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MICROWAVE AND MILLIMETER-WAVE TECHNIQUES

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Notes and problems from UofT ECE1236H 2016

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DOCUMENT VERSION

Sources for this notes compilation can be found in the github repository

<https://github.com/peeterjoot/physicsplay>

The last commit (Feb/12/2016), associated with this pdf was

f91cbf4ab4b9835f385e5bd1e6a23a59220dae43

Dedicated to:
Aurora and Lance, my awesome kids, and
Sofia, who not only tolerates and encourages my studies, but is also awesome enough to think
that math is sexy.

PREFACE

This document was produced while taking the Spring 2016, University of Toronto Microwave and Millimeter-Wave Techniques (aka Microwave Circuits) course (ECE1236H), taught by Prof. G. V. Eleftheriades.

Course Syllabus This course outlines the principles of designing modern microwave and RF circuits.

- Signal-integrity issues in high-speed digital circuits.
- The wave equation.
- Ideal transmission lines.
- Transients on transmission-lines.
- Planar transmission lines and introduction to MMIC's.
- Designing with scattering parameters.
- Planar power dividers.
- Directional couplers.
- Microwave filters.
- Solid-state microwave amplifiers.
- Noise.
- Diode-mixers.
- RF receiver chains.
- Oscillators.

The text for this course was [5]. It was not a required text, and the course was taught off of slides.

This document contains:

- Lecture notes (or transcriptions from the class slides when they went too fast).
- Personal notes exploring auxiliary details.
- Worked practice problems.
- Assigned problems.
- Links to Matlab, Mathematica, and Julia notebooks associated with problems and course material.

My thanks go to Professor Eleftheriades for teaching this course.

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Part I

READING AND LECTURE NOTES

ELECTROMAGNETIC FIELDS

1.1 MAXWELL'S EQUATIONS IN INTEGRAL FORM

Faraday's law For the integration surface sketched in fig. 1.1 Faraday's law is

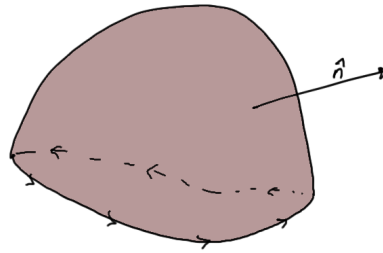


Figure 1.1: Stokes integration surface

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}} dS \quad (1.1)$$

Ampere's law

$$\int_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot \hat{\mathbf{n}} dS + \frac{\partial}{\partial t} \iint_S \mathbf{D} \cdot \hat{\mathbf{n}} dS \quad (1.2)$$

On the RHS we have the sum of the conduction and displacement currents respectively.

Units

- \mathbf{E} V/m : Electric field
- \mathbf{H} A/m : Magnetic field
- \mathbf{B} Weber/m² : Magnetic flux density
- \mathbf{D} Cb/m² : Electric flux density (Coloumbs/meter-squared)
- \mathbf{J} A/m² : Electric current density
- ρ Cb/m³ : Electric charge density

Charge conservation law Consider a closed surface S with interior volume V , as sketched in fig. 1.2. The charge conservation relation is

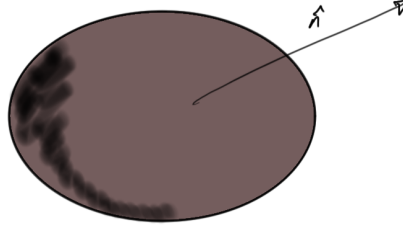


Figure 1.2: Divergence theorem integration volume

$$\oiint \mathbf{J} \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \int \rho dV \quad (1.3)$$

Maxwell's equations in integral form Faraday's

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}} dS \quad (1.4)$$

Ampere-Maxwell's law:

$$\int \mathbf{H} \cdot d\mathbf{l} = \iint \mathbf{J} \cdot \hat{\mathbf{n}} dS + \frac{\partial}{\partial t} \iint \mathbf{D} \cdot \hat{\mathbf{n}} dS \quad (1.5)$$

Charge conservation:

$$\oiint \mathbf{J} \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \int \rho dV \quad (1.6)$$

1.2 MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM

Useful calculus theorems

Stokes' theorem For the integration surface sketched in fig. 1.1 the 3D Stokes theorem statement is

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dS \quad (1.7)$$

Gauss's theorem (divergence theorem) Integrating over a closed surface

$$\oiint \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iiint \nabla \cdot \mathbf{F} dV \quad (1.8)$$

- ∇f : vector (f is a scalar) : Gradient of a scalar.
- $\nabla \cdot \mathbf{F}$: vector (\mathbf{F} is a vector) : Divergence of a vector.
- $\nabla \times \mathbf{F}$: vector (\mathbf{F} is a vector) : Curl of a vector.

Faraday's law Referring again to fig. 1.1, and Stokes theorem, Faraday's law

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}} dS \quad (1.9)$$

can be expressed using Stokes's theorem as

$$\begin{aligned} \oint_C \mathbf{E} \cdot d\mathbf{l} &= \iint_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} dS \\ &= -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}} dS. \end{aligned} \quad (1.10)$$

Hence

$$\oint_C \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot \hat{\mathbf{n}} dS = 0, \quad (1.11)$$

or

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (1.12)$$

which is Faraday's law in differential form.

Similarly, Ampere's law in differential form is

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \quad (1.13)$$

Charge conservation law Starting with

$$\oiint \mathbf{J} \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \int \rho dV, \quad (1.14)$$

and using Gauss's theorem

$$\begin{aligned} \oiint \mathbf{J} \cdot \hat{\mathbf{n}} dS &= \int_V \nabla \cdot \mathbf{J} dV \\ &= -\frac{\partial}{\partial t} \int \rho dV, \end{aligned} \quad (1.15)$$

or

$$\int_V \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dV = 0. \quad (1.16)$$

Hence

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \quad (1.17)$$

Summary: Maxwell's equations in differential form Faraday's law, Ampere-Maxwell law, and charge conservation law respectively:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.18a)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (1.18b)$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad (1.18c)$$

Derived laws Provides a vector \mathbf{F} has second derivatives that commute, we must have

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0, \quad (1.19)$$

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.20)$$

for a continuous electric field \mathbf{E} gives

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} \\ &= 0. \end{aligned} \quad (1.21)$$

The divergence must be independent of time or constant. If constant, we can probably assume that a non-zero constant isn't physically relevant, in which case we would have

$$\nabla \cdot \mathbf{B} = 0. \quad (1.22)$$

Similarly, Ampere's law gives

$$\begin{aligned}
0 &= \nabla \cdot (\nabla \times \mathbf{H}) \\
&= \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \\
&= -\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D},
\end{aligned} \tag{1.23}$$

where the last line follows from charge conservation. This means that the quantity

$$-\rho + \nabla \cdot \mathbf{D}, \tag{1.24}$$

is constant or independent of time. Again, assuming a non-zero constant value or time independent value isn't physically relevant, we have

$$\nabla \cdot \mathbf{D} = \rho. \tag{1.25}$$

Comment: Peeter: It is interesting to see $\nabla \cdot \mathbf{B} = 0$, and $\nabla \cdot \mathbf{D} = \rho$ presented as derived values, but it seems to me that some hand waving through the time independent and non-zero constant cases is required to get there.

Summary Independent equations, Faraday's, Ampere-Maxwell, and charge conservation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.26a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.26b}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{1.26c}$$

Derived equations (no magnetic charges, and Gauss's law respectively),

$$\nabla \cdot \mathbf{B} = 0 \tag{1.27a}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{1.27b}$$

Note can consider Gauss's law as the third independent equation, in which case the charge conservation law becomes a derived law.

1.3 CONSTITUTIVE RELATIONS

Consider the number of unknowns in this mix of equations: $\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{J}$, and ρ . Five vectors and one scalar: 16 unknowns. We have only $3 + 3 + 1 = 7$ equations, so can consider Maxwell's equations to have an indeterminate form.

Further information is provided by the nature of the medium in which the fields exist.

$$\begin{aligned}\mathbf{B} &= \mathbf{F}_1 \left(\mathbf{E}, \frac{\partial \mathbf{E}}{\partial t}, \left| \frac{\partial \mathbf{E}}{\partial t} \right|^2, \dots, \mathbf{H}, \frac{\partial \mathbf{H}}{\partial t}, \left| \frac{\partial \mathbf{H}}{\partial t} \right|^2, \dots \right) \\ \mathbf{D} &= \mathbf{F}_2 \left(\mathbf{E}, \frac{\partial \mathbf{E}}{\partial t}, \left| \frac{\partial \mathbf{E}}{\partial t} \right|^2, \dots, \mathbf{H}, \frac{\partial \mathbf{H}}{\partial t}, \left| \frac{\partial \mathbf{H}}{\partial t} \right|^2, \dots \right) \\ \mathbf{J} &= \mathbf{F}_3 \left(\mathbf{E}, \frac{\partial \mathbf{E}}{\partial t}, \left| \frac{\partial \mathbf{E}}{\partial t} \right|^2, \dots, \mathbf{H}, \frac{\partial \mathbf{H}}{\partial t}, \left| \frac{\partial \mathbf{H}}{\partial t} \right|^2, \dots \right),\end{aligned}\tag{1.28}$$

where $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ are designated vector functions which characterize the medium where the fields exist.

Assume

- (i) The medium is stationary. i.e. $\mathbf{F}_1, \mathbf{F}_2$ and \mathbf{F}_3 do not depend on time.
- (ii) The medium is linear. i.e. there is no dependence upon $\mathbf{E}^2, \mathbf{H}^2, \left| \frac{\partial \mathbf{E}}{\partial t} \right|^2, \left| \frac{\partial \mathbf{H}}{\partial t} \right|^2$ and higher order power terms.
- (iii) The medium is homogeneous. i.e. there is no x, y, z variation of $\mathbf{F}_1, \mathbf{F}_2$ and \mathbf{F}_3 .
- (iv) The medium responds independently to \mathbf{E} and \mathbf{H} . Then one can write

$$\begin{aligned}\mathbf{B} &= \bar{\boldsymbol{\mu}} \cdot \mathbf{H} \\ \mathbf{D} &= \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E} \\ \mathbf{J} &= \bar{\boldsymbol{\sigma}} \cdot \mathbf{E}\end{aligned}\tag{1.29}$$

where $\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\sigma}}$ are the magnetic permeability, electric permittivity, and electric conductivity tensors respectively. The last equation above is Ohm's law.

- (v) If the medium is also isotropic, meaning that $\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\sigma}}$ are scalars, then we have simple media described by

$$\begin{aligned}\mathbf{B} &= \bar{\mu} \mathbf{H} \\ \mathbf{D} &= \bar{\epsilon} \mathbf{E} \\ \mathbf{J} &= \bar{\sigma} \mathbf{E}.\end{aligned}\tag{1.30}$$

In simple media we now have another $3 + 3 + 3 = 9$ equations. With 7 equations from Maxwell's laws we have $9 + 7 = 16$ equations and 16 unknowns.

An example of a simple medium is free space where we have

$$\begin{aligned}\mu &= \mu_0 = 4\pi \times 10^{-7} \text{H/m} \\ \epsilon &= \epsilon_0 = 8.854 \times 10^{-12} \text{F/m} \\ \sigma &= 0.\end{aligned}\tag{1.31}$$

Maxwell's equations in simple media are

$$\begin{aligned}\nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} &= \left(\sigma + \epsilon \frac{\partial}{\partial t} \right) \mathbf{E} \\ \sigma \nabla \cdot \mathbf{E} &= -\frac{\partial \rho}{\partial t}.\end{aligned}\tag{1.32}$$

Example 1.1: Anisotropic media

Here \mathbf{D} is not parallel with \mathbf{E} , and \mathbf{B} is not parallel with \mathbf{H} .

- Crystals in the principle axis coordinate system can be described by diagonal tensors of the form

$$\bar{\epsilon} = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}.\tag{1.33}$$

This is electric anisotropy.

- Uniaxial crystals where $\epsilon_x = \epsilon_y$, so that

$$\bar{\epsilon} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}.\tag{1.34}$$

The z axis is the optical axis. Positive uniaxial crystal if $\epsilon_z > \epsilon$ and negative uniaxial crystal if $\epsilon_z < \epsilon$.

- Biaxial crystals when $\epsilon_x \neq \epsilon_y \neq \epsilon_z$.

Example 1.2: Bi-anisotropic media

$$\begin{aligned}\mathbf{D} &= \bar{\epsilon} \cdot \mathbf{E} + \bar{\zeta} \cdot \mathbf{H} \\ \mathbf{B} &= \bar{\mathbf{J}} \cdot \mathbf{E} + \bar{\mu} \cdot \mathbf{H}\end{aligned}\tag{1.35}$$

Example 1.3

This is the case when $\bar{\epsilon}$, $\bar{\zeta}$, $\bar{\mathbf{J}}$ and $\bar{\mu}$ are all scalar quantities. i.e.

$$\begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} + \zeta \mathbf{H} \\ \mathbf{B} &= \mathbf{J} \mathbf{E} + \mu \mathbf{H}\end{aligned}\tag{1.36}$$

An example is Chiral media like sugars, and DNA, where we have

$$\begin{aligned}\mathbf{D} &= \epsilon \mathbf{E} - \chi \frac{\partial \mathbf{H}}{\partial t} \\ \mathbf{B} &= \mu \mathbf{H} + \chi \frac{\partial \mathbf{E}}{\partial t},\end{aligned}\tag{1.37}$$

where χ is the chiral parameter.

Simple medium When we have a stationary, linear, homogeneous, and isotropic medium where

$$\begin{aligned}\mathbf{B} &= \mu \mathbf{H} \\ \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{J} &= \sigma \mathbf{E}\end{aligned}\tag{1.38}$$

then

(i) If $\sigma = 0$: dielectric.

(ii) If $\sigma \rightarrow \infty$: conductor.

(iii) ϵ (F/m): Permittivity. $\epsilon_r = \epsilon/\epsilon_0$ is the relative permittivity (dielectric constant).

(iv) μ (H/m) : Permeability. $\mu_r = \mu/\mu_0$ is the relative permeability.

Some material types:

- Non-magnetic materials, $\mu_r = 1$.
- Diamagnetic materials, $\mu_r < 1$. This is not common.
- Paramagnetic materials, $\mu_r > 1$.
- Ferromagnetic materials, $\mu_r \gg 1$.

1.4 PLANE WAVES

In simple media with no sources

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad (1.39a)$$

$$\nabla \cdot \mathbf{H} = 0. \quad (1.39b)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \left(\sigma + \epsilon \frac{\partial}{\partial t} \right) \mathbf{E} \quad (1.39c)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} = 0 \quad (1.39d)$$

Here $\mathbf{J} = \sigma \mathbf{E}$ is considered to be an induced current, and the ρ/ϵ term vanishes since we have no sources.

