Peeter Joot
peeter.joot@gmail.com

A symmetric real Hamiltonian

Exercise 1.1 A symmetric real Hamiltonian ([1] pr. 2.9)

Find the time evolution for the state $\left|a'\right\rangle$ for a Hamiltian of the form

$$H = \delta \left(\left|a'\right\rangle \left\langle a'\right| + \left|a''\right\rangle \left\langle a''\right| \right) \tag{1.1}$$

Answer for Exercise 1.1

This Hamiltonian has the matrix representation

$$H = \begin{bmatrix} 0 & \delta \\ \delta & 0 \end{bmatrix}, \tag{1.2}$$

which has a characteristic equation of

$$\lambda^2 - \delta^2 = 0, \tag{1.3}$$

so the energy eigenvalues are $\pm\delta$.
The diagonal basis states are respectively

$$\left|\pm\delta\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix}. \tag{1.4}$$

The time evolution operator is

$$\begin{aligned} U &= e^{-iHt/\hbar} \\ &= e^{-i\delta t/\hbar} \left|+\delta\right\rangle \left\langle +\delta\right| + e^{i\delta t/\hbar} \left|-\delta\right\rangle \left\langle -\delta\right| \\ &= \frac{e^{-i\delta t/\hbar}}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{e^{i\delta t/\hbar}}{2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{e^{-i\delta t/\hbar}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{e^{i\delta t/\hbar}}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\delta t/\hbar) & -i\sin(\delta t/\hbar) \\ -i\sin(\delta t/\hbar) & \cos(\delta t/\hbar) \end{bmatrix}. \end{aligned} \tag{1.5}$$

The desired time evolution in the original basis is

$$\begin{aligned} \left|a', t\right\rangle &= e^{-iHt/\hbar} \left|a', 0\right\rangle \\ &= \begin{bmatrix} \cos(\delta t/\hbar) & -i\sin(\delta t/\hbar) \\ -i\sin(\delta t/\hbar) & \cos(\delta t/\hbar) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\delta t/\hbar) \\ -i\sin(\delta t/\hbar) \end{bmatrix} \\ &= \cos(\delta t/\hbar) \left|a', 0\right\rangle - i\sin(\delta t/\hbar) \left|a'', 0\right\rangle . \end{aligned} \tag{1.6}$$

This evolution has the same structure as left circularly polarized light.
The probability of finding the system in state $|a''\rangle$ given an initial state of $|a', 0\rangle$ is

$$\begin{aligned} P &= \left|\left\langle a'' \middle| a', t\right\rangle\right|^2 \\ &= \sin^2\left(\delta t/\hbar\right) . \end{aligned} \tag{1.7}$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1