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On Tai and Pereira's half power beamwidth approximations

Exercise 1.1 Directivities for a short horizontal electrical dipole

In [2] a field for which directivities can be calculated exactly was used in comparisons of some directivity approximations

$$\mathbf{E} = E_0 \left(\cos\theta \cos\phi \hat{\boldsymbol{\theta}} - \sin\phi \hat{\boldsymbol{\phi}} \right). \tag{1.1}$$

(Observe that an inverse radial dependence in E_0 must be implied here for this to be a valid far-field representation of the field.)

Show that Tai & Pereira's formula gives $D_1 = 3$, and $D_2 = 1$ respectively for this field.

Calculate the exact directivity for this field.

Answer for Exercise 1.1

The field components are

$$E_\theta = E_0 \cos\theta \cos\phi \tag{1.2a}$$

$$E_\phi = -E_0 \sin\phi \tag{1.2b}$$

Using eq. (1.11) from the paper, the directivities are

$$\begin{aligned} D_1 &= \frac{2}{\int_0^\pi \cos^2\theta \sin\theta d\theta} \\ &= \frac{2}{-\frac{1}{3}\cos^3\theta \big|_0^\pi} \\ &= 3, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} D_2 &= \frac{2}{\int_0^\pi \sin\theta d\theta} \\ &= \frac{2}{-\cos\theta \big|_0^\pi} \\ &= 1. \end{aligned} \tag{1.4}$$

To find the exact directivity, first the Poynting vector is required. That is

$$\begin{aligned}
\mathbf{P} &= \frac{|E_0|^2}{2c\mu_0} \left(\cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}\right) \times \left(\hat{\mathbf{r}} \times \left(\cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}\right)\right) \\
&= \frac{|E_0|^2}{2c\mu_0} \left(\cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}\right) \times \left(\cos\theta\cos\phi\hat{\boldsymbol{\phi}} + \sin\phi\hat{\boldsymbol{\theta}}\right) \\
&= \frac{|E_0|^2\hat{\mathbf{r}}}{2c\mu_0} \left(\cos^2\theta\cos^2\phi + \sin^2\phi\right),
\end{aligned} \tag{1.5}$$

so the radiation intensity is

$$U(\theta, \phi) \propto \cos^2\theta\cos^2\phi + \sin^2\phi. \tag{1.6}$$

The $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ contributions to this intensity, and the total intensity are all plotted in fig. 1.1, fig. 1.2, and fig. 1.3 respectively.

FIXME: did I save these under the right paths? Recall thetacap and phicap reversed.

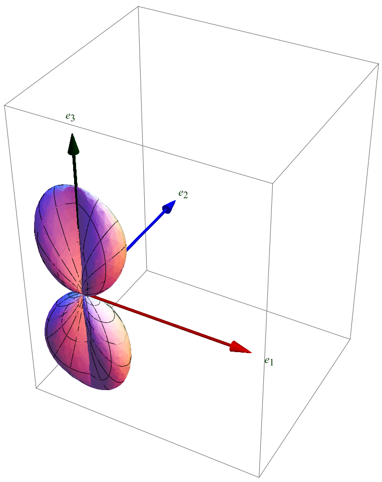

Figure 1.1: The $\hat{\boldsymbol{\theta}}$ contribution to the radiation intensity.

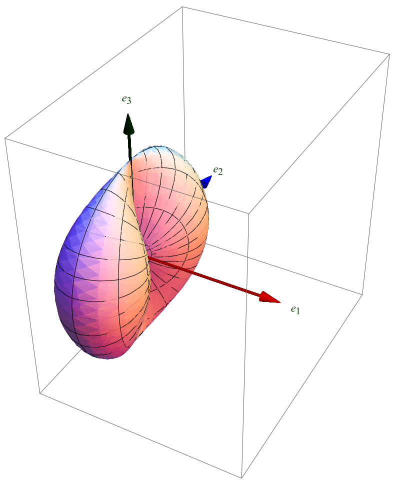

Figure 1.2: The $\hat{\boldsymbol{\phi}}$ contribution to the radiation intensity.