From Faraday's law eq. (1.39a)

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) \\
&= -\mu \frac{\partial}{\partial t} \left(\sigma + \epsilon \frac{\partial}{\partial t} \right) \mathbf{E} \\
&= -\mu \sigma \frac{\partial \mathbf{E}}{\partial t} - \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}.
\end{aligned} \tag{1.40}$$

We can also use the vector identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}, \tag{1.41}$$

so

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \tag{1.42}$$

or

$$0 = \nabla^2 \mathbf{E} - \mu \sigma \frac{\partial \mathbf{E}}{\partial t} - \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{1.43}$$

The $\mu \sigma \partial \mathbf{E} / \partial t$ term is a damping contribution.

Similarly for the magnetic field

$$0 = \nabla^2 \mathbf{H} - \mu \sigma \frac{\partial \mathbf{H}}{\partial t} - \epsilon \mu \frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{1.44}$$

This is the wave or Helmholtz equation.

Note that for the no loss condition $\sigma = 0$, we have the undamped wave equations

$$\nabla^2 \mathbf{E} = \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{1.45a}$$

$$\nabla^2 \mathbf{H} = \epsilon \mu \frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{1.45b}$$

This is one wave equation for each of E_x, E_y, E_z, H_x, H_y and H_z .

Propagation in one direction Consider the undamped propagation along the z direction of $E_x(z, t)$, which must satisfy

$$\nabla^2 E_x = \epsilon \mu \frac{\partial^2 E_x}{\partial t^2}, \tag{1.46}$$

or

$$\frac{\partial^2 E_x}{\partial z^2} = \epsilon \mu \frac{\partial^2 E_x}{\partial t^2}. \tag{1.47}$$

This is analogous to the voltage transmission line equation

$$\frac{\partial^2 V}{\partial z^2} = LC \frac{\partial^2 V}{\partial t^2}, \quad (1.48)$$

With solution

$$V(z, t) = V_0^+ f(z - v_\phi t) + V_0^- f(z + v_\phi t), \quad (1.49)$$

where $v_\phi = 1/\sqrt{LC}$ is the phase velocity, $f(z - v_\phi t)$ is the forward wave, and $f(z + v_\phi t)$ is the reflected wave. By analogy the solution to the 1D wave equation is

$$E_x(z, t) = E_0^+ f(z - v_\phi t) + E_0^- f(z + v_\phi t), \quad (1.50)$$

where $v_\phi = 1/\sqrt{\epsilon\mu}$ is the phase velocity.

For myself, the transmission line equation is something that I only encountered in this class (in a later lecture), so this analogy isn't a great one. That said, wave equation solutions are very familiar, so not much motivation for the structure of the solution is really required.

More rigorous derivation of the 1D solution Suppose that we assume the form of the solution is

$$\mathbf{E} = \mathbf{E}_0 f(z - v_\phi t), \quad (1.51)$$

where \mathbf{E}_0 is a constant vector, so this describes a wave propagation in the z -direction, but without a-priori knowledge of the direction of this vector in space.

For source free media, we have

$$\begin{aligned} \nabla \cdot (\mathbf{E}_0 f) &= \nabla f \cdot \mathbf{E}_0 + f \nabla \cdot \mathbf{E}_0 \\ &= (\hat{\mathbf{z}} f') \cdot \mathbf{E}_0. \end{aligned} \quad (1.52)$$

Because $\mathbf{E}_0 \cdot \hat{\mathbf{z}} = 0$, the vector \mathbf{E}_0 is perpendicular to the direction of propagation (z). The next task is to find the magnetic field that couples to this electric field solution. Suppose the coordinate axis are picked so that

$$\mathbf{E}(z, t) = \mathbf{E}_0 E_x(z \pm v_\phi t). \quad (1.53)$$

Using Faraday's law $\nabla \times \mathbf{E} = -\mu \partial \mathbf{H} / \partial t$ gives

$$\begin{aligned}\nabla \times (\mathbf{E}_0 E_x) &= (\nabla \times \mathbf{E}_0) E_x - \mathbf{E}_0 \times \nabla E_x \\ &= -\mathbf{E}_0 \times \nabla E_x \\ &= -\mathbf{E}_0 \times \hat{\mathbf{z}} \frac{\partial E_x}{\partial z},\end{aligned}\tag{1.54}$$

so

$$-\mu \frac{\partial \mathbf{H}}{\partial t} = (\hat{\mathbf{z}} \times \mathbf{E}_0) \frac{\partial E_x}{\partial z}.\tag{1.55}$$

Because of the $E_x(z \pm v_\phi t)$ dependence on z, t , we must have

$$\frac{\partial E_x}{\partial z} = \pm \frac{1}{v_\phi} \frac{\partial E_x}{\partial t},\tag{1.56}$$

or

$$\begin{aligned}-\mu \frac{\partial \mathbf{H}}{\partial t} &= (\hat{\mathbf{z}} \times \mathbf{E}_0) \\ &= \pm \frac{1}{v_\phi} \frac{\partial E_x}{\partial t}.\end{aligned}\tag{1.57}$$

Integrating with respect to t and assuming a zero integration constant (physical justification for that?), we have

$$\mathbf{H} = \mp E_x(z, t) \frac{\hat{\mathbf{z}} \times \mathbf{E}_0}{v_\phi \mu}.\tag{1.58}$$

The $v_\phi \mu$ product is the intrinsic wave impedance of the medium

$$\begin{aligned}v_\phi \mu &= \frac{\mu}{\sqrt{\epsilon \mu}} \\ &= \sqrt{\frac{\mu}{\epsilon}} \\ &= \eta.\end{aligned}\tag{1.59}$$

The magnetic field can be written in terms of the propagation direction $\hat{\mathbf{n}} = \mp \hat{\mathbf{z}}$ has

$$\mathbf{H} = \mp \frac{\hat{\mathbf{z}} \times \mathbf{E}}{\eta} = \frac{\hat{\mathbf{n}} \times \mathbf{E}}{\eta}.\tag{1.60}$$

Note that in free space we have $\eta \approx 377\Omega$.

summary For 1D wave propagation along the z axis we have

$$\begin{aligned}\nabla^2 \mathbf{E} &= \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla^2 \mathbf{H} &= \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2}\end{aligned}\tag{1.61}$$

where

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_0 f(z \mp v_\phi t) \\ 0 &= \mathbf{E}_0 \cdot \hat{\mathbf{z}} \\ \mathbf{H} &= \pm \frac{\hat{\mathbf{z}} \times \mathbf{E}_0}{\eta} f(z \mp v_\phi t)\end{aligned}\tag{1.62}$$

Note that $\pm \hat{\mathbf{z}}$ is the direction of propagation, with $+z$ being the forward wave, and $-z$ the backwards wave.

The phase velocity is

$$v_\phi = \frac{1}{\sqrt{\mu\epsilon}},\tag{1.63}$$

and the intrinsic wave impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}}.\tag{1.64}$$

Notes:

- \mathbf{E}, \mathbf{H} are perpendicular to the direction of propagation $\hat{\mathbf{z}}$. These are transverse waves.
- $|\mathbf{E}|/|\mathbf{H}| = \eta$. An analogy with transmission lines is

$$\begin{aligned}V &\leftrightarrow E \\ I &\leftrightarrow H \\ Z_0 &\leftrightarrow \eta\end{aligned}\tag{1.65}$$

- \mathbf{E}, \mathbf{H} are orthogonal to each other as sketched in fig. 1.3.

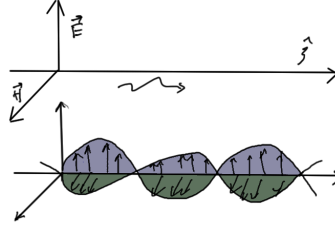


Figure 1.3: Transverse electric and magnetic propagation.

1.5 TIME HARMONIC FIELDS

Assume that the fields vary sinusoidally and use phasor notation

$$\mathbf{E} = \text{Re}(\mathbf{E}(\mathbf{r})e^{j\omega t}), \quad (1.66)$$

where $\mathbf{E}(\mathbf{r})$ is a (complex) vector-valued phasor. Notes:

- (i) Physicists use a $e^{-j\omega t}$ notation for the time dependence.
- (ii) Sometimes RMS values are used instead of peak values, so that

$$\mathbf{E} = \sqrt{2} \text{Re}(\mathbf{E}(\mathbf{r})e^{j\omega t}), \quad (1.67)$$

and $\mathbf{E}(\mathbf{r})$ is an RMS phasor.

- (iii) The time derivative $\partial/\partial t \rightarrow j\omega$ in Maxwell's equations.
- (iv) The $e^{j\omega t}$ variation is dropped for convenience.

The $e^{j\omega t}$ time harmonic dependence is not restrictive because one can synthesize any function of t using a Fourier transform

$$\mathcal{F}(f(t)) = \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \quad (1.68)$$

where $\tilde{f}(\omega)$ is the spectrum. If the spectrum is known then the inverse Fourier transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{j\omega t} d\omega, \quad (1.69)$$

If the response to a time harmonic signal $e^{j\omega t}$ is known then the response to an arbitrary waveform $f(t)$ can be synthesized, as shown in fig. 1.4 where the waveform input is $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{j\omega t} d\omega$ and the waveform output is $\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) G(\omega) e^{j\omega t} d\omega$

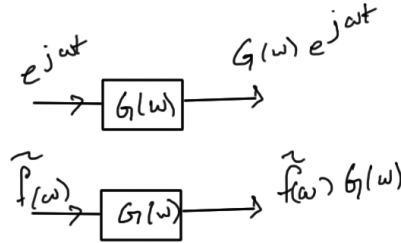


Figure 1.4: System frequency response.

If the fields are known in phasor form $\mathbf{E}(\mathbf{r}, \omega)$ then the response to an arbitrary signal with spectrum $\tilde{f}(\omega)$ is

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) \tilde{f}(\omega) e^{j\omega t} d\omega. \quad (1.70)$$

FIXME: unsure what $\tilde{f}(\omega)$ is here? Isn't $\mathbf{E}(\mathbf{r}, \omega)$ the spectrum of the field? Is this supposed to be the field response to the system $G(\omega)$?

Maxwell's equations in phasor form Faraday's law, the Ampere-Maxwell law, the continuity equation, and the (linear, homogeneous, isotropic) constitutive relations respectively are

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} \quad (1.71a)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} \quad (1.71b)$$

$$\nabla \cdot \mathbf{J} = -j\omega \rho \quad (1.71c)$$

$$\begin{aligned} \mathbf{B} &= \mu \mathbf{H} \\ \mathbf{D} &= \epsilon \mathbf{E} \end{aligned} \quad (1.71d)$$

Note that $\mathbf{D}(\mathbf{r}, \omega) = \epsilon(\omega)\mathbf{E}(\mathbf{r}, \omega)$ becomes a convolution in the time domain

$$\mathbf{D}(\mathbf{r}, t) = \int \mathbf{E}(\mathbf{r}, t - \tau) \epsilon(\tau) d\tau. \quad (1.72)$$

This describes the delay between the \mathbf{E} field and the \mathbf{D} field, the polarization.

1D propagation, and sinusoidal waves In a source free, loss less ($\sigma = 0$) simple medium the electric field satisfies

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (1.73)$$

In a phasor form the field is

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}(\mathbf{E}(\mathbf{r})e^{j\omega t}), \quad (1.74)$$

so the wave equation becomes

$$\nabla^2 \mathbf{E} - \epsilon\mu(j\omega)^2 \mathbf{E} = 0, \quad (1.75)$$

or

$$\boxed{\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0.} \quad (1.76)$$

where the propagation constant, or wave number is

$$\beta = \omega \sqrt{\epsilon\mu} = \frac{\omega}{v_\phi} = \frac{2\pi}{\lambda}, \quad (1.77)$$

where $\omega = 2\pi f$ is the angular frequency and $T = 1/f$ is the period. For 1D propagation along the z-axis the fields are

$$\begin{aligned} E_x(z, t) &= E_0^+ \cos(\omega t - \beta z) \\ H_x(z, t) &= \frac{E_0^+}{\eta} \cos(\omega t - \beta z). \end{aligned} \quad (1.78)$$

These correspond to phasors

$$\begin{aligned} E_x(z) &= E_0^+ e^{-jkz} \\ H_x(z) &= \frac{E_0^+}{\eta} e^{-jkz}. \end{aligned} \quad (1.79)$$

The peak velocity of either the electric or magnetic fields travels with the phase velocity as sketched in fig. 1.5.

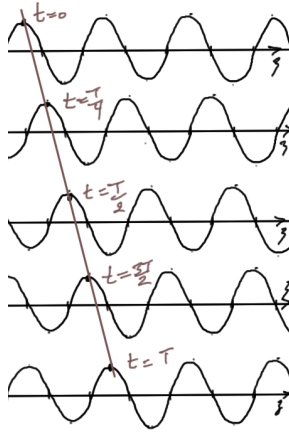


Figure 1.5: Phase velocity propagation.

Plane waves Consider

$$E_x(z, t) = E_0^+ \cos(\omega t - \beta z), \quad (1.80)$$

which had the phasor form $E_x = E_0^+ e^{-j\beta z}$. The phase of the wave is given by

$$\Phi = \omega t - \beta z. \quad (1.81)$$

At a fixed time t , the locus of the points with the same phase is given by the equation

$$\beta z = \omega t - \Phi = \text{constant}. \quad (1.82)$$

$\beta z = \text{constant}$ describes planes perpendicular to the z axis, as sketched in fig. 1.6.

Notes

- Since the equiphase surfaces are planes, we deal with plane waves.
- $\beta(z + \lambda) = \beta z + 2\pi$. i.e. wavefronts are separated in space by one wavelength λ .
- velocity of wavefront $\beta dz = \omega dt$ means that phase velocity is

$$\frac{dz}{dt} = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}, \quad (1.83)$$

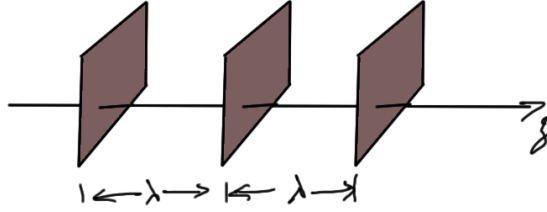


Figure 1.6: Equiphase surfaces.

or

$$v_\phi = \frac{\omega}{\beta}. \quad (1.84)$$

Since $\beta z - \omega t = \beta(z - v_\phi t) = \text{constant}$ we also have

$$\frac{dz}{dt} - v_\phi = 0, \quad (1.85)$$

or

$$v_\phi = \frac{\lambda}{T}. \quad (1.86)$$

These can be combined to give

$$\begin{aligned} \beta &= \frac{\omega}{v_\phi} \\ &= \frac{\omega T}{\lambda} \\ &= \frac{2\pi f T}{\lambda} \\ &= \frac{2\pi}{\lambda}. \end{aligned} \quad (1.87)$$

1.6 POYNTING THEOREM

Poynting vector Consider a volume V , surrounded by a closed surface S as sketched in fig. 1.7.

