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ECE1236H Microwave and Millimeter-Wave Techniques. Lecture 8: Continuum and other transformers. Taught by Prof. G.V. Eleftheriades

Disclaimer Peeter's lecture notes from class. These may be incoherent and rough.

These are notes for the UofT course ECE1236H, Microwave and Millimeter-Wave Techniques, taught by Prof. G.V. Eleftheriades, covering ch. 1 [1] content.

1.1 Continuum transformer

A non-discrete impedance matching transformation, as sketched in fig. 1.1, is also possible.

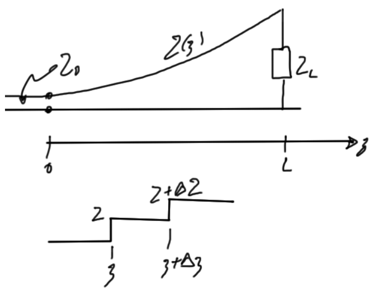

Figure 1.1: Tapered impedance matching.

$$\Delta\Gamma = \frac{(Z + \Delta Z) - Z}{(Z + \Delta Z) + Z} = \frac{\Delta Z}{2Z} \tag{1.1}$$

$$\Delta Z \rightarrow 0 \tag{1.2}$$

$$\begin{aligned} d\Gamma &= \frac{dZ}{2Z} \\ &= \frac{1}{2}\frac{d(\ln Z)}{dz} \\ &= \frac{Z_0}{Z}\frac{d(Z/Z_0)}{dz} \\ &= \frac{1}{Z}\frac{dZ}{dz}. \end{aligned} \tag{1.3}$$

Hence as we did for multisection transformers, associate $\Delta\Gamma$ with $e^{-2j\beta z}$ as sketched in fig. 1.2.

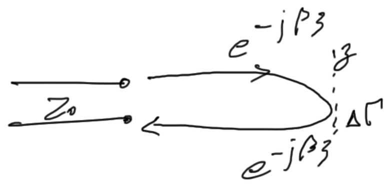

Figure 1.2: Reflection coefficient over an interval

assuming small reflections (i.e. $Z(z)$ is a slowly varying (adiabatic). Then

$$\begin{aligned} \Gamma(\omega) &= \int_0^L e^{-2j\beta z} d\Gamma \\ &= \frac{1}{2}\int_0^L e^{-2j\beta z}\frac{d(\ln Z)}{dz} dz \end{aligned} \tag{1.4}$$

This supplies the means to calculate the reflection coefficient for any impedance curve. As with the step impedance matching process, it is assumed that only first order reflections are of interest.

1.2 Exponential taper

Let

$$Z(z) = Z_0 e^{az}, \qquad 0 < z < L \tag{1.5}$$

subject to

$$\begin{aligned} Z(0) &= Z_0 \\ Z(L) &= Z_0 e^{aL} = Z_L, \end{aligned} \tag{1.6}$$

which gives

$$\ln \frac{Z_L}{Z_0} = aL, \tag{1.7}$$

or

$$a = \frac{1}{L}\ln\frac{Z_\mathrm{L}}{Z_0} \tag{1.8}$$

Also

$$\frac{d}{dz}\ln\frac{Z_\mathrm{L}}{Z_0} = \frac{d}{dz}(az) = a, \tag{1.9}$$

Hence

$$\begin{aligned}\Gamma(\omega) &= \frac{1}{2}\int_0^L e^{-2j\beta z}\frac{d}{dz}\ln\frac{Z_\mathrm{L}}{Z_0}dz \\ &= \frac{a}{2}\int_0^L e^{-2j\beta z}dz \\ &= \frac{1}{2L}\ln\frac{Z_\mathrm{L}}{Z_0}\left.\frac{e^{-2j\beta z}}{-2j\beta}\right|_0^L \\ &= \frac{1}{2L\beta}\ln\frac{Z_\mathrm{L}}{Z_0}\frac{1-e^{-2j\beta L}}{2j} \\ &= \frac{1}{2}\ln\frac{Z_\mathrm{L}}{Z_0}e^{-j\beta L}\frac{\sin(\beta L)}{\beta L},\end{aligned} \tag{1.10}$$

or

$$\Gamma(\omega) = \frac{1}{2}\ln\frac{Z_\mathrm{L}}{Z_0}e^{-j\beta L}\operatorname{sinc}(\beta L). \tag{1.11}$$

