partial t}\Psi_\alpha(p') = \left(\frac{(p')^2}{2m} + V\left(i\hbar\partial/\partial p'\right)\right)\Psi_\alpha(p').} \tag{1.11}$$

For the SHO Hamiltonian the potential is $V(x) = (1/2)m\omega^2x^2$, so the Schrödinger equation is

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi_\alpha(p') &= \left(\frac{(p')^2}{2m} - \frac{1}{2}m\omega^2\hbar^2\frac{\partial^2}{\partial(p')^2}\right)\Psi_\alpha(p') \\ &= \frac{1}{2m}\left((p')^2 - m^2\omega^2\hbar^2\frac{\partial^2}{\partial(p')^2}\right)\Psi_\alpha(p'). \end{aligned} \tag{1.12}$$

To determine the wave functions, let's non-dimensionalize this and compare to the position space Schrödinger equation. Let

$$p_0^2 = m\omega\hbar, \tag{1.13}$$

so

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi_\alpha(p') &= \frac{p_0^2}{2m}\left(\left(\frac{p'}{p_0}\right)^2 - \frac{\partial^2}{\partial(p'/p_0)^2}\right)\Psi_\alpha(p') \\ &= \frac{\omega\hbar}{2}\left(-\frac{\partial^2}{\partial(p'/p_0)^2} + \left(\frac{p'}{p_0}\right)^2\right)\Psi_\alpha(p'). \end{aligned} \tag{1.14}$$

Compare this to the position space equation with $x_0^2 = m\omega/\hbar$,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi_\alpha(x') &= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial(x')^2} + \frac{1}{2}m\omega^2(x')^2\right)\Psi_\alpha(x') \\ &= \frac{\hbar^2}{2m}\left(-\frac{\partial^2}{\partial(x')^2} + \frac{m^2\omega^2}{\hbar^2}(x')^2\right)\Psi_\alpha(x') \\ &= \frac{\hbar^2x_0^2}{2m}\left(-\frac{\partial^2}{\partial(x'/x_0)^2} + \left(\frac{x'}{x_0}\right)^2\right)\Psi_\alpha(x') \\ &= \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial(x'/x_0)^2} + \left(\frac{x'}{x_0}\right)^2\right)\Psi_\alpha(x'). \end{aligned} \tag{1.15}$$

It's clear that there is a straightforward duality relationship between the respective wave functions. Since

$$\langle x'|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}x_0^{n+1/2}}\left(x' - x_0^2\frac{d}{dx'}\right)^n\exp\left(-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right), \tag{1.16}$$

the momentum space wave functions are

$$\langle p'|n\rangle = \frac{1}{\pi^{1/4}\sqrt{2^n n!}p_0^{n+1/2}}\left(p' - p_0^2\frac{d}{dp'}\right)^n\exp\left(-\frac{1}{2}\left(\frac{p'}{p_0}\right)^2\right). \tag{1.17}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1