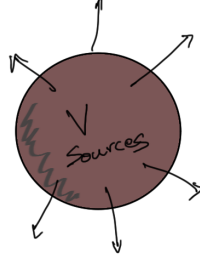


Figure 1.7: Poynting volume.

The instantaneous power density outflowing the surfaces S is given by the Poynting vector

$$\mathbf{P} = \mathcal{E} \times \mathcal{H}, \quad (\text{W/m}^2), \quad (1.88)$$

where \mathcal{E} and \mathcal{H} are time-dependent fields. For time harmonic fields

$$\begin{aligned} \mathcal{E} &= \text{Re}(\mathbf{E}e^{j\omega t}) \\ \mathcal{H} &= \text{Re}(\mathbf{H}e^{j\omega t}) \end{aligned} \quad (1.89)$$

Hence

$$\begin{aligned} \mathbf{P} &= \mathcal{E} \times \mathcal{H} \\ &= \frac{1}{4} (\mathbf{E}e^{j\omega t} + \mathbf{E}^*e^{-j\omega t}) \times (\mathbf{H}e^{j\omega t} + \mathbf{H}^*e^{-j\omega t}) \\ &= \frac{1}{4} ((\mathbf{E} \times \mathbf{H})e^{2j\omega t} + (\mathbf{E}^* \times \mathbf{H}) + (\mathbf{E} \times \mathbf{H}^*)(\mathbf{E}^* \times \mathbf{H}^*)e^{-2j\omega t}). \end{aligned} \quad (1.90)$$

Therefore the time-average Poynting vector is

$$\boxed{\mathbf{S} = \frac{1}{T} \int_0^T \mathbf{P} dt = \frac{1}{2} \text{Re}(\mathbf{E} \times \mathbf{H}^*) \quad (\text{W/m}^2).} \quad (1.91)$$

The total time-average power outflowing S is

$$P = \frac{1}{2} \oint \text{Re}(\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S}. \quad (1.92)$$

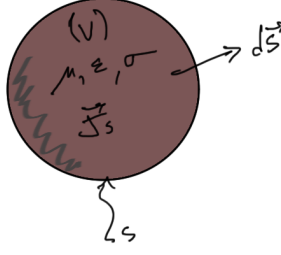


Figure 1.8: Poynting volume with sources.

Poynting theorem Consider again a volume V enclosed in a surface S . Assume that electric currents may exist in V and that V is filled with a material characterized by ϵ, μ and a conductivity σ , where $\epsilon = \epsilon' - j\epsilon''$, and $\mu = \mu' - j\mu''$ as illustrate in fig. 1.8.

From Faraday's law $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$, and Ampere's law $\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E}$, we have

$$\begin{aligned}\mathbf{H}^* \cdot (\nabla \times \mathbf{E}) &= -j\omega\mu|\mathbf{H}|^2 \\ \mathbf{E} \cdot (\nabla \times \mathbf{H}^*) &= \mathbf{J}^* \cdot \mathbf{E} - j\omega\epsilon^*|\mathbf{E}|^2\end{aligned}\tag{1.93}$$

We assume that the current is a sum of sources and induced currents $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_{\text{ind}}$, where $\mathbf{J}_{\text{ind}} = \sigma\mathbf{E}$. Recall that

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E}^* \cdot (\nabla \times \mathbf{H}^*).\tag{1.94}$$

Hence

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega\mu|\mathbf{H}|^2 - \mathbf{J}_s^* \cdot \mathbf{E} - \sigma|\mathbf{E}|^2 + j\omega\epsilon^*|\mathbf{E}|^2.\tag{1.95}$$

Now integrate over V , using the divergence theorem

$$\begin{aligned}\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) dV &= \oint (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} \\ &= -\sigma \int_V |\mathbf{E}|^2 dV + j\omega \int_V (\epsilon^*|\mathbf{E}|^2 - \mu|\mathbf{H}|^2) dV - \int_V \mathbf{J}_s^* \cdot \mathbf{E} dV,\end{aligned}\tag{1.96}$$

or

$$\begin{aligned}
 -\frac{1}{2} \int_V \mathbf{J}_s^* \cdot \mathbf{E} dV &= +\frac{1}{2} \oint (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} + \frac{\sigma}{2} \int_V |\mathbf{E}|^2 dV - j\frac{\omega}{2} \int_V (\epsilon^* |\mathbf{E}|^2 - \mu |\mathbf{H}|^2) dV \\
 &= +\frac{1}{2} \oint (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} + \frac{\sigma}{2} \int_V |\mathbf{E}|^2 dV + \frac{\omega}{2} \int_V (\epsilon'' |\mathbf{E}|^2 \\
 &\quad + \mu'' |\mathbf{H}|^2) dV + j\frac{\omega}{2} \int_V (-\epsilon' |\mathbf{E}|^2 + \mu' |\mathbf{H}|^2) dV
 \end{aligned} \tag{1.97}$$

This is the mathematical manifestation of Poynting's theorem which is nothing else but the conservation of power with volume V . We can now identify the following terms:

- Power generated by current sources

$$P_s = -\frac{1}{2} \int_V \mathbf{J}_s^* \cdot \mathbf{E} dV \tag{1.98}$$

- Complex power flowing out of V

$$P_o = \frac{1}{2} \oint (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} \tag{1.99}$$

- Power loss due to materials

$$P_m = \frac{\omega}{2} \int_V (\epsilon'' |\mathbf{E}|^2 + \mu'' |\mathbf{H}|^2) dV \tag{1.100}$$

- Ohmic losses

$$P_c = \frac{\sigma}{2} \int_V |\mathbf{E}|^2 dV \tag{1.101}$$

- Time average stored electric energy

$$W_e = \frac{1}{4} \epsilon' \int_V |\mathbf{E}|^2 dV \tag{1.102}$$

- Time average stored magnetic energy

$$W_m = \frac{1}{4} \mu' \int_V |\mathbf{H}|^2 dV \tag{1.103}$$

The Poynting theorem is a complex power balance of the form

$$P_s = P_o + P_m + P_c + 2j\omega (W_m - W_e) \quad (1.104)$$

The real part of the above equation is related to the time average power. i.e.

$$\text{Re}(P_s) = \text{Re}(P_o) + P_m + P_c. \quad (1.105)$$

On the other hand the imaginary part is related to the net stored energy with V. i.e.

$$\text{Im}(P_s) = 2\omega (W_m - W_e) + \text{Im}(P_o). \quad (1.106)$$

We can now use Poynting theorem to calculate losses in waveguides.

1.7 LOSSY MEDIA

In a lossy medium characterized by $\gamma = \alpha + j\beta$ such that the wave behaves like $e^{-\gamma z}$ for propagation along $+z$, or

$$e^{-\gamma z} = e^{-\alpha z} e^{-j\beta z} \quad (1.107)$$

The $e^{-\alpha z}$ term introduces an exponential attenuation with z . The wave equation in the phasor domain is then

$$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = 0, \quad (1.108)$$

where

$$\gamma^2 = -\omega^2 \mu \epsilon_{\text{eff}}. \quad (1.109)$$

The effective complex permittivity $\epsilon_{\text{eff}} = \epsilon' - j\epsilon''$ is obtained from Ampere's law

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J} + j\omega \epsilon \mathbf{E} \\ &= \sigma \mathbf{E} + j\omega \epsilon \mathbf{E} \\ &= (\sigma + j\omega \epsilon) \mathbf{E} \\ &= j\omega \left(\epsilon - j \frac{\sigma}{\omega} \right) \mathbf{E}, \end{aligned} \quad (1.110)$$

so

$$\begin{aligned}\epsilon_{\text{eff}} &= \epsilon' - j\epsilon'' \\ \epsilon' &= \epsilon \\ \epsilon'' &= \frac{\sigma}{\omega}\end{aligned}\tag{1.111}$$

Since $\gamma = j\omega\sqrt{\mu\epsilon_{\text{eff}}}$ and $\gamma^2 = -\omega^2\mu\epsilon_{\text{eff}}$ we see that

$$\begin{aligned}(\alpha + j\beta)^2 &= (\alpha^2 - \beta^2) + j2\alpha\beta \\ &= -\omega^2\mu\epsilon_{\text{eff}} \\ &= -\omega^2\mu\epsilon' + j\omega^2\mu\epsilon'',\end{aligned}\tag{1.112}$$

therefore

$$\begin{aligned}\alpha^2 - \beta^2 &= -\omega^2\mu\epsilon' \\ 2\alpha\beta &= \omega^2\mu\epsilon''.\end{aligned}\tag{1.113}$$

Solving for α and β yields

$$\begin{aligned}\alpha &= \omega\sqrt{\frac{\mu\epsilon'}{2}\left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right)}, & \text{Np/m} \\ \beta &= \omega\sqrt{\frac{\mu\epsilon'}{2}\left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right)}, & \text{rad/m}\end{aligned}\tag{1.114}$$

Assuming propagation along $+\hat{\mathbf{z}}$

$$\begin{aligned}\mathbf{E}(z) &= \hat{\mathbf{x}}E_x(z) \\ &= \hat{\mathbf{x}}E_{x0}e^{-\gamma z} \\ &= \hat{\mathbf{x}}E_{x0}e^{-\alpha z}e^{-j\beta z}\end{aligned}\tag{1.115}$$

The magnetic field can be determined from

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},\tag{1.116}$$

or

$$\begin{aligned}\mathbf{H} &= \hat{\mathbf{y}}\frac{E_x}{\eta_c} \\ &= \hat{\mathbf{y}}\frac{E_{x0}}{\eta_c}e^{-\alpha z}e^{-j\beta z},\end{aligned}\tag{1.117}$$

where

$$\begin{aligned}\eta_c &= \sqrt{\frac{\mu}{\epsilon_{\text{eff}}}} \\ &= \sqrt{\frac{\mu}{\epsilon'}} \left(1 - j \frac{\epsilon''}{\epsilon'}\right)^{-1/2} \quad \Omega,\end{aligned}\tag{1.118}$$

is the complex intrinsic impedance of the medium. Note that since η_c is complex the electric and magnetic fields are no longer in phase.

1.8 SKIN DEPTH

Note that

$$\begin{aligned}|E_x| &= |E_{x0}|e^{-\alpha z} \\ |H_y| &= |H_{y0}|e^{-\alpha z}.\end{aligned}\tag{1.119}$$

The skin depth δ_s is defined as the distance that the wave needs to travel to reduce its magnitude by $1/e$. Hence

$$e^{-\alpha\delta_s} = e^{-1},\tag{1.120}$$

or

$$\boxed{\delta_s = \frac{1}{\alpha}}.\tag{1.121}$$

This is sketched in fig. 1.9.

Low loss dielectrics A low loss dielectric medium is characterized by $\epsilon'' \ll \epsilon'$, so $\epsilon \approx \epsilon'$. This gives

$$\begin{aligned}\gamma &= j\omega \sqrt{\mu\epsilon'} \left(1 - j \frac{\epsilon''}{\epsilon'}\right)^{1/2} \\ &\approx j\omega \sqrt{\mu\epsilon'} \left(1 - j \frac{\epsilon''}{2\epsilon'}\right).\end{aligned}\tag{1.122}$$

Hence

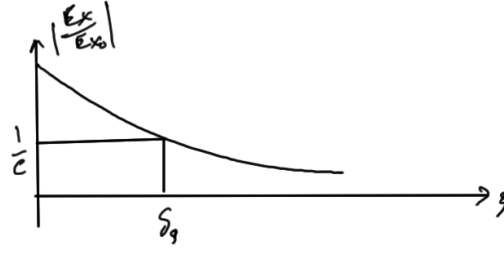


Figure 1.9: Skin depth.

$$\begin{aligned}
 \alpha &\approx \omega \sqrt{\mu \epsilon'} \frac{\epsilon''}{2\epsilon'} \\
 &= \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon'}} \\
 &\approx \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}.
 \end{aligned} \tag{1.123}$$

$$\begin{aligned}
 \beta &\approx \omega \sqrt{\mu \epsilon'} \\
 &\approx \omega \sqrt{\mu \epsilon},
 \end{aligned} \tag{1.124}$$

which is the same as in the lossless medium ϵ . Also in this case

$$\begin{aligned}
 \eta_c &= \sqrt{\frac{\mu}{\epsilon'}} \left(1 - j \frac{\epsilon''}{\epsilon'}\right)^{-1/2} \\
 &\approx \sqrt{\frac{\mu}{\epsilon'}} \left(1 + j \frac{\epsilon''}{2\epsilon'}\right) \\
 &= \sqrt{\frac{\mu}{\epsilon'}} \left(1 + j \frac{\sigma}{2\omega \epsilon'}\right) \\
 &\approx \sqrt{\frac{\mu}{\epsilon'}},
 \end{aligned} \tag{1.125}$$

(for $\sigma \ll \omega \epsilon$, (a good dielectric), so

$$\eta_c \approx \sqrt{\frac{\mu}{\epsilon}}, \tag{1.126}$$

which is the same as if the medium is lossless.

Good conductors In this case $\sigma \gg \omega\epsilon$ so that

$$\begin{aligned}
 \gamma &= j\omega \sqrt{\epsilon_{\text{eff}}\mu} \\
 &= j\omega \sqrt{\epsilon\mu} \left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{1/2} \\
 &\approx j\omega \sqrt{\epsilon\mu} \sqrt{-j\frac{\sigma}{\omega\epsilon}} \\
 &= j\omega \sqrt{\epsilon\mu} \sqrt{\frac{\sigma}{\omega\epsilon}} e^{-j\pi/4} \\
 &= j\omega \sqrt{\frac{\mu\sigma}{2\omega}} (1 - j) \\
 &= \sqrt{\frac{\mu\sigma\omega}{2}} (1 + j) \\
 &= \alpha + j\beta,
 \end{aligned} \tag{1.127}$$

Hence

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}. \tag{1.128}$$

Since $\nabla \times \mathbf{E} = -\mu(j\omega)\mathbf{H}$, and with a $\mathbf{E} \propto e^{-\gamma z}$ dependence, we have

$$\begin{aligned}
 \eta_c &= \frac{|\mathbf{E}|}{|\mathbf{H}|} \\
 &= \frac{-\mu(j\omega)}{-\gamma} \\
 &= \frac{j\omega\mu}{\gamma}
 \end{aligned} \tag{1.129}$$

so

$$\begin{aligned}
 \eta_c &= \frac{j\omega\mu}{\gamma} \\
 &= \frac{j\omega\mu}{\sqrt{\frac{\mu\sigma\omega}{2}}(1 + j)} \\
 &= \frac{j(1 - j)\omega\mu}{2\sqrt{\frac{\mu\sigma\omega}{2}}} \\
 &= (j + 1)\sqrt{\frac{\omega\mu}{2\sigma}} \\
 &= (j + 1)\frac{\alpha}{\sigma}.
 \end{aligned} \tag{1.130}$$

The skin depth for a good conductor is

$$\begin{aligned}\delta_s &= \frac{1}{\alpha} \\ &= \sqrt{\frac{2}{\omega\mu\sigma}} \\ &= \frac{1}{\sqrt{\pi f\mu\sigma}}.\end{aligned}\tag{1.131}$$

Example 1.4: Copper skin depth.