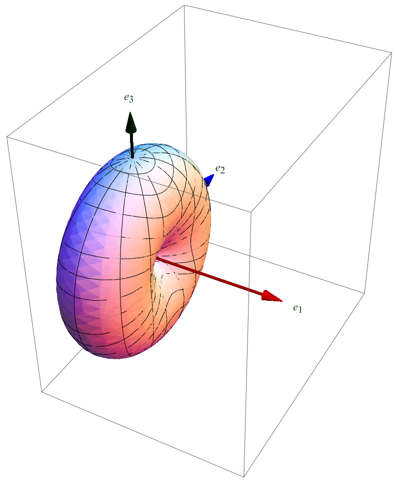

Figure 1.3: Radiation intensity.

Given this the total radiated power is

$$\begin{aligned} P_{\text{rad}} &= \int_0^{2\pi} \int_0^{\pi} \left(\cos^2\theta \cos^2\phi + \sin^2\phi \right) \sin\theta d\theta d\phi \\ &= \frac{8\pi}{3}. \end{aligned} \tag{1.7}$$

Observe that the radiation intensity U can also be decomposed into two components, one for each component of the original $\mathbf{E}$ phasor.

$$U_\theta = \cos^2\theta \cos^2\phi \tag{1.8a}$$

$$U_\phi = \sin^2\phi \tag{1.8b}$$

This decomposition allows for expression of the partial directivities in these respective (orthogonal) directions

$$D_\theta = \frac{4\pi U_\theta}{P_{\text{rad}}} = \frac{3}{2} \cos^2\theta \cos^2\phi \tag{1.9a}$$

$$D_\phi = \frac{4\pi U_\phi}{P_{\text{rad}}} = \frac{3}{2} \sin^2\phi \tag{1.9b}$$

The maximum of each of these partial directivities is both $3/2$, giving a maximum directivity of

$$D_0 = D_\theta\big|_{\text{max}} + D_\phi\big|_{\text{max}} = 3, \tag{1.10}$$

the exact value from the paper.

Exercise 1.2 E and H plane directivities

In [2] directivities associated with the half power beamwidths are given as

$$D_1 = \frac{\left|E_\theta\right|^2_{\max}}{\frac{1}{2}\int_0^\pi \left|E_\theta(\theta,0)\right|^2 \sin\theta d\theta} \tag{1.11a}$$

$$D_2 = \frac{\left|E_\phi\right|^2_{\max}}{\frac{1}{2}\int_0^\pi \left|E_\phi(\theta,\pi/2)\right|^2 \sin\theta d\theta}, \tag{1.11b}$$

whereas [1] lists these as

$$\frac{1}{D_1} = \frac{1}{2\ln 2}\int_0^{\Theta_{1r}/2} \sin\theta d\theta \tag{1.12a}$$

$$\frac{1}{D_2} = \frac{1}{2\ln 2}\int_0^{\Theta_{2r}/2} \sin\theta d\theta. \tag{1.12b}$$

Reconcile these pairs of relations.

Answer for Exercise 1.2

TODO.

Exercise 1.3 Arithmetic mean formula

Equation (1.11) and the associated arithmetic mean formula

$$\frac{1}{D_0} = \frac{1}{2}\left(\frac{1}{D_1} + \frac{1}{D_2}\right), \tag{1.13}$$

should follow from the far field approximation formula for U. Derive that result.

Answer for Exercise 1.3

TODO.

Exercise 1.4 For narrow beams

It is claimed that for narrow beams better approximations are

$$D_1 \approx \frac{16\ln 2}{\Theta_{1r}^2} \tag{1.14a}$$

$$D_1 \approx \frac{16\ln 2}{\Theta_{2r}^2}, \tag{1.14b}$$

where these are derived by considering "the asypmtotic expression for the directivity of an antenna with a rotationally symmetrical power pattern of the form $U(\theta) = \cos^m\theta$, $\theta \in [0,\pi/2]$, with a large value of m".

Derive these results. What does it mean to have a rotationally symmetric power pattern in two different directions?

Answer for Exercise 1.4

TODO.

Bibliography

[1] Constantine A Balanis. *Antenna theory: analysis and design*. John Wiley & Sons, 3rd edition, 2005. 1.2

[2] C-T Tai and CS Pereira. An approximate formula for calculating the directivity of an antenna. *IEEE Transactions on Antennas and Propagation*, 24:235, 1976. 1.1, 1.2