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Two spin time evolution

1.1 Motivation

Our midterm posed a (low mark "quick question") that I didn't complete (or at least not properly). This shouldn't have been a difficult question, but I spend way too much time on it, costing me time that I needed for other questions.

It turns out that there isn't anything fancy required for this question, just perseverance and careful work.

1.2 Guts

The question asked for the time evolution of a two particle state

$$\psi = \frac{1}{\sqrt{2}} \left(\left| \uparrow\downarrow \right\rangle - \left| \downarrow\uparrow \right\rangle \right) \tag{1.1}$$

under the action of the Hamiltonian

$$H = -B S_{z,1} + 2 B S_{x,2} = \frac{\hbar B}{2} \left(-\sigma_{z,1} + 2 \sigma_{x,2} \right). \tag{1.2}$$

We have to know the action of the Hamiltonian on all the states

$$\begin{aligned}
H \left| \uparrow\uparrow \right\rangle &= \frac{B\hbar}{2} \left(- \left| \uparrow\uparrow \right\rangle + 2 \left| \uparrow\downarrow \right\rangle \right) \\
H \left| \uparrow\downarrow \right\rangle &= \frac{B\hbar}{2} \left(- \left| \uparrow\downarrow \right\rangle + 2 \left| \uparrow\uparrow \right\rangle \right) \\
H \left| \downarrow\uparrow \right\rangle &= \frac{B\hbar}{2} \left(\left| \downarrow\uparrow \right\rangle + 2 \left| \downarrow\downarrow \right\rangle \right) \\
H \left| \downarrow\downarrow \right\rangle &= \frac{B\hbar}{2} \left(\left| \downarrow\downarrow \right\rangle + 2 \left| \downarrow\uparrow \right\rangle \right)
\end{aligned} \tag{1.3}$$

With respect to the basis $\left\{ \left| \uparrow\uparrow \right\rangle, \left| \uparrow\downarrow \right\rangle, \left| \downarrow\uparrow \right\rangle, \left| \downarrow\downarrow \right\rangle \right\}$, the matrix of the Hamiltonian is

$$
H = \frac{\hbar B}{2} \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \tag{1.4}
$$

Utilizing the block diagonal form (and ignoring the $\hbar B/2$ factor for now), the characteristic equation is

$$
\begin{aligned}
0 &= \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \\
&= \left((1+\lambda)^2 - 4\right)\left((1-\lambda)^2 - 4\right).
\end{aligned} \tag{1.5}
$$

This has solutions

$$
1 \pm \lambda = \pm 2, \tag{1.6}
$$

or, with the $\hbar B/2$ factors put back in

$$
\lambda = \pm \hbar B/2, \pm 3\hbar B/2. \tag{1.7}
$$

I was thinking that we needed to compute the time evolution operator

$$
U = e^{-iHt/\hbar}, \tag{1.8}
$$

but we actually only need the eigenvectors, and the inverse relations. We can find the eigenvectors by inspection in each case from

$$
\begin{aligned}
H - (1)\frac{\hbar B}{2} &= \frac{\hbar B}{2} \begin{bmatrix} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \\
H - (-1)\frac{\hbar B}{2} &= \frac{\hbar B}{2} \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \\
H - (3)\frac{\hbar B}{2} &= \frac{\hbar B}{2} \begin{bmatrix} -4 & 2 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \\
H - (-3)\frac{\hbar B}{2} &= \frac{\hbar B}{2} \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}.
\end{aligned} \tag{1.9}
$$

The eigenkets are

$$
\begin{aligned}
|1\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
|-1\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \\
|3\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\
|-3\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix},
\end{aligned}
\tag{1.10}
$$

or

$$
\begin{aligned}
\sqrt{2}\,|1\rangle &= |\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle \\
\sqrt{2}\,|-1\rangle &= |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle \\
\sqrt{2}\,|3\rangle &= |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle \\
\sqrt{2}\,|-3\rangle &= |\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle .
\end{aligned}
\tag{1.11}
$$