Copper has a conductivity of $\sigma = 5.8 \times 10^7 \text{ S/m}$. What is the skin depth $\delta_s = 1 / \sqrt{\pi f\mu_0\sigma}$?

- At 1 GHz, $\delta_s \approx 2.1 \mu\text{m}$
- At 3 GHz, $\delta_s \approx 1.2 \mu\text{m}$
- At 10 GHz, $\delta_s \approx 0.6 \mu\text{m}$

We can conclude that a very thin metallic film can effectively shield RF energy.

1.9 SKIN EFFECT

At DC the current through a wire flows uniformly throughout the entire cross section of a wire as sketched in fig. 1.10.

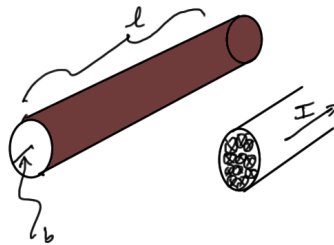


Figure 1.10: DC current distribution in wire.

The current occupies the entire cross section πb^2 , so the DC resistance is

$$R_{DC} = \frac{1}{\sigma} \frac{l}{\pi b^2}. \quad (1.132)$$

However, with RF (like microwave) signals the current only exists effectively within on skin depth, as sketched in fig. 1.11.

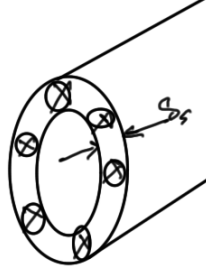


Figure 1.11: RF current distribution in wire.

therefore the RF resistance is

$$R_{RF} = \frac{1}{\sigma} \frac{l}{2\pi b \delta_s}. \quad (1.133)$$

Since $A_{RF} \ll A = \pi b^2$,

$$R_{RF} \gg R_{DC}. \quad (1.134)$$

Wires become very lossy at microwave frequencies.

Example 1.5: Skin effect for copper.

Consider a copper wire with $\sigma = 5.8 \times 10^7 \text{ S/m}$ of radius $b = 0.5 \text{ mm}$ and length 10m. Find the DC and RF resistance at 1, 3 and 10GHz. The DC resistance is

$$\begin{aligned} R_{DC} &= \frac{1}{\sigma} \frac{l}{\pi b^2} \\ &= \frac{10}{5.8 \times 10^7 \times \pi (0.0005)^2} \\ &= 0.22 \Omega. \end{aligned} \quad (1.135)$$

On the other hand

$$R_{RF} = \frac{1}{\sigma} \frac{l}{2\pi b \delta_s}. \quad (1.136)$$

So, at

- 1GHz, $R_{\text{RF}} = 26\Omega$
- 3GHz, $R_{\text{RF}} = 45.7\Omega$
- 10GHz, $R_{\text{RF}} = 91.5\Omega$

Compare these values with $R_{\text{DC}} = 0.22\Omega$.

Exercise 1.1 Solve for the α, β constants.

Prove eq. (1.114).

1.10 PROBLEMS

Answer for Exercise 1.1

Eliminating α , and letting $a = \omega^2\mu$, we have

$$\left(\frac{a\epsilon''}{2\beta}\right)^2 - \beta^2 = -a\epsilon', \quad (1.137)$$

or

$$\frac{a^2(\epsilon'')^2}{4} - \beta^4 = -a\epsilon'\beta^2 \quad (1.138)$$

or

$$\frac{a^2(\epsilon'')^2}{4} = \beta^4 - a\epsilon'\beta^2 \quad (1.139)$$

or

$$\begin{aligned} \left(\beta^2 - \frac{a\epsilon'}{2}\right)^2 &= \frac{a^2(\epsilon'')^2}{4} + \frac{a^2(\epsilon')^2}{4} \\ &= \frac{a^2}{4} ((\epsilon'')^2 + (\epsilon')^2) \end{aligned} \quad (1.140)$$

This gives

$$\begin{aligned} \beta^2 &= \frac{a\epsilon'}{2} + \frac{a}{2} \sqrt{(\epsilon'')^2 + (\epsilon')^2} \\ &= \frac{a\epsilon'}{2} \left(1 + \sqrt{1 + \frac{(\epsilon'')^2}{(\epsilon')^2}}\right), \end{aligned} \quad (1.141)$$

or

$$\beta = \omega^2 \mu \sqrt{\frac{a\epsilon'}{2} \left(1 + \sqrt{1 + \frac{(\epsilon'')^2}{(\epsilon')^2}} \right)}, \quad (1.142)$$

as stated. Solving for α can be done by inspection from eq. (1.141) and eq. (1.137).

TRANSMISSION LINES

2.1 REQUIREMENTS

A transmission line requires two conductors as sketched in fig. 2.1, which shows a 2-wire line such a telephone line, a coaxial cable as found in cable TV distribution, and a microstrip line as found in cell phone RF interconnects.

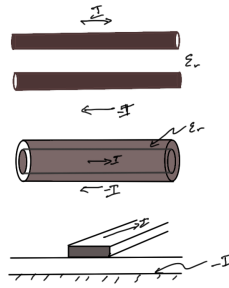


Figure 2.1: Transmission line examples.

A two-wire line becomes a transmission line when the wavelength of operation becomes comparable to the size of the line (or higher spectral component for pulses). In general a transmission line much support (TEM) transverse electromagnetic modes.

2.2 TIME HARMONIC SOLUTIONS ON TRANSMISSION LINES

In fig. 2.2, an electronic representation of a transmission line circuit is sketched.

In this circuit all the elements have per-unit length units. With $I = CdV/dt \sim j\omega CV$, $v = IR$, and $V = LdI/dt \sim j\omega LI$, the KVL equation is

$$V(z) - V(z + \Delta z) = I(z)\Delta z(R + j\omega L), \quad (2.1)$$

or in the $\Delta z \rightarrow 0$ limit

$$\frac{\partial V}{\partial z} = -I(z)(R + j\omega L). \quad (2.2)$$

The KCL equation at the interior node is

$$-I(z) + I(z + \Delta z) + (j\omega C + G)V(z + \Delta z) = 0, \quad (2.3)$$

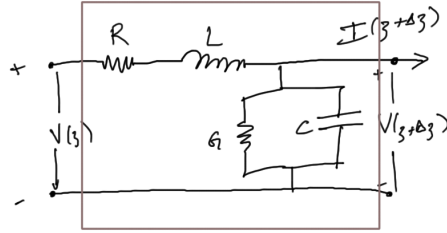


Figure 2.2: Transmission line equivalent circuit.

or

$$\frac{\partial I}{\partial z} = -V(z)(j\omega C + G). \quad (2.4)$$

This pair of equations is known as the telegrapher's equations

$$\begin{aligned} \frac{\partial V}{\partial z} &= -I(z)(R + j\omega L) \\ \frac{\partial I}{\partial z} &= -V(z)(j\omega C + G). \end{aligned} \quad (2.5)$$

The second derivatives are

$$\begin{aligned} \frac{\partial^2 V}{\partial z^2} &= -\frac{\partial I}{\partial z}(R + j\omega L) \\ \frac{\partial^2 I}{\partial z^2} &= -\frac{\partial V}{\partial z}(j\omega C + G), \end{aligned} \quad (2.6)$$

which allow the V, I to be decoupled

$$\begin{aligned} \frac{\partial^2 V}{\partial z^2} &= V(z)(j\omega C + G)(R + j\omega L) \\ \frac{\partial^2 I}{\partial z^2} &= I(z)(R + j\omega L)(j\omega C + G), \end{aligned} \quad (2.7)$$

With a complex propagation constant

$$\begin{aligned} \gamma &= \alpha + j\beta \\ &= \sqrt{(j\omega C + G)(R + j\omega L)} \\ &= \sqrt{RG - \omega^2 LC + j\omega(LG + RC)}, \end{aligned} \quad (2.8)$$

the decouple equations have the structure of a wave equation for a lossy line in the frequency domain

$$\boxed{\begin{aligned}\frac{\partial^2 V}{\partial z^2} - \gamma^2 V &= 0 \\ \frac{\partial^2 I}{\partial z^2} - \gamma^2 I &= 0.\end{aligned}} \quad (2.9)$$

We write the solutions to these equations as

$$\begin{aligned}V(z) &= V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \\ I(z) &= I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z}\end{aligned} \quad (2.10)$$

Only one of V or I is required since they are dependent through eq. (2.5), as can be seen by taking derivatives

$$\begin{aligned}\frac{\partial V}{\partial z} &= \gamma (-V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}) \\ &= -I(z)(R + j\omega L),\end{aligned} \quad (2.11)$$

so

$$I(z) = \frac{\gamma}{R + j\omega L} (V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}). \quad (2.12)$$

Introducing the characteristic impedance Z_0 of the line

$$\begin{aligned}Z_0 &= \frac{R + j\omega L}{\gamma} \\ &= \sqrt{\frac{R + j\omega L}{G + j\omega C}},\end{aligned} \quad (2.13)$$

we have

$$\begin{aligned}I(z) &= \frac{1}{Z_0} (V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}) \\ &= I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z},\end{aligned} \quad (2.14)$$

where

$$\begin{aligned}I_0^+ &= \frac{V_0^+}{Z_0} \\ I_0^- &= \frac{V_0^-}{Z_0}.\end{aligned} \quad (2.15)$$

2.3 MAPPING TL GEOMETRY TO PER UNIT LENGTH C AND L ELEMENTS**Example 2.1: Coaxial cable.**

From electrostatics and magnetostatics the per unit length induction and capacitance constants for a co-axial cable can be calculated. For the cylindrical configuration sketched in fig. 2.3

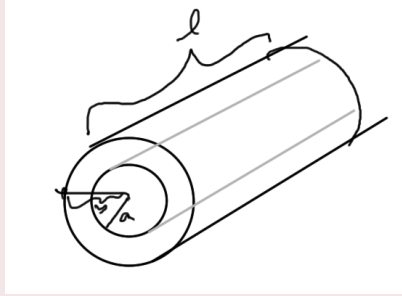


Figure 2.3: Coaxial cable.

From Gauss' law the total charge can be calculated assuming that the ends of the cable can be neglected

$$\begin{aligned}
 Q &= \int \nabla \cdot \mathbf{D} dV \\
 &= \oint \mathbf{D} \cdot d\mathbf{A} \\
 &= \epsilon_0 \epsilon_r E (2\pi r) l,
 \end{aligned} \tag{2.16}$$

This provides the radial electric field magnitude, in terms of the total charge

$$E = \frac{Q/l}{\epsilon_0 \epsilon_r (2\pi r)}, \tag{2.17}$$

which must be a radial field as sketched in fig. 2.4.

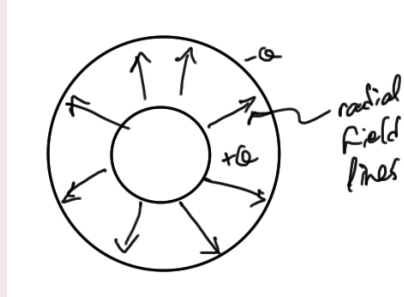


Figure 2.4: Radial electric field for coaxial cable.

The potential difference from the inner transmission surface to the outer is

$$\begin{aligned}
 V &= \int_a^b E dr \\
 &= \frac{Q/l}{2\pi\epsilon_0\epsilon_r} \int_a^b \frac{dr}{r} \\
 &= \frac{Q/l}{2\pi\epsilon_0\epsilon_r} \ln \frac{b}{a}.
 \end{aligned} \tag{2.18}$$

Therefore the capacitance per unit length is

$$C = \frac{Q/l}{V} = \frac{2\pi\epsilon_0\epsilon_r}{\ln \frac{b}{a}}. \tag{2.19}$$

The inductance per unit length can be calculated from Ampere's law

$$\begin{aligned}
 \int (\nabla \times \mathbf{H}) \cdot d\mathbf{S} &= \int \mathbf{J} \cdot d\mathbf{S} + \frac{\partial}{\partial t} \int \mathbf{D} \cdot d\mathbf{A} \\
 &= I \\
 &= \oint \mathbf{H} \cdot d\mathbf{l} \\
 &= H(2\pi r) \\
 &= \frac{B}{\mu_0}(2\pi r)
 \end{aligned} \tag{2.20}$$

The flux is

$$\begin{aligned}
\Phi &= \int \mathbf{B} \cdot d\mathbf{A} \\
&= \frac{\mu_0 I}{2\pi} \int_A \frac{1}{r} ddr \\
&= \frac{\mu_0 I}{2\pi} \int_a^b \frac{1}{r} l ddr \\
&= \frac{\mu_0 I l}{2\pi} \ln \frac{b}{a}.
\end{aligned} \tag{2.21}$$

The inductance per unit length is

$$L = \frac{\Phi/l}{I} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}. \tag{2.22}$$

For a lossless line where $R = G = 0$, we have $\gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega \sqrt{LC}$, so the phase velocity for a (lossless) coaxial cable is

$$\begin{aligned}
v_\phi &= \frac{\omega}{\beta} \\
&= \frac{\omega}{\text{Im}(\gamma)} \\
&= \frac{\omega}{\omega \sqrt{LC}} \\
&= \frac{1}{\sqrt{LC}}.
\end{aligned} \tag{2.23}$$

This gives

$$\begin{aligned}
v_\phi^2 &= \frac{1}{L} \frac{1}{C} \\
&= \frac{2\pi}{\mu_0 \ln \frac{b}{a}} \frac{\ln \frac{b}{a}}{2\pi \epsilon_0 \epsilon_r} \\
&= \frac{1}{\mu_0 \epsilon_0 \epsilon_r} \\
&= \frac{1}{\mu_0 \epsilon}.
\end{aligned} \tag{2.24}$$

So

$$v_\phi = \frac{1}{\sqrt{\epsilon\mu_0}}, \quad (2.25)$$

which is the speed of light in the medium (ϵ_r) that fills the co-axial cable.

This is not a coincidence. In any two-wire homogeneously filled transmission line, the phase velocity is equal to the speed of light in the unbounded medium that fills the line.

The characteristic impedance (again assuming the lossless $R = G = 0$ case) is

$$\begin{aligned} Z_0 &= \sqrt{\frac{\bar{R} + j\omega L}{\bar{G} + j\omega C}} \\ &= \sqrt{\frac{L}{C}} \\ &= \sqrt{\frac{\mu_0}{2\pi} \ln \frac{b}{a} \frac{2\pi \epsilon_0 \epsilon_r}{\ln \frac{b}{a}}} \\ &= \sqrt{\frac{\mu_0}{\epsilon} \ln \frac{b}{a} \frac{1}{2\pi}}. \end{aligned} \quad (2.26)$$

Note that $\eta = \sqrt{\mu_0/\epsilon_0} = 120\pi\Omega$ is the intrinsic impedance of free space. The values a, b in eq. (2.26) can be used to tune the characteristic impedance of the transmission line.

