# Peeter Joot <br> peeterjoot@protonmail.com 

## Simplest perturbation two by two Hamiltonian

Q: two state Hamiltonian. Given a two-state system

$$
\begin{align*}
H & =H_{0}+\lambda V \\
& =\left[\begin{array}{ll}
E_{1} & \lambda \Delta \\
\lambda \Delta & E_{2}
\end{array}\right] \tag{1.1}
\end{align*}
$$

a Solve the system exactly.
b Find the first order perturbed states and second order energy shifts, and compare to the exact solution.
c Solve the degenerate case for $E_{1}=E_{2}$, and compare to the exact solution.
A: part (a) The energy eigenvalues $\epsilon$ are given by

$$
\begin{equation*}
0=\left(E_{1}-\epsilon\right)\left(E_{2}-\epsilon\right)-(\lambda \Delta)^{2}, \tag{1.2}
\end{equation*}
$$

or

$$
\begin{equation*}
\epsilon^{2}-\epsilon\left(E_{1}+E_{2}\right)+E_{1} E_{2}=(\lambda \Delta)^{2} . \tag{1.3}
\end{equation*}
$$

After rearranging this is

$$
\begin{equation*}
\epsilon=\frac{E_{1}+E_{2}}{2} \pm \sqrt{\left(\frac{E_{1}-E_{2}}{2}\right)^{2}+(\lambda \Delta)^{2}} . \tag{1.4}
\end{equation*}
$$

Notice that for $E_{2}=E_{1}$ we have

$$
\begin{equation*}
\epsilon=E_{1} \pm \lambda \Delta . \tag{1.5}
\end{equation*}
$$

Since a change of basis can always put the problem in a form so that $E_{1}>E_{2}$, let's assume that to make an approximation of the energy eigenvalues for $|\lambda \Delta| \ll\left(E_{1}-E_{2}\right) / 2$

$$
\begin{align*}
\epsilon & =\frac{E_{1}+E_{2}}{2} \pm \frac{E_{1}-E_{2}}{2} \sqrt{1+\frac{(2 \lambda \Delta)^{2}}{\left(E_{1}-E_{2}\right)^{2}}} \\
& \approx \frac{E_{1}+E_{2}}{2} \pm \frac{E_{1}-E_{2}}{2}\left(1+2 \frac{(\lambda \Delta)^{2}}{\left(E_{1}-E_{2}\right)^{2}}\right)  \tag{1.6}\\
& =\frac{E_{1}+E_{2}}{2} \pm \frac{E_{1}-E_{2}}{2} \pm \frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}} \\
& =E_{1}+\frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}}, E_{2}+\frac{(\lambda \Delta)^{2}}{E_{2}-E_{1}} .
\end{align*}
$$

For the perturbed states, starting with the plus case, if
we must have

$$
\begin{align*}
0 & =\left(E_{1}-\left(E_{1}+\frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}}\right)\right) a+\lambda \Delta b  \tag{1.8}\\
& =\left(-\frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}}\right) a+\lambda \Delta b
\end{align*}
$$

so

Similarily for the minus case we must have

$$
\begin{align*}
0 & =\lambda \Delta a+\left(E_{2}-\left(E_{2}+\frac{(\lambda \Delta)^{2}}{E_{2}-E_{1}}\right)\right) b  \tag{1.10}\\
& =\lambda \Delta b+\left(-\frac{(\lambda \Delta)^{2}}{E_{2}-E_{1}}\right) b
\end{align*}
$$

for

A: part (b) For the pertubation the first energy shift for pertubation of the $|+\rangle$ state is

The first order energy shift for the pertubation of the $|-\rangle$ state is also zero. The perturbed $|+\rangle$ state is

The numerator matrix element is

$$
\begin{align*}
\langle-| V|+\rangle & =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{ll}
0 & \Delta \\
\Delta & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{l}
0 \\
\Delta
\end{array}\right]  \tag{1.14}\\
& =\Delta
\end{align*}
$$

so

Observe that this matches the first order series expansion of the exact value above.
For the perturbation of $|-\rangle$ we need the matrix element

$$
\begin{align*}
\langle+| V|-\rangle & =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{ll}
0 & \Delta \\
\Delta & 0
\end{array}\right]\left[\begin{array}{l}
0 \\
1
\end{array}\right] \\
& =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{l}
\Delta \\
0
\end{array}\right]  \tag{1.16}\\
& =\Delta
\end{align*}
$$

so it's clear that the perturbed ket is
also matching the approximation found from the exact computation. The second order energy shifts can now be calculated

$$
\begin{align*}
\langle+| V|+\rangle^{\prime} & =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{ll}
0 & \Delta \\
\Delta & 0
\end{array}\right]\left[\begin{array}{c}
1 \\
\frac{\Delta}{E_{1}-E_{2}}
\end{array}\right] \\
& =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{c}
\frac{\Delta^{2}}{E_{1}-E_{2}} \\
\Delta
\end{array}\right]  \tag{1.18}\\
& =\frac{\Delta^{2}}{E_{1}-E_{2}}
\end{align*}
$$

and

$$
\begin{align*}
\langle-| V|-\rangle^{\prime} & =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{ll}
0 & \Delta \\
\Delta & 0
\end{array}\right]\left[\begin{array}{c}
\frac{\Delta}{E_{2}-E_{1}} \\
1
\end{array}\right] \\
& =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{c}
\Delta \\
\frac{\Delta^{2}}{E_{2}-E_{1}}
\end{array}\right]  \tag{1.19}\\
& =\frac{\Delta^{2}}{E_{2}-E_{1}},
\end{align*}
$$

The energy perturbations are therefore

$$
\begin{align*}
& E_{1} \rightarrow E_{1}+\frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}} \\
& E_{2} \rightarrow E_{2}+\frac{(\lambda \Delta)^{2}}{E_{2}-E_{1}} . \tag{1.20}
\end{align*}
$$

This is what we had by approximating the exact case.
A: part (c) For the $E_{2}=E_{1}$ case, we'll have to diagonalize the perturbation potential. That is

$$
\begin{align*}
& V=U \bigwedge U^{+} \\
& \bigwedge=\left[\begin{array}{cc}
\Delta & 0 \\
0 & -\Delta
\end{array}\right]  \tag{1.21}\\
& U=U^{\dagger}=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right] .
\end{align*}
$$

A change of basis for the Hamiltonian is

$$
\begin{align*}
H^{\prime} & =U^{\dagger} H U \\
& =U^{\dagger} H_{0} U+\lambda U^{\dagger} V U  \tag{1.22}\\
& =E_{1} U^{\dagger}+\lambda U^{\dagger} V U \\
& =H_{0}+\lambda \bigwedge .
\end{align*}
$$

We can now calculate the perturbation energy with respect to the new basis, say $\{|1\rangle,|2\rangle\}$. Those energy shifts are

$$
\begin{align*}
& \Delta^{(1)}=\langle 1| V|1\rangle=\Delta  \tag{1.23}\\
& \Delta^{(2)}=\langle 2| V|2\rangle=-\Delta .
\end{align*}
$$

The perturbed energies are therefore

$$
\begin{align*}
& E_{1} \rightarrow E_{1}+\lambda \Delta  \tag{1.24}\\
& E_{2} \rightarrow E_{2}-\lambda \Delta,
\end{align*}
$$

which matches eq. (1.5), the exact result.

Bibliography