We can invert these

$$
\begin{aligned}
|\uparrow\uparrow\rangle &= \frac{1}{\sqrt{2}} \left(|1\rangle + |-3\rangle \right) \\
|\uparrow\downarrow\rangle &= \frac{1}{\sqrt{2}} \left(|1\rangle - |-3\rangle \right) \\
|\downarrow\uparrow\rangle &= \frac{1}{\sqrt{2}} \left(|3\rangle + |-1\rangle \right) \\
|\downarrow\downarrow\rangle &= \frac{1}{\sqrt{2}} \left(|3\rangle - |-1\rangle \right)
\end{aligned}
\tag{1.12}
$$

The original state of interest can now be expressed in terms of the eigenkets

$$
\psi = \frac{1}{2} \left(|1\rangle - |-3\rangle - |3\rangle - |-1\rangle \right)
\tag{1.13}
$$

The time evolution of this ket is

$$
\begin{aligned}
\psi(t) &= \frac{1}{2}\left(e^{-iBt/2}\left|1\right\rangle - e^{3iBt/2}\left|-3\right\rangle - e^{-3iBt/2}\left|3\right\rangle - e^{iBt/2}\left|-1\right\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\Bigg(e^{-iBt/2}\left(\left|\uparrow\uparrow\right\rangle + \left|\uparrow\downarrow\right\rangle\right) - e^{3iBt/2}\left(\left|\uparrow\uparrow\right\rangle - \left|\uparrow\downarrow\right\rangle\right) \\
&\quad - e^{-3iBt/2}\left(\left|\downarrow\uparrow\right\rangle + \left|\downarrow\downarrow\right\rangle\right) - e^{iBt/2}\left(\left|\downarrow\uparrow\right\rangle - \left|\downarrow\downarrow\right\rangle\right)\Bigg) \\
&= \frac{1}{2\sqrt{2}}\Bigg(\left(e^{-iBt/2} - e^{3iBt/2}\right)\left|\uparrow\uparrow\right\rangle + \left(e^{-iBt/2} + e^{3iBt/2}\right)\left|\uparrow\downarrow\right\rangle \\
&\quad - \left(e^{-3iBt/2} + e^{iBt/2}\right)\left|\downarrow\uparrow\right\rangle + \left(e^{iBt/2} - e^{-3iBt/2}\right)\left|\downarrow\downarrow\right\rangle\Bigg) \\
&= \frac{1}{2\sqrt{2}}\Bigg(e^{iBt/2}\left(e^{-2iBt/2} - e^{2iBt/2}\right)\left|\uparrow\uparrow\right\rangle + e^{iBt/2}\left(e^{-2iBt/2} + e^{2iBt/2}\right)\left|\uparrow\downarrow\right\rangle \\
&\quad - e^{-iBt/2}\left(e^{-2iBt/2} + e^{2iBt/2}\right)\left|\downarrow\uparrow\right\rangle + e^{-iBt/2}\left(e^{2iBt/2} - e^{-2iBt/2}\right)\left|\downarrow\downarrow\right\rangle\Bigg) \\
&= \frac{1}{\sqrt{2}}\left(i\sin(Bt)\left(e^{-iBt/2}\left|\downarrow\downarrow\right\rangle - e^{iBt/2}\left|\uparrow\uparrow\right\rangle\right) + \cos(Bt)\left(e^{iBt/2}\left|\uparrow\downarrow\right\rangle - e^{-iBt/2}\left|\downarrow\uparrow\right\rangle\right)\right)
\end{aligned} \tag{1.14}
$$

Note that this returns to the original state when $t = \frac{2\pi n}{B}, n \in \mathbb{Z}$. I think I've got it right this time (although I got a slightly different answer on paper before typing it up.)

This doesn't exactly seem like a quick answer question, at least to me. Is there some easier way to do it?