2.4 LOSSLESS LINE.

The lossless case where $R = G = 0$ was considered above. The results were

$$\gamma = j\omega\sqrt{LC}, \quad (2.27)$$

so $\alpha = 0$ and $\beta = \omega\sqrt{LC}$, and the phase velocity was

$$v_\phi = \frac{1}{\sqrt{LC}}, \quad (2.28)$$

the characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}}, \quad (2.29)$$

and the signals are

$$\begin{aligned} V(z) &= V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ I(z) &= \frac{1}{Z_0} (V_0^+ e^{-j\beta z} - V_0^- e^{j\beta z}) \end{aligned} \quad (2.30)$$

In the time domain for an infinite line, we have

$$\begin{aligned}
 v(z, t) &= \text{Re} \left(V(z) e^{j\omega t} \right) \\
 &= V_0^+ \text{Re} \left(e^{-j\beta z} e^{j\omega t} \right) \\
 &= V_0^+ \cos(\omega t - \beta z).
 \end{aligned} \tag{2.31}$$

In this case the shape and amplitude of the waveform are preserved as sketched in fig. 2.5.

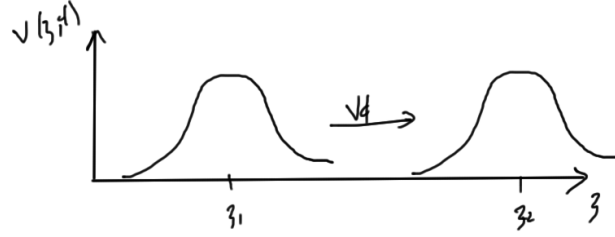


Figure 2.5: Lossless line signal preservation.

2.5 LOW LOSS LINE.

Assume $R \ll \omega L$ and $G \ll \omega C$. In this case we have

$$\begin{aligned}
 \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\
 &= j\omega \sqrt{LC} \sqrt{\left(1 + \frac{R}{j\omega L}\right) \left(1 + \frac{G}{j\omega C}\right)} \\
 &\approx j\omega \sqrt{LC} \left(1 + \frac{R}{2j\omega L}\right) \left(1 + \frac{G}{2j\omega C}\right) \\
 &\approx j\omega \sqrt{LC} \left(1 + \frac{R}{2j\omega L} + \frac{G}{2j\omega C}\right) \\
 &= j\omega \sqrt{LC} + j\omega \frac{R\sqrt{C/L}}{2j\omega} + j\omega \frac{G\sqrt{L/C}}{2j\omega} \\
 &= j\omega \sqrt{LC} + \frac{1}{2} \left(R \sqrt{\frac{C}{L}} + G \sqrt{\frac{L}{C}} \right),
 \end{aligned} \tag{2.32}$$

so

$$\begin{aligned}
 \alpha &= \frac{1}{2} \left(R \sqrt{\frac{C}{L}} + G \sqrt{\frac{L}{C}} \right) \\
 \beta &= \omega \sqrt{LC}.
 \end{aligned} \tag{2.33}$$

Observe that this value for β is the same as the lossless case to first order. We also have

$$\begin{aligned} Z_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\ &\approx \sqrt{\frac{L}{C}}, \end{aligned} \quad (2.34)$$

also the same as the lossless case. We must also have $v_\phi = 1/\sqrt{LC}$. To consider a time domain signal note that

$$\begin{aligned} V(z) &= V_0^+ e^{-\gamma z} \\ &= V_0^+ e^{-\alpha z} e^{-j\beta z}, \end{aligned} \quad (2.35)$$

so

$$\begin{aligned} v(z, t) &= \text{Re} \left(V(z) e^{j\omega t} \right) \\ &= \text{Re} \left(V_0^+ e^{-\alpha z} e^{-j\beta z} e^{j\omega t} \right) \\ &= V_0^+ e^{-\alpha z} \cos(\omega t - \beta z). \end{aligned} \quad (2.36)$$

The phase factor can be written

$$\omega t - \beta z = \omega \left(t - \frac{\beta}{\omega} z \right) = \omega \left(t - z/v_\phi \right), \quad (2.37)$$

so the signal still moves with the phase velocity $v_\phi = 1/\sqrt{LC}$, but in a diminishing envelope as sketched in fig. 2.6.

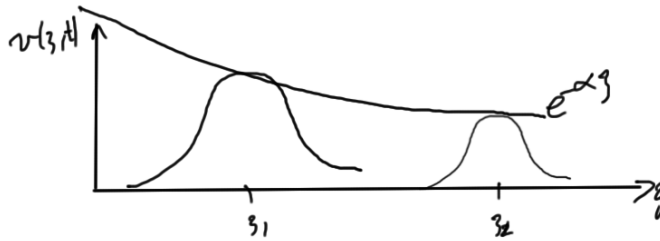


Figure 2.6: Time domain envelope for lossy line.

Notes

- The shape is preserved but the amplitude has an exponential attenuation along the line.

- In this case, since $\beta(\omega)$ is a linear function to first order, we have no dispersion. All of the Fourier components of a pulse travel with the same phase velocity since $v_\phi = \omega/\beta$ is constant. i.e. $v(z, t) = e^{-\alpha z} f(t - z/v_\phi)$. We should expect dispersion when the $R/\omega L$ and $G/\omega C$ start becoming more significant.

2.6 DISTORTIONLESS LINE.

Motivated by the early telegraphy days, when low loss materials were not available. Therefore lines with a constant attenuation and constant phase velocity (i.e. no dispersion) were required in order to eliminate distortion of the signals. This can be achieved by setting

$$\frac{R}{L} = \frac{G}{C}. \quad (2.38)$$

When that is done we have

$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= j\omega \sqrt{LC} \sqrt{\left(1 + \frac{R}{j\omega L}\right) \left(1 + \frac{G}{j\omega C}\right)} \\ &= j\omega \sqrt{LC} \sqrt{\left(1 + \frac{R}{j\omega L}\right) \left(1 + \frac{R}{j\omega L}\right)} \\ &= j\omega \sqrt{LC} \left(1 + \frac{R}{j\omega L}\right) \\ &= R \sqrt{\frac{C}{L}} + j\omega \sqrt{LC} \\ &= \sqrt{RG} + j\omega \sqrt{LC}. \end{aligned} \quad (2.39)$$

We have

$$\begin{aligned} \alpha &= \sqrt{RG} \\ \beta &= \omega \sqrt{LC}. \end{aligned} \quad (2.40)$$

The phase velocity is the same as that of the lossless and low-loss lines

$$v_\phi = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}. \quad (2.41)$$

2.7 TERMINATED LOSSLESS LINE.

Consider the load configuration sketched in fig. 2.7.

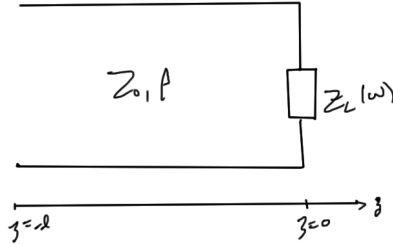


Figure 2.7: Terminated line.

Recall that

$$\begin{aligned} V(z) &= V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \\ I(z) &= \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z} \end{aligned} \quad (2.42)$$

At the load ($z = 0$), we have

$$\begin{aligned} V(0) &= V_0^+ + V_0^- \\ I(0) &= \frac{1}{Z_0} (V_0^+ - V_0^-) \end{aligned} \quad (2.43)$$

So

$$\begin{aligned} Z_L &= \frac{V(0)}{I(0)} \\ &= Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \\ &= Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L}, \end{aligned} \quad (2.44)$$

where

$$\Gamma_L \equiv \frac{V_0^-}{V_0^+}, \quad (2.45)$$

is the reflection coefficient at the load.

The phasors for the signals take the form

$$\begin{aligned} V(z) &= V_0^+ (e^{-j\beta z} + \Gamma_L e^{+j\beta z}) \\ I(z) &= \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma_L e^{+j\beta z}). \end{aligned} \quad (2.46)$$

Observe that we can rearranging for Γ_L in terms of the impedances

$$(1 - \Gamma_L) Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L}, \quad (2.47)$$

or

$$\Gamma_L (Z_0 + Z_L) = Z_L - Z_0, \quad (2.48)$$

or

$$\Gamma_L = \frac{Z_L - Z_0}{Z_0 + Z_L}. \quad (2.49)$$

Power The average (time) power on the line is

$$\begin{aligned} P_{\text{av}} &= \frac{1}{2} \text{Re} (V(z) I^*(z)) \\ &= \frac{1}{2} \text{Re} \left(V_0^+ (e^{-j\beta z} + \Gamma_L e^{+j\beta z}) \left(\frac{V_0^+}{Z_0} \right)^* (e^{j\beta z} - \Gamma_L^* e^{-j\beta z}) \right) \\ &= \frac{|V_0^+|^2}{2Z_0} \text{Re} (1 + \Gamma_L e^{2j\beta z} - \Gamma_L^* e^{-2j\beta z} - |\Gamma_L|^2) \\ &= \frac{|V_0^+|^2}{2Z_0} (1 - |\Gamma_L|^2). \end{aligned} \quad (2.50)$$

where we've made use of the fact that $Z_0 = \sqrt{L/C}$ is real for the lossless line, and the fact that a conjugate difference $A - A^* = 2j \text{Im}(A)$ is purely imaginary.

This can be written as

$$P_{\text{av}} = P^+ - P^-, \quad (2.51)$$

where

$$\begin{aligned} P^+ &= \frac{|V_0^+|^2}{2Z_0} \\ P^- &= \frac{|V_0^+|^2}{2Z_0} |\Gamma_L|^2. \end{aligned} \quad (2.52)$$

This difference is the power delivered to the load. This is not z -dependent because we are considering the lossless case. Maximum power is delivered to the load when $\Gamma_L = 0$, which occurs when the impedances are matched.

2.8 RETURN LOSS AND INSERTION LOSS, DEFINED.

Return loss (dB) is defined as

$$\begin{aligned} \text{RL} &= 10 \log_{10} \frac{P_{\text{inc}}}{P_{\text{refl}}} \\ &= 10 \log_{10} \frac{1}{|\Gamma|^2} \\ &= -20 \log_{10} |\Gamma|. \end{aligned} \tag{2.53}$$

Insertion loss (dB) is defined as

$$\begin{aligned} \text{IL} &= 10 \log_{10} \frac{P_{\text{inc}}}{P_{\text{trans}}} \\ &= 10 \log_{10} \frac{P^+}{P^+ - P^-} \\ &= 10 \log_{10} \frac{1}{1 - |\Gamma|^2} \\ &= -10 \log_{10} (1 - |\Gamma|^2). \end{aligned} \tag{2.54}$$

2.9 STANDING WAVE RATIO

Consider again the lossless loaded configuration of fig. 2.7. Now let $z = -l$, where l is the distance from the load. The phasors at this point on the line are

$$\begin{aligned} V(-l) &= V_0^+ (e^{j\beta l} + \Gamma_L e^{-j\beta l}) \\ I(-l) &= \frac{V_0^+}{Z_0} (e^{j\beta l} - \Gamma_L e^{-j\beta l}) \end{aligned} \tag{2.55}$$

The absolute voltage at this point is

$$\begin{aligned} |V(-l)| &= |V_0^+| |e^{j\beta l} + \Gamma_L e^{-j\beta l}| \\ &= |V_0^+| |1 + \Gamma_L e^{-2j\beta l}| \\ &= |V_0^+| |1 + |\Gamma_L| e^{j\angle \Gamma_L} e^{-2j\beta l}|, \end{aligned} \tag{2.56}$$

where the complex valued Γ_L is given by $\Gamma_L = |\Gamma_L|e^{j\Theta_L}$.

This gives

$$|V(-l)| = |V_0^+| |1 + |\Gamma_L|e^{j(\Theta_L - 2\beta l)}|. \quad (2.57)$$

The voltage magnitude oscillates as one moves along the line. The maximum occurs when $e^{j(\Theta_L - 2\beta l)} = 1$

$$V_{\max} = |V_0^+| |1 + |\Gamma_L||. \quad (2.58)$$

This occurs when $\Theta_L - 2\beta l = 2k\pi$ for $k = 0, 1, 2, \dots$. The minimum occurs when $e^{j(\Theta_L - 2\beta l)} = -1$

$$V_{\min} = |V_0^+| |1 - |\Gamma_L||, \quad (2.59)$$

which occurs when $\Theta_L - 2\beta l = (2k - 1)\pi$ for $k = 1, 2, \dots$. The standing wave ratio is defined as

$$\text{SWR} = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}. \quad (2.60)$$

This is a measure of the mismatch of a line. This is sketched in fig. 2.8.

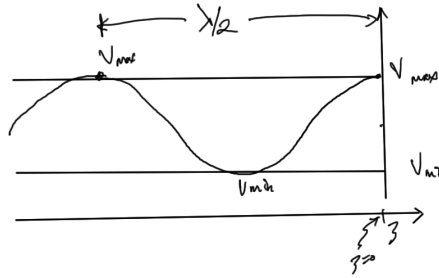


Figure 2.8: SWR extremes.

Notes:

- Since $0 \leq |\Gamma_L| \leq 1$, we have $1 \leq \text{SWR} \leq \infty$. The lower bound is for a matched line, and open, short, or purely reactive termination leads to the infinities.

- The distance between two successive maxima (or minima) can be determined by setting $\Theta_L - 2\beta l = 2k\pi$ for two consecutive values of k . For $k = 0$, suppose that V_{\max} occurs at d_1

$$\Theta_L - 2\beta d_1 = 2(0)\pi, \quad (2.61)$$

or

$$d_1 = \frac{\Theta_L}{2\beta}. \quad (2.62)$$

For $k = 1$, let the max occur at d_2

$$\Theta_L - 2\beta d_2 = 2(1)\pi, \quad (2.63)$$

or

$$d_2 = \frac{\Theta_L - 2\pi}{2\beta}. \quad (2.64)$$

The difference is

$$\begin{aligned} d_1 - d_2 &= \frac{\Theta_L}{2\beta} - \frac{\Theta_L - 2\pi}{2\beta} \\ &= \frac{\pi}{\beta} \\ &= \frac{\pi}{2\pi/\lambda} \\ &= \frac{\lambda}{2}. \end{aligned} \quad (2.65)$$

The distance between two consecutive maxima (or minima) of the SWR is $\lambda/2$.

2.10 IMPEDANCE TRANSFORMATION.

Referring to fig. 2.9, let's solve for the impedance at the load where $z = 0$ and at $z = -l$.