1. β is constant with Z varying: this is good only for TEM lines.
2. $|\Gamma|$ decreases with increasing length.
3. An electrical length $\beta L > \pi$, is required to minimize low frequency mismatch ($L > \lambda/2$).

This is sketched in fig. 1.3.

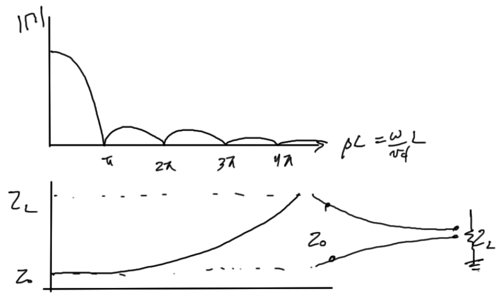

Figure 1.3: Exponential taper reflection coefficient.

Want:

$$\beta L = \pi, \tag{1.12}$$

or

$$\frac{\omega_c}{v_\phi} L = \pi \tag{1.13}$$

where ω_c is the cutoff frequency. This gives

$$\omega_c = \frac{\pi v_\phi}{L}. \tag{1.14}$$

Triangular Taper

$$Z(z) = \begin{cases} Z_0 e^{2(z/L)^2 \ln(Z_L/Z_0)} & 0 \leq z \leq L/2 \\ Z_0 e^{(4z/L - 2z^2 - 1)\ln(Z_L/Z_0)} & L/2 \leq z \leq L \end{cases} \tag{1.15}$$

$$\frac{d}{dz}\ln(Z/Z_0) = \begin{cases} (4z/L^2)\ln(Z_L/Z_0) & 0 \leq z \leq L/2 \\ (4/L - 4z/L^2)\ln(Z_L/Z_0) & L/2 \leq z \leq L \end{cases} \tag{1.16}$$

In this case

$$\Gamma(\omega) = \frac{1}{2} e^{-\beta L} \ln \frac{Z_L}{Z_0} e^{-j\beta L} \operatorname{sinc}^2(\beta L/2). \tag{1.17}$$

Compared to the exponential taper $\operatorname{sinc}(\beta L)$ for the $\beta L > 2\pi$ the peaks of $|\Gamma|$ are lower, but the first null occurs at $\beta L = 2\pi$ whereas for the exponential taper it occurs at $\beta L = \pi$. This is sketched in fig. 1.4. The price to pay for this is that the zero is at 2π so we have to make it twice as long to get the ripple down.

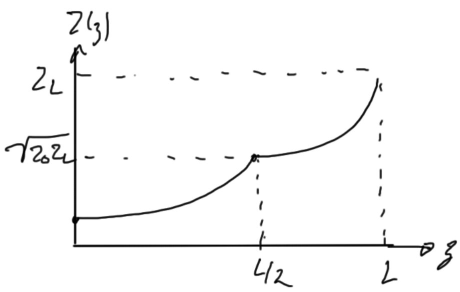

Figure 1.4: Triangular taper impedance curve.

Klopfenstein Taper For a given taper length L, the Klopfenstein taper is optimum in the sense that the reflection coefficient in the passband is minimum. Alternatively, for a given minimum reflection coefficient in the passband, the Klopfenstein taper yields the shortest length L.

