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Unimodular transformation

Q: Show that ([1] pr. 3.3)
Given the matrix

$$U = \frac{a_0 + i\sigma \cdot \mathbf{a}}{a_0 - i\sigma \cdot \mathbf{a}}, \tag{1.1}$$

where $a_0, \mathbf{a}$ are real valued constant and vector respectively.

- Show that this is a unimodular and unitary transformation.
- A unitary transformation can represent an arbitary rotation. Determine the rotation angle and direction in terms of $a_0, \mathbf{a}$.

A: unimodular Let's call these factors $A_\pm$, which expand to

$$\begin{aligned} A_\pm &= a_0 \pm i\sigma \cdot \mathbf{a} \\ &= \begin{bmatrix} a_0 \pm i a_z & \pm\left(a_y + i a_x\right) \\ \mp(a_y - i a_x) & a_0 \mp i a_z \end{bmatrix}, \end{aligned} \tag{1.2}$$

or with $z = a_0 + i a_z$, and $w = a_y + i a_x$, these are

$$A_+ = \begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} \tag{1.3}$$

$$A_- = \begin{bmatrix} z^* & -w \\ w^* & z \end{bmatrix}. \tag{1.4}$$

These both have a determinant of

$$\begin{aligned} |z|^2 + |w|^2 &= |a_0 + i a_z|^2 + \left|a_y + i a_x\right|^2 \\ &= a_0^2 + \mathbf{a}^2. \end{aligned} \tag{1.5}$$

The inverse of the latter is

$$A_-^{-1} = \frac{1}{a_0^2 + \mathbf{a}^2} \begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} \tag{1.6}$$

Noting that the numerator and denominator commute the inverse can be applied in either order. Picking one, the transformation of interest, after writing $A = a_0^2 + \mathbf{a}^2$, is

$$
\begin{aligned}
U &= \frac{1}{A} \begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} \begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix} \\
&= \frac{1}{A} \begin{bmatrix} z^2 - |w|^2 & w(z+z^*) \\ -w^*(z^*+z) & (z^*)^2 - |w|^2 \end{bmatrix}.
\end{aligned}
\tag{1.7}
$$

Recall that a unimodular transformation is one that has the form

$$
\begin{bmatrix} z & w \\ -w^* & z^* \end{bmatrix},
\tag{1.8}
$$

provided $|z|^2 + |w|^2 = 1$, so eq. (1.7) is unimodular if the following sum is unity, which is the case

$$
\begin{aligned}
\frac{\left|z^2 - |w|^2\right|^2}{\left(|z|^2 + |w|^2\right)^2} + |w|^2 \frac{|z+z^*|^2}{\left(|z|^2 + |w|^2\right)^2} &= \frac{\left(z^2 - |w|^2\right)\left((z^*)^2 - |w|^2\right) + |w|^2 (z+z^*)^2}{\left(|z|^2 + |w|^2\right)^2} \\
&= \frac{|z|^4 + |w|^4 - |w|^2 \left(\cancel{z^2 + (z^*)^2}\right) + |w|^2 \left(\cancel{z^2 + (z^*)^2} + 2|z|^2\right)}{\left(|z|^2 + |w|^2\right)^2} \\
&= 1.
\end{aligned}
\tag{1.9}
$$

A: rotation The most general rotation of a vector $\mathbf{a}$, described by Pauli matrices is

$$
e^{i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\theta/2} \boldsymbol{\sigma} \cdot \mathbf{a} e^{-i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\theta/2} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} + (\boldsymbol{\sigma} \cdot \mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{n}})\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \cos\theta + \boldsymbol{\sigma} \cdot (\mathbf{a} \times \hat{\mathbf{n}}) \sin\theta.
\tag{1.10}
$$

If the unimodular matrix above, applied as $\boldsymbol{\sigma} \cdot \mathbf{a}' = U^\dagger \boldsymbol{\sigma} \cdot \mathbf{a} U$ is to also describe this rotation, we want the equivalence

$$
U = e^{-i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\theta/2},
\tag{1.11}
$$

or

$$
\frac{1}{a_0^2 + \mathbf{a}^2} \begin{bmatrix} a_0^2 - \mathbf{a}^2 + 2ia_0a_z & 2a_0(a_y + ia_x) \\ -2a_0(a_y - ia_x) & a_0^2 - \mathbf{a}^2 - 2ia_0a_z \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) - in_z \sin(\theta/2) & (-n_y - in_x)\sin(\theta/2) \\ -(-n_y + in_x)\sin(\theta/2) & \cos(\theta/2) + in_z \sin(\theta/2) \end{bmatrix}.
\tag{1.12}
$$

Equating components, that is

$$
\begin{aligned}
\cos(\theta/2) &= \frac{a_0^2 - \mathbf{a}^2}{a_0^2 + \mathbf{a}^2} \\
-n_x \sin(\theta/2) &= \frac{2a_0a_x}{a_0^2 + \mathbf{a}^2} \\
-n_y \sin(\theta/2) &= \frac{2a_0a_y}{a_0^2 + \mathbf{a}^2} \\
-n_z \sin(\theta/2) &= \frac{2a_0a_y}{a_0^2 + \mathbf{a}^2}
\end{aligned}
\tag{1.13}
$$

Noting that

$$
\begin{aligned}
\sin(\theta/2) &= \sqrt{1 - \frac{(a_0^2 - \mathbf{a}^2)^2}{(a_0^2 + \mathbf{a}^2)^2}} \\
&= \frac{\sqrt{(a_0^2 + \mathbf{a}^2)^2 - (a_0^2 - \mathbf{a}^2)^2}}{a_0^2 + \mathbf{a}^2} \\
&= \frac{\sqrt{4a_0^2\mathbf{a}^2}}{a_0^2 + \mathbf{a}^2} \\
&= \frac{2a_0|\mathbf{a}|}{a_0^2 + \mathbf{a}^2}
\end{aligned}
\tag{1.14}
$$

The vector normal direction can be written

$$
\mathbf{n} = -\frac{2a_0}{(a_0^2 + \mathbf{a}^2)\sin(\theta/2)}\mathbf{a}, \tag{1.15}
$$

or

$$
\boxed{\mathbf{n} = -\frac{\mathbf{a}}{|\mathbf{a}|}.} \tag{1.16}
$$

The angle of rotation is

$$
\boxed{\theta = 2\,\mathrm{atan}\,\frac{2a_0|\mathbf{a}|}{a_0^2 - \mathbf{a}^2}.} \tag{1.17}
$$

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1