At any point on the line we have

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{2j\beta z}), \quad (2.66)$$

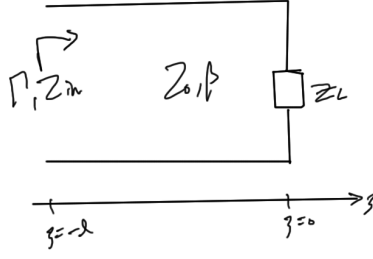


Figure 2.9: Configuration for impedance transformation.

so at the load and input we have

$$\begin{aligned} V_L &= V_0^+ (1 + \Gamma_L) \\ V(-l) &= V^+ (1 + \Gamma_L(-1)), \end{aligned} \quad (2.67)$$

where

$$\begin{aligned} V^+ &= V_0^+ e^{j\beta l} \\ \Gamma_L(-1) &= \Gamma_L e^{-2j\beta l} \end{aligned} \quad (2.68)$$

Similarly

$$I(-l) = \frac{V^+}{Z_0} (1 - \Gamma_L(-1)). \quad (2.69)$$

Define an input impedance as

$$\begin{aligned} Z_{\text{in}} &= \frac{V(-l)}{I(-l)} \\ &= Z_0 \frac{1 + \Gamma_L(-1)}{1 - \Gamma_L(-1)} \end{aligned} \quad (2.70)$$

This is analogous to

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \quad (2.71)$$

From eq. (2.49), we have

$$\begin{aligned}
Z_{\text{in}} &= Z_0 \frac{Z_0 + Z_L + (Z_L - Z_0) e^{-2j\beta l}}{Z_0 + Z_L - (Z_L - Z_0) e^{-2j\beta l}} \\
&= Z_0 \frac{(Z_0 + Z_L) e^{j\beta l} + (Z_L - Z_0) e^{-j\beta l}}{(Z_0 + Z_L) e^{j\beta l} - (Z_L - Z_0) e^{-j\beta l}} \\
&= Z_0 \frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)},
\end{aligned} \tag{2.72}$$

or

$$Z_{\text{in}} = \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}. \tag{2.73}$$

This can be thought of as providing a reflection coefficient function along the line to the load at any point as sketched in fig. 2.10.

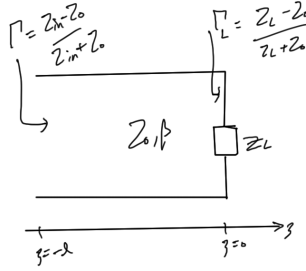


Figure 2.10: Impedance transformation reflection on the line.

2.11 SHORT CIRCUITED LINE

A short circuited line, also called a shorted stub, is sketched in fig. 2.11.

With

$$Z_L = 0, \tag{2.74}$$

the input impedance is

$$Z_{\text{in}} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} = jZ_0 \tan(\beta l) \tag{2.75}$$

For short line sections $\beta l \ll \pi/2$, or $l \ll \lambda/4$, the input impedance is approximately

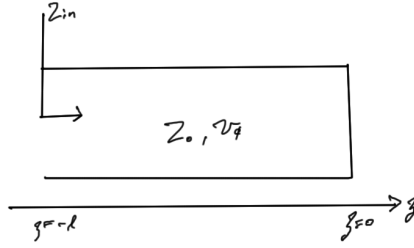


Figure 2.11: Short circuited line.

$$\begin{aligned}
 Z_{\text{in}} &= jZ_0 \tan(\beta l) \\
 &\approx jZ_0 \sin(\beta l) \\
 &\approx jZ_0 \beta l
 \end{aligned} \tag{2.76}$$

Introducing an equivalent inductance defined by $Z_{\text{in}} = j\omega L_{\text{eq}}$, we have

$$\begin{aligned}
 L_{\text{eq}} &= \frac{Z_0}{\omega} \beta l \\
 &= \frac{Z_0}{\omega} \frac{\omega}{v_\phi} l \\
 &= \frac{Z_0 l}{v_\phi}.
 \end{aligned} \tag{2.77}$$

The inductance per unit length of the line is $C = Z_0/v_\phi$. An application for this result is that instead of using inductors, shorted stubs can be used in high frequency applications.

This is also the case for short sections of high impedance line.

2.12 OPEN CIRCUITED LINE

An open circuited line is sketched in fig. 2.12.

This time with $Z_L \rightarrow \infty$ we have

$$\begin{aligned}
 Z_{\text{in}} &= Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \\
 &= -jZ_0 \cot(\beta l).
 \end{aligned} \tag{2.78}$$

This time we have an equivalent capacitance. For short sections with $\beta l \ll \pi/2$

$$Z_{\text{in}} \approx -j \frac{Z_0}{\beta l} \tag{2.79}$$

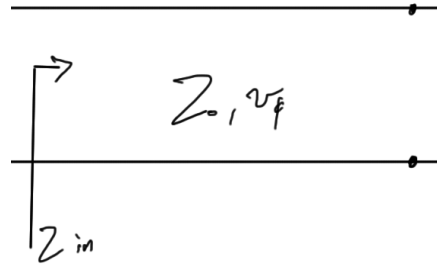


Figure 2.12: Open circuited line.

Introducing an equivalent capacitance defined by $Z_{in} = 1/(j\omega C_{eq})$, we have

$$\begin{aligned}
 C_{eq} &= \frac{\beta l}{\omega Z_0} \\
 &= \frac{\omega/v_\phi l}{\omega Z_0} \\
 &= \frac{l}{v_\phi Z_0}
 \end{aligned} \tag{2.80}$$

The capacitance per unit length of the line is $C = 1/(Z_0 v_\phi)$.

This is also the case for short sections of low impedance line.

2.13 HALF WAVELENGTH TRANSFORMER.

A half wavelength transmission line equivalent circuit is sketched in fig. 2.13.

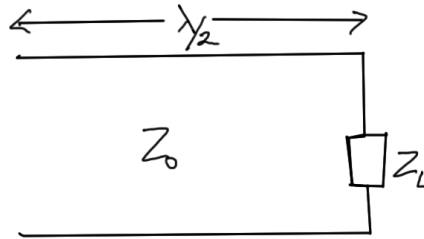


Figure 2.13: Half wavelength transmission line.

With $l = \lambda/2$

$$\begin{aligned}\beta l &= \frac{2\pi}{\lambda} \frac{\lambda}{2} \\ &= \pi.\end{aligned}\tag{2.81}$$

Since $\tan \pi = 0$, the input impedance is

$$\begin{aligned}Z_{\text{in}} &= Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \\ &= Z_L.\end{aligned}\tag{2.82}$$

2.14 QUARTER WAVELENGTH TRANSFORMER.

A quarter wavelength transmission line equivalent circuit is sketched in fig. 2.14.

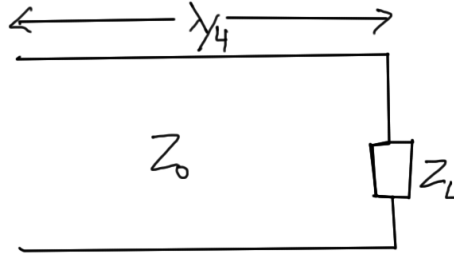


Figure 2.14: Quarter wavelength transmission line.

With $l = \lambda/4$

$$\begin{aligned}\beta l &= \frac{2\pi}{\lambda} \frac{\lambda}{4} \\ &= \frac{\pi}{2}.\end{aligned}\tag{2.83}$$

We have $\tan \beta l \rightarrow \infty$, so the input impedance is

$$\begin{aligned}Z_{\text{in}} &= Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \\ &= \frac{Z_0^2}{Z_L}.\end{aligned}\tag{2.84}$$

This relation

$$Z_{\text{in}} = \frac{Z_0^2}{Z_L}, \quad (2.85)$$

is called the impedance inverter.

- A large impedance is transformed into a small one and vice-versa.
- A short becomes an open and vice-versa.
- A capacitive load becomes inductive and vice-versa.
- If Z_L is a series resonant circuit then Z_{in} becomes parallel resonant.

See [3] for an explanation of the term series resonant.

Matching with a $\lambda/4$ transformer. Matching for a quarter wavelength transmission line equivalent circuit is sketched in fig. 2.15.

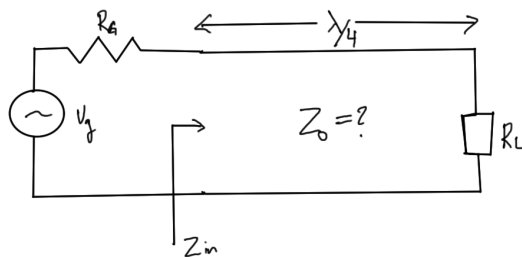


Figure 2.15: Quarter wavelength transmission line matching.

For maximum power transfer

$$Z_{\text{in}} = \frac{Z_0^2}{R_L} = R_G, \quad (2.86)$$

so

$$Z_0 = \sqrt{R_G R_L}. \quad (2.87)$$

We have

$$|\Gamma_L| = \frac{R_L - Z_0}{R_L + Z_0} \neq 0, \quad (2.88)$$

and still maximum power is transferred.

2.15 SMITH CHART

A Smith chart is a graphical tool for making the transformation $\Gamma \leftrightarrow Z_{\text{in}}$. Given

$$Z_{\text{in}} = Z_0 \frac{1 + \Gamma}{1 - \Gamma}, \quad (2.89)$$

where $\Gamma = \Gamma_L e^{-2j\beta l}$, we begin by normalizing the input impedance, using an overbar to denote that normalization

$$Z_{\text{in}} \rightarrow \bar{Z}_{\text{in}} = \frac{Z_{\text{in}}}{Z_0}, \quad (2.90)$$

so

$$\begin{aligned} \bar{Z}_{\text{in}} &= \frac{1 + \Gamma}{1 - \Gamma} \\ &= \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i} \\ &= \frac{((1 + \Gamma_r) + j\Gamma_i)((1 - \Gamma_r) + j\Gamma_i)}{(1 - \Gamma_r)^2 + \Gamma_i^2} \\ &= \frac{(1 - \Gamma_r^2 - \Gamma_i^2) + j\Gamma_i(1 - \Gamma_r + 1 + \Gamma_r)}{(1 - \Gamma_r)^2 + \Gamma_i^2} \\ &= \frac{(1 - |\Gamma|^2) + 2j\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}. \end{aligned} \quad (2.91)$$

If we let $\bar{Z}_{\text{in}} = \bar{\Gamma}_L + j\bar{X}_L$, and equate real and imaginary parts we have

$$\begin{aligned} \bar{\Gamma}_L &= \frac{1 - |\Gamma|^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \\ \bar{X}_L &= \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \end{aligned} \quad (2.92)$$

It is left as an exercise to demonstrate that these can be rearranged into

$$\begin{aligned} \left(\Gamma_r - \frac{\bar{\Gamma}_L}{1 + \bar{\Gamma}_L} \right)^2 + \Gamma_i^2 &= \left(\frac{1}{1 + \bar{\Gamma}_L} \right)^2 \\ (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{\bar{X}_L} \right)^2 &= \frac{1}{\bar{X}_L^2}, \end{aligned} \quad (2.93)$$

which trace out circles in the Γ_r, Γ_i plane, one for the real part of Γ and one for the imaginary part. This provides a graphical way for implementing the impedance transformation.

Real impedance circle The circle for the real part is centered at

$$\left(\frac{\bar{\Gamma}_L}{1 + \bar{\Gamma}_L}, 0 \right), \quad (2.94)$$

with radius

$$\frac{1}{1 + \bar{\Gamma}_L}. \quad (2.95)$$

All these circles pass through the point (1, 0), since

$$\frac{\bar{\Gamma}_L}{1 + \bar{\Gamma}_L} + \frac{1}{1 + \bar{\Gamma}_L} = \frac{1 + \bar{\Gamma}_L}{1 + \bar{\Gamma}_L} = 1. \quad (2.96)$$

For reactive loads where $\bar{\Gamma}_L = 0$, we have $\Gamma_r^2 + \Gamma_i^2 = 1$, a circle through the origin with unit radius.

For matched loads where $\bar{\Gamma}_L = 1$ the circle is centered at (1/2, 0), with radius 1/2.

Imaginary impedance circle The circle obtained by equating imaginary parts are constant reactance circles with center

$$\left(1, \frac{1}{\bar{X}_L} \right), \quad (2.97)$$

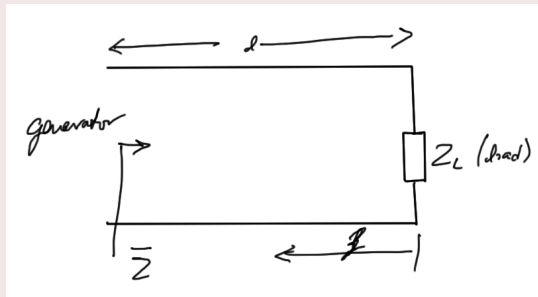
with radius

$$\frac{1}{\bar{X}_L}. \quad (2.98)$$

These circles also pass through the point (1, 0). These circles are orthogonal to the constant resistance circles. Some of the features of a Smith chart are sketched in fig. 2.16.

A matlab produced blank Smith chart can be found in fig. 2.17.

Example 2.2: Perform a transformation along a lossless line.



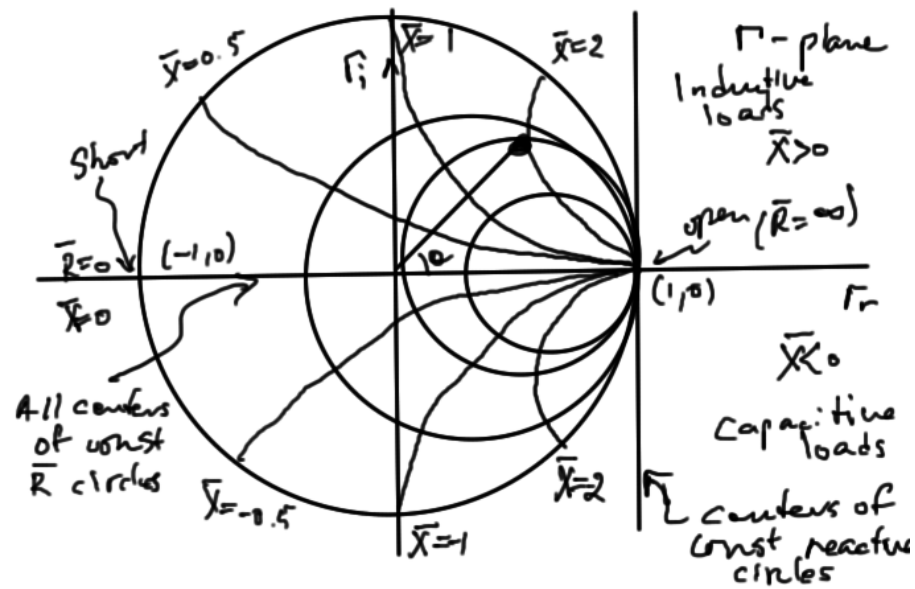


Figure 2.16: Hand sketched Smith chart.