Definition:

$$\ln Z = \frac{1}{2}\ln(Z_0 Z_L) + \frac{\Gamma_0}{\cosh A} A^2 \phi(2z/L - 1, A), \qquad 0 \leq z \leq L, \tag{1.18}$$

where

$$\phi(x, A) = \int_0^x \frac{I_1(A\sqrt{1-y^2})}{A\sqrt{1-y^2}} dy, \qquad |x| \leq 1. \tag{1.19}$$

Here $I_1(x)$ is the modified Bessel function. Note that

$$\begin{aligned} \phi(0, A) &= 0 \\ \phi(x, 0) &= x/2 \\ \phi(1, A) &= \frac{\cosh A - 1}{A^2} \end{aligned} \tag{1.20}$$

The resulting reflection coefficient is

$$\Gamma(\omega) = \begin{cases} \Gamma_0 e^{-j\beta L} \frac{\cos\sqrt{(\beta L)^2 - A^2}}{\cosh A} & \beta L > A \\ \Gamma_0 e^{-j\beta L} \frac{\cos\sqrt{A^2 - (\beta L)^2}}{\cosh A} & \beta L < A \end{cases}, \tag{1.21}$$

where as usual

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} \approx \frac{1}{2} \ln(Z_L/Z_0). \tag{1.22}$$

The passband is defined by $\beta L \geq A$ and the maximum ripple in the passband is

$$\Gamma_m = \frac{\Gamma_0}{\cosh A}. \tag{1.23}$$

Example 1.1: Triangular taper vs. exponential taper vs. Klopfenstein taper.

Design a triangular taper, an exponential taper, and a Klopfenstein taper (with $\Gamma_m = 0.02$) to match a 50Ω load to a 100Ω line.

- Triangular taper:

$$Z(z) = \begin{cases} Z_0 e^{2(z/L)^2 \ln Z_L/Z_0} & 0 \leq z \leq L/2 \\ Z_0 e^{(4z/L - 2z^2/L^2 - 1) \ln Z_L/Z_0} & L/2 \geq z \geq L \end{cases} \tag{1.24}$$

  The resulting Γ is

$$|\Gamma| = \frac{1}{2} \ln(Z_L/Z_0) \operatorname{sinc}^2(\beta L/2). \tag{1.25}$$

- Exponential taper:

$$
\begin{aligned}
Z(z) &= Z_0 e^{az}, \qquad 0 \le z \le L \\
a &= \frac{1}{L}\ln(Z_L/Z_0) = \frac{0.693}{L} \\
|\Gamma| &= \frac{1}{2}\ln(Z_L/Z_0)\,\mathrm{sinc}(\beta L)
\end{aligned} \tag{1.26}
$$

- Klopfenstein taper:

$$
\begin{aligned}
Z(z) &= \frac{1}{2}\ln(Z_L/Z_0) = 0.346 \\
A &= \cosh^{-1}\left(\frac{\Gamma_0}{\Gamma_m}\right) = \cosh^{-1}\left(\frac{0.346}{0.02}\right) = 3.543 \\
|\Gamma| &= \Gamma_0 \frac{\cos\sqrt{(\beta L)^2 - A^2}}{\cosh A},
\end{aligned} \tag{1.27}
$$

  The passband $\beta L > A = 3.543 = 1.13\pi$. The impedance $Z(z)$ must be evaluated numerically.

To illustrate some of the differences, we are referred to fig. 5.21 [1]. It is noted that

1. The exponential taper has the lowest cutoff frequency $\beta L = \pi$. Then is the Klopfenstein taper which is close $\beta L = 1.13\pi$. Last is the triangular with $\beta L = 2\pi$.
2. The Klopfenstein taper has the lowest $|\Gamma|$ in the passband and meets the spec of $\Gamma_m = 0.02$. The worst $|\Gamma|$ in the passband is from the exponential taper and the triangular ripple is between the two others.

Bibliography

[1] David M Pozar. *Microwave engineering*. John Wiley & Sons, 2009. 1, 1.1