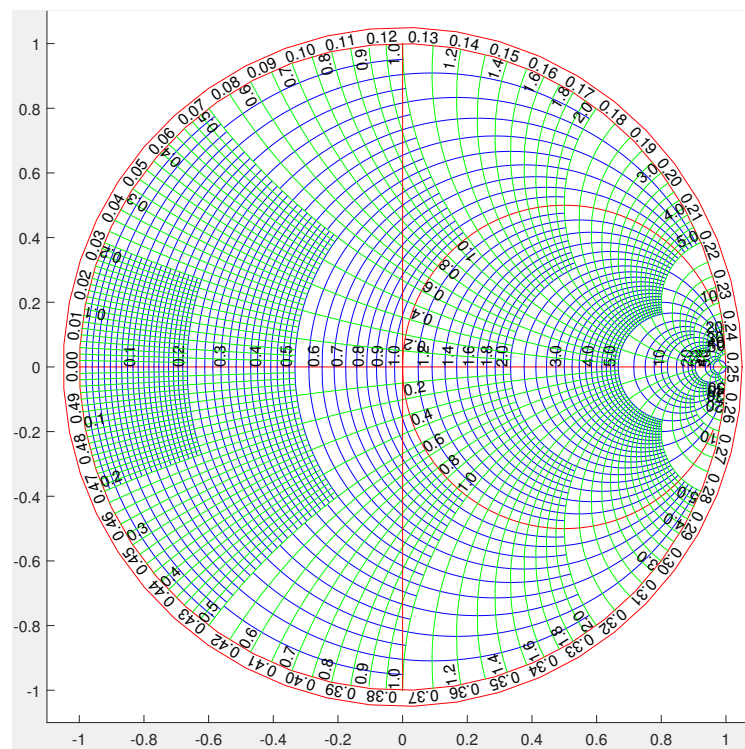


Figure 2.17: Blank Smith chart.

Figure 2.18: Impedance transformation along lossless line.

Given

$$\bar{Z} = \frac{1 + \Gamma}{1 - \Gamma}, \quad (2.99)$$

$$\Gamma = \Gamma_L e^{-2j\beta l}, \quad (2.100)$$

and

$$\Gamma_L = |\Gamma_L| e^{j\Theta_L} \quad (2.101)$$

The total reflection coefficient is

$$\Gamma = |\Gamma_L| e^{j(\Theta_L - 2\beta l)} \quad (2.102)$$

If $\Gamma_L = |\Gamma_L| e^{j\Theta_L}$ is plotted on the Smith chart, then in order to move towards the generator, a subtraction from Θ_L of $2\beta l$ is required.

Some worked examples that demonstrate this can be found in fig. 2.19, fig. 2.20, and fig. 2.21.

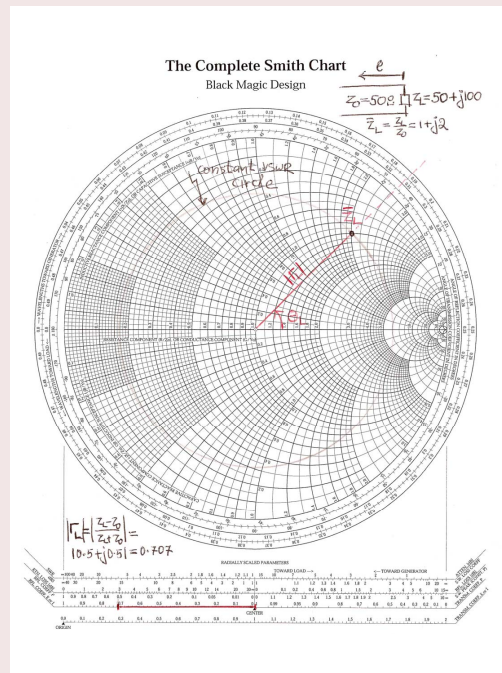


Figure 2.19: Mapping an impedance value onto a Smith chart.

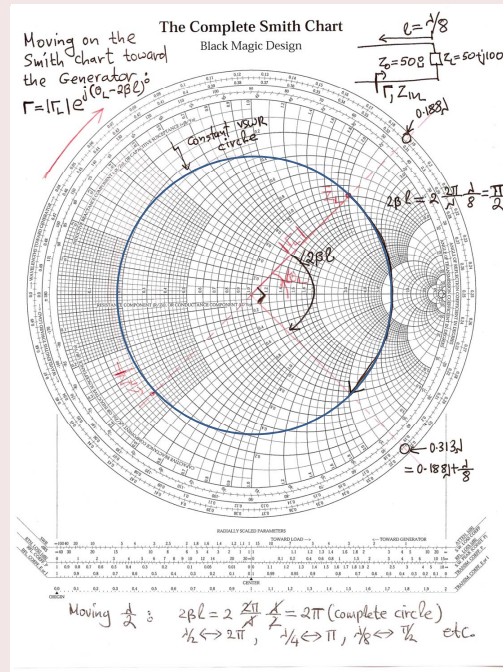


Figure 2.20: Moving on the Smith chart towards the generator.

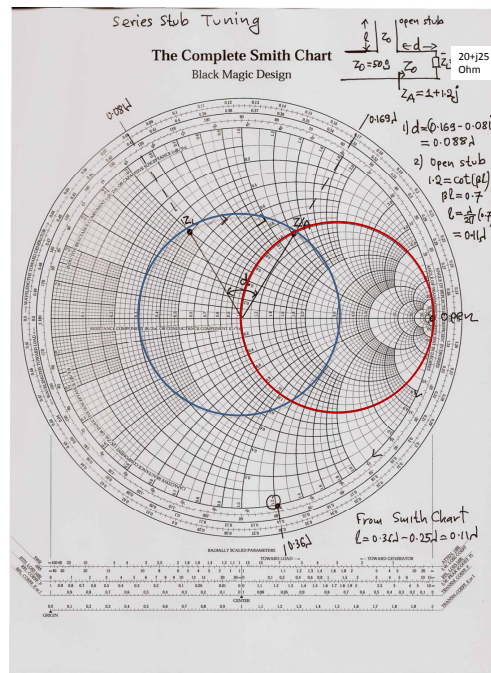


Figure 2.22: Single stub tuning example

Notes:

- (a) In step (3) there are two points where the $R=1$ circle is intersected . Usually we chose the shortest one
- (b) By adding multiples of half-wavelength lengths to either d or l an infinite number of solutions can be constructed.

2.16 PROBLEMS

Exercise 2.1 Find the Smith chart circle equations

Prove eq. (2.93).

Answer for Exercise 2.1

We can write

$$(1 - \Gamma_r)^2 + \Gamma_i^2 = \frac{2\Gamma_i}{\bar{X}_L}, \quad (2.103)$$

or

$$(1 - \Gamma_r)^2 + \left(\Gamma_i - \frac{1}{\bar{X}_L}\right)^2 = \frac{1}{(\bar{X}_L)^2}, \quad (2.104)$$

which is one of the circular equations. For the other, putting the Γ_r, Γ_i terms in the numerator, we have

$$\begin{aligned} \frac{1 - \Gamma_r^2 - \Gamma_i^2}{\bar{\Gamma}_L} &= (1 - \Gamma_r)^2 + \Gamma_i^2 \\ &= 1 - 2\Gamma_r + \Gamma_r^2 + \Gamma_i^2, \end{aligned} \quad (2.105)$$

or

$$\Gamma_r^2 \left(1 + \frac{1}{\bar{\Gamma}_L}\right) - 2\Gamma_r + \Gamma_i^2 \left(1 + \frac{1}{\bar{\Gamma}_L}\right) = \frac{1}{\bar{\Gamma}_L} - 1. \quad (2.106)$$

Dividing through by $1 + 1/\bar{\Gamma}_L = (\bar{\Gamma}_L + 1)/\bar{\Gamma}_L$, we have

$$\begin{aligned} \Gamma_r^2 - 2\Gamma_r \frac{\bar{\Gamma}_L}{\bar{\Gamma}_L + 1} + \Gamma_i^2 &= \frac{1 - \bar{\Gamma}_L}{\bar{\Gamma}_L} \frac{\bar{\Gamma}_L}{\bar{\Gamma}_L + 1} \\ &= \frac{1 - \bar{\Gamma}_L}{\bar{\Gamma}_L + 1}, \end{aligned} \quad (2.107)$$

or

$$\begin{aligned}
 \left(\Gamma_r - \frac{\bar{\Gamma}_L}{\bar{\Gamma}_L + 1} \right)^2 + \Gamma_i^2 &= \frac{1 - \bar{\Gamma}_L}{\bar{\Gamma}_L + 1} + \left(\frac{\bar{\Gamma}_L}{\bar{\Gamma}_L + 1} \right)^2 \\
 &= \frac{1 - \bar{\Gamma}_L^2 + \bar{\Gamma}_L^2}{(\bar{\Gamma}_L + 1)^2} \\
 &= \frac{1}{(\bar{\Gamma}_L + 1)^2}.
 \end{aligned} \tag{2.108}$$

Exercise 2.2 **Transmission line basics.**

A two-wire line has distributed parameters $L' = 6.5\mu\text{H/m}$ and $C' = 8.7\text{pF/m}$.

- What is the characteristic impedance (resistance) of the line?
- What is the phase velocity this transmission line?
- If a voltage wave with time waveform $V^-(t)$ is traveling along the line in the $-z$ direction, and has an amplitude of 10V, write a time-domain expression for the current wave associated with this voltage wave, using numerical parameters determined in part [a](#) and part [b](#).

Answer for Exercise 2.2

Part a. In this case we have

$$\begin{aligned}
 \gamma &= \sqrt{(j\omega L')(j\omega C')} \\
 &= j\omega \sqrt{L'C'}
 \end{aligned} \tag{2.109}$$

and impedance

$$\begin{aligned}
Z_0 &= \frac{R' + j\omega L'}{\gamma} \\
&= \frac{L'}{\sqrt{L'C'}} \\
&= \sqrt{\frac{L'}{C'}} \\
&= \sqrt{\frac{6.5\mu\text{H/m}}{8.7\text{pF/m}}} \\
&= \sqrt{\frac{6.5 \times 10^{-6}\text{H}}{8.7 \times 10^{-12}\text{F}}} \\
&= \sqrt{\frac{6.5}{8.7}}\text{k}\Omega. \\
&= 0.86\text{k}\Omega.
\end{aligned} \tag{2.110}$$

Part b. The phase velocity is

$$\begin{aligned}
v_\phi &= \frac{1}{\sqrt{L'C'}} \\
&= \frac{1}{\sqrt{(6.5)(8.7)10^{-6-12}(\text{H/m})(\text{F/m})}} \\
&= 1.3 \times 10^8 \text{m/s},
\end{aligned} \tag{2.111}$$

a factor of 23 less than the speed of light in vacuum.

A verification that $\sqrt{\text{H/F}}$ is an Ohm, and that $(1/\sqrt{(\text{H/m})(\text{F/m})}) = \text{m/s}$ can be found in [transmissionLineUnits.nb](#).

Part c. The phasor voltage is

$$V = 10 \times e^{\beta z}, \tag{2.112}$$

so the current phasor is

$$I = -\frac{10V}{0.86\text{k}\Omega} \times e^{\beta z}, \tag{2.113}$$

We have

$$\begin{aligned}
\beta &= \omega \sqrt{LC} \\
&= \frac{\omega}{v_\phi},
\end{aligned} \tag{2.114}$$

so

$$i(z, t) = -11.6 \cos(\omega(z/v_\phi + t)) \text{ mA}, \quad (2.115)$$

where v_ϕ is given by eq. (2.111).

Exercise 2.3 Low-loss transmission line with non-magnetic dielectric.

A transmission line is filled with a non-magnetic dielectric of $\epsilon_r = 2.5$. The line has a capacitance per unit length of $C = 200 \text{ pF/m}$ and a resistance per unit length of $R = 2 \Omega/\text{m}$. Calculate the corresponding phase velocity, characteristic impedance and attenuation constant α (assume $G = 0$).

Answer for Exercise 2.3

The impedance of the line is

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}, \quad (2.116)$$

we know R, C, G , but not L . We'd found that

$$v_\phi = \frac{1}{\sqrt{LC}}, \quad (2.117)$$

for lossless, low-loss, and distortionless lines. If we assume that is the case here too, and

$$\begin{aligned} v_\phi &= \frac{c}{\sqrt{\epsilon_r}} \\ &= \frac{3 \times 10^8}{\sqrt{2.5}} \\ &= 1.9 \times 10^8 \text{ m/s}, \end{aligned} \quad (2.118)$$

we have

$$\begin{aligned} L &= \frac{1}{v_\phi^2 C} \\ &= \frac{1}{(1.9 \times 10^8)^2 (200 \times 10^{-12})} \text{ H/m} \\ &= 14 \text{ nH/m}. \end{aligned} \quad (2.119)$$

For frequencies in the GHz range are the low-loss conditions satisfied? With $G = 0$, the condition $G \ll \omega C$ is clearly true. How about $R \ll \omega L$? For f in GHz we have

$$\begin{aligned}\omega L &= 2(14)\pi f \\ &= 88f.\end{aligned}\tag{2.120}$$

We have $2 < 88f$, so the low-loss approximation is definitely valid for frequencies above 1 GHz.

Without such approximations, our impedance expression is a bit of an ugly beast

$$\begin{aligned}Z_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\ &= \sqrt{\frac{2 + j\omega 14 \times 10^{-9}}{j\omega 200 \times 10^{-12}}} \Omega.\end{aligned}\tag{2.121}$$

The low-loss approximation for the impedance, with $R = 0$, is

$$Z_0 = 26.35.\tag{2.122}$$

Compare this to the exact expansion of eq. (2.121) at frequencies starting at 1 GHz

$$\begin{aligned}Z_0(1 \text{ GHz}) &= 26.35/\underline{-0.065\,66^\circ} \\ Z_0(2 \text{ GHz}) &= 26.35/\underline{-0.032\,83^\circ} \\ Z_0(3 \text{ GHz}) &= 26.35/\underline{-0.021\,89^\circ} \\ Z_0(10 \text{ GHz}) &= 26.35/\underline{-0.006\,566^\circ}\end{aligned}\tag{2.123}$$

At all of these frequencies, the impedance matches the low-loss approximation.

For the attenuation factor first note that

$$\begin{aligned}\gamma &= \left((2 + j\omega 1.4 \times 10^{-9}) (j\omega 200 \times 10^{-12}) \right)^{1/2} \\ &= ((2 + j2\pi f 1.4) (j0.4\pi f))^{1/2} \\ &= f \sqrt{0.4\pi} ((2/f)j - 2.8\pi)^{1/2},\end{aligned}\tag{2.124}$$

where f is in GHz. We see that for $f \in [1, 10]$, this γ is close to purely imaginary, as is also the case for $R = 0$. Specifically

$$\begin{aligned}\gamma(1 \text{ GHz})/f &= 33.12/\underline{89.93^\circ} \\ \gamma(2 \text{ GHz})/f &= 33.12/\underline{89.97^\circ} \\ \gamma(3 \text{ GHz})/f &= 33.12/\underline{89.98^\circ} \\ \gamma(10 \text{ GHz})/f &= 33.12/\underline{89.99^\circ}\end{aligned}\tag{2.125}$$

We want the real component of this. For the same frequencies, some specific values of the real part, are

$$\begin{aligned}\alpha(1 \text{ GHz}) &= 0.037947307 \\ \alpha(2 \text{ GHz}) &= 0.037947326 \\ \alpha(3 \text{ GHz}) &= 0.037947329 \\ \alpha(10 \text{ GHz}) &= 0.037947332.\end{aligned}\tag{2.126}$$

How much attenuation is this, relative to the wavelength in question? Even for lower frequencies, where the wavelength is larger, there is essentially zero attenuation over one wavelength. For the same respective frequencies we have

$$\begin{aligned}e^{-\lambda\alpha} &= 0.9928 \\ e^{-\lambda\alpha} &= 0.9964 \\ e^{-\lambda\alpha} &= 0.9976 \\ e^{-\lambda\alpha} &= 0.9993.\end{aligned}\tag{2.127}$$

Let's see how the low-loss line approximation fares for both the real (α) and imaginary (β) components of γ ? For all of these frequencies, we have

$$\gamma = 0.377 + 0.5305j\omega.\tag{2.128}$$

which compares exactly with the low-loss approximations using $Z_0 = \sqrt{L/C}$, $\alpha = (R/Z_0 + GZ_0)/2$, $\beta = \omega\sqrt{LC}$.

The numerical calculations made for this problem can be found in [ps1:ps1_2.jl](#).

Exercise 2.4 Lossless transmission line with load.

A lossless transmission line of characteristic impedance $Z_0 = 75\Omega$ is terminated to an unknown complex impedance Z_L . The transmission line is filled with air. The first voltage maximum is measured at a distance $d = 9.7\text{mm}$ from the load and the voltage standing-wave ratio on the line is measured to be $\text{VSWR} = 2.33$. The operating frequency is $f = 3.0\text{GHz}$.

- Determine the location of the first current minimum.
- What is the magnitude (amplitude) of the reflection coefficient $|\Gamma_L|$.
- What is the phase of the reflection coefficient Θ_L ?
- Determine the load Z_L .

Answer for Exercise 2.4

Part a. Because the current has a different sign in the superposition sum, the current min/max will be where there is a voltage max/min, so the first minimum is at $d = 9.7\text{mm}$.

Showing this explicitly, the current amplitude is

$$\begin{aligned} |I(-l)| &= \left| \frac{V_0^+}{Z_0} \right| |e^{-j\beta(-l)} - \Gamma_L e^{j\beta(-l)}| \\ &= \left| \frac{V_0^+}{Z_0} \right| |1 - |\Gamma_L| e^{j\Theta_L - 2j\beta l}|. \end{aligned} \quad (2.129)$$

The current minimum and maximums are respectively

$$\begin{aligned} |I(-l)|_{\min} &= \left| \frac{V_0^+}{Z_0} \right| (1 - |\Gamma_L|) \\ |I(-l)|_{\max} &= \left| \frac{V_0^+}{Z_0} \right| (1 + |\Gamma_L|) \end{aligned} \quad (2.130)$$

and occur when $\Theta_L - 2\beta l = 2\pi k$, and $\Theta_L - 2\beta l = (2k - 1)\pi$ respectively.

The first current minimum occurs when $\Theta_L = 2\beta l$, which is exactly where the first voltage maximum also occurs.

Part b. We have $R = G = 0$, so

$$Z_0 = \sqrt{\frac{L}{C}} = 75\Omega. \quad (2.131)$$

We also have

$$v_\phi = c = \frac{1}{\sqrt{LC}}. \quad (2.132)$$

The VSWR is

$$\begin{aligned} V_{\text{SWR}} &= \frac{V_{\max}}{V_{\min}} \\ &= \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \\ &= 2.33, \end{aligned} \quad (2.133)$$

so

$$1 + |\Gamma_L| = (1 - |\Gamma_L|) V_{\text{SWR}} \quad (2.134)$$

or

$$\begin{aligned}
|\Gamma_L| &= \frac{V_{\text{SWR}} - 1}{V_{\text{SWR}} + 1} \\
&= \frac{2.33 - 1}{2.33 + 1} \\
&= \frac{1.33}{3.33} \\
&= \frac{4/3}{10/3} \\
&= 0.4 \Omega.
\end{aligned} \tag{2.135}$$

Part c. The absolute voltage at $z = -l$ is

$$|V(-l)| = |V_0^+| |1 + |\Gamma_L| e^{j(\Theta_L - 2\beta l)}|, \tag{2.136}$$

so the maximum was measured at

$$\Theta_L - 2\beta d = 2\pi(0), \tag{2.137}$$

or

$$\begin{aligned}
\Theta_L &= 2\beta d \\
&= 2 \frac{\omega}{v_\phi} d \\
&= \frac{4\pi f d}{c} \\
&= \frac{4\pi(3 \times 10^9)(9.7 \times 10^{-3})}{3 \times 10^8} \\
&= 1.219 \text{ rad}.
\end{aligned} \tag{2.138}$$

The reflection coefficient can now be written out explicitly

$$\Gamma_L = 0.1377 + 0.3749j \Omega. \tag{2.139}$$

Part d. The load impedance is

$$\begin{aligned}
Z_L &= Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \\
&= 75 \frac{1.4 + 0.049j}{0.6 - 0.049j} \\
&= 71.29 + 63.6j \Omega.
\end{aligned} \tag{2.140}$$

The numerical results are computed in [ps1:ps1_3.jl](#).

Exercise 2.5 Parallel-plate transmission line.

Consider a parallel-plate transmission line of height d and width w ($w \gg d$). The plates are filled with a lossless dielectric ϵ .

- Calculate the capacitance per unit length C' .
- Calculate the inductance per unit length L' .
- Determine an expression for the characteristic impedance Z_0 .
- Calculate the resistance per unit length R' . Assume that the conductivity of the plates is σ and the skin depth is δ_s .
- Calculate an expression for the attenuation constant α in terms of d .
- Now consider that $d/w = 0.1$ and $\epsilon_r = 1$. The plates have a conductivity of $\sigma = 3.538 \times 10^7$ siemens/m. and the frequency of operation is $f = 30$ GHz. Calculate the characteristic impedance Z_0 .
- Calculate the attenuation constant α numerically, and in dB/m when $d = 1$ cm, 1mm and $1\mu\text{m}$.

Answer for Exercise 2.5

Part a. To get some quick results, the transmission line can be treated as a large parallel plate capacitor. First recall that for a single plate with linear charge density $+(Q/l)$ on it, the electric field is normal and outwards from the plate as in sketched in fig. 2.23.

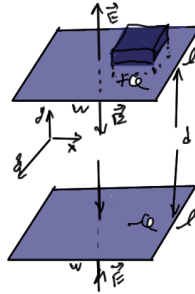


Figure 2.23: Electric field between two plates

Using a Gaussian volume the width of the plate we have

$$\int \mathbf{D} \cdot \hat{\mathbf{n}} dA = 2\epsilon E w l = Q, \quad (2.141)$$

so the magnitude of the field for one plate is

$$E = \frac{Q}{2\epsilon w l} \quad (2.142)$$

Introducing a second plate with equal but opposite charge density on it, we have cancelation of the electric field outside of the plates, but a doubling within. The magnitude of the total electric field between the plates is therefore

$$\mathbf{E} = -\frac{Q}{\epsilon w l} \hat{\mathbf{y}}. \quad (2.143)$$

The voltage difference between the plates is

$$\begin{aligned} V &= \int_0^d E dl \\ &= \frac{Qd}{\epsilon w l}. \end{aligned} \quad (2.144)$$

The capacitance per unit length is

$$\begin{aligned} C' &= \frac{Q/l}{V} \\ &= \frac{(Q/l)(\epsilon w l)}{Qd} \end{aligned} \quad (2.145)$$

$$C' = \frac{\epsilon w}{d}. \quad (2.146)$$

Here edge effects and charge distribution on the plates has been completely ignored. This is also only one of the possible field modes in the cavity (TEM mode). This mode and the others are covered in greater detail in [5] ch. 3.

Part b. For the magnetic field the situation is similar. Suppose the magnetic field is oriented as sketched in fig. 2.24.

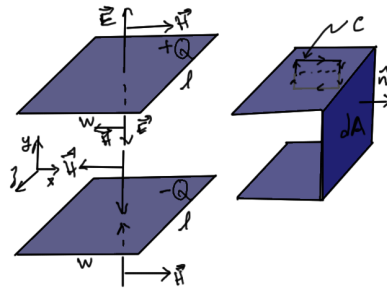


Figure 2.24: Magnetic field and flux between two plates

Integrating over the loop C that surrounds the top plate we have

$$\oint \mathbf{D} \cdot d\mathbf{l} = 2Hw \quad (2.147)$$

$$= I,$$

or

$$H = \frac{I}{2w}. \quad (2.148)$$

Like the electric field, an opposite charge on the other plate leads to a doubled magnetic field in the cavity, and no magnetic field outside the plates, so the total magnetic field magnitude in the cavity is

$$H = \frac{I}{w}. \quad (2.149)$$

Calculating the flux through the side we have

$$\begin{aligned} \Phi &= \int \mathbf{B} \cdot d\mathbf{A} \\ &= \mu H l d \\ &= \frac{\mu I l d}{w} \end{aligned} \quad (2.150)$$

The inductance per unit length is

$$L' = \frac{\Phi/l}{I}, \quad (2.151)$$

or

$$L' = \frac{\mu d}{w}. \quad (2.152)$$

Part c. Assuming a low-loss condition, the impedance is

$$\begin{aligned} Z_0 &= \sqrt{\frac{L}{C}} \\ &= \sqrt{\frac{\mu d}{w} \frac{d}{\epsilon w}} \\ &= \frac{d}{w} \sqrt{\frac{\mu}{\epsilon}}, \end{aligned} \quad (2.153)$$

or

$$Z_0 = \frac{d}{w} \eta. \quad (2.154)$$

Part d. Referring to fig. 2.25, the RF resistance can be to the effective area that the field acts on.

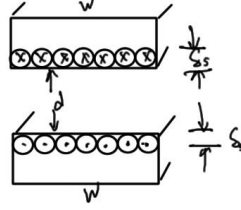


Figure 2.25: Skin depth for parallel waveguide.

The resistance per plate in a length l is

$$R = \frac{l}{\sigma A_{\text{eff}}}, \quad (2.155)$$

so the total resistance per unit length is

$$R' = \frac{2}{\sigma w \delta_s}. \quad (2.156)$$

Part e. Using the low-loss approximation, we have

$$\begin{aligned} \alpha &= \frac{1}{2} \left(\frac{R'}{Z_0} + GZ_0 \right) \\ &= \frac{1}{2} \left(\frac{2}{\sigma w \delta_s} \frac{w}{d\eta} \right) \\ &= \frac{1}{\sigma \delta_s} \frac{1}{d\eta}. \end{aligned} \quad (2.157)$$

The leading quantity is identified in [5] (1.125) as the surface resistance

$$R_s = \frac{1}{\sigma \delta_s} = \sqrt{\frac{\omega \mu}{2\sigma}}, \quad (2.158)$$

so we can write

$$\alpha = \frac{R_s}{d\eta}. \quad (2.159)$$

Part f. At these values of d/w we have

$$Z_0 = 37.8\Omega. \quad (2.160)$$

Part g. With

$$\begin{aligned} R_s &= 0.05786\Omega \\ \eta &= 377.9\Omega, \end{aligned} \quad (2.161)$$

we have

$$\begin{aligned} \alpha(1 \text{ cm}) &= 0.0153108\text{m}^{-1} \\ \alpha(1 \text{ mm}) &= 0.153108\text{m}^{-1} \\ \alpha(1\mu\text{m}) &= 153.108\text{m}^{-1} \end{aligned} \quad (2.162)$$

This by itself isn't a reasonable quantity to convert to dB, since it isn't a ratio of two quantities. Asking about this, it turns out that the quantity of interest is the loss ratio

$$L \equiv e^{-2\alpha} \quad (2.163)$$

This is of interest because the voltage is of the form

$$V = V_0 e^{-\alpha z} e^{-j\beta z}, \quad (2.164)$$

so the ratio of powers relative to the $z = 0$ power is

$$\begin{aligned} \frac{P}{P_0} &= \frac{|V|^2}{|V_0|^2} \\ &= e^{-2\alpha z}. \end{aligned} \quad (2.165)$$

The loss factor for one wavelength (something that could be expressed as dB/m) would be

$$e^{-2\alpha\lambda}, \quad (2.166)$$

whereas the loss factor for one meter ($z = 1$), is

$$L = e^{-2\alpha}. \quad (2.167)$$

In dB losses are

$$\begin{aligned} -10 \log_{10} L &= -10 \log_{10} e^{-2\alpha} \\ &= 20\alpha \log_{10} e. \end{aligned} \quad (2.168)$$

Note that the sign has been toggled to express the result as dB loss (as opposed to dB gain, where we'd normally want a positive sign)

This is really the quantity that was desired in this part of the problem. That is

$$\begin{aligned} L(1 \text{ cm}) &= 0.132988 \text{ dB}/m \\ L(1 \text{ mm}) &= 1.32988 \text{ dB}/m \\ L(1 \mu\text{m}) &= 1329.88 \text{ dB}/m. \end{aligned} \tag{2.169}$$

Should this loss factor have been desired in Np/m (Nepers per meter), the loss factor would be

$$\ln e^{-\alpha} = -\alpha, \tag{2.170}$$

so the raw values of α , up to a sign, can be considered to be in “Neper” units.

Numeric substitutions for this problem were performed in [ps1:ps1_4.jl](#)

IMPEDANCE MATCHING TRANSFORMERS

3.1 MULTISECTION TRANSFORMERS

Using a transformation of the form fig. 3.2 it is possible to optimize for maximum power delivery, using for example a matching transformation $Z_{\text{in}} = Z_1^2/R_L = Z_0$, or $Z_1 = \sqrt{R_L Z_0}$. Unfortunately, such a transformation does not allow any control over the bandwidth. This results in a pinched frequency response for which the standard solution is to add more steps as sketched in fig. 3.1.

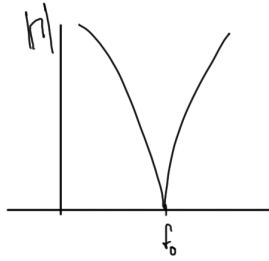


Figure 3.1: Pinched frequency response.

This can be implemented in electronics, or potentially geometrically as in this sketch of a microwave stripline transformer implementation fig. 3.3.

To find a multistep transformation algebraically can be hard, but it is easy to do on a Smith chart. The rule of thumb is that we want to stay near the center of the chart with each transformation.

There is however, a closed form method of calculating a specific sort of multisection transformation that is algebraically tractable. That method uses a chain of $\lambda/4$ transformers to increase the bandwidth as sketched in fig. 3.4.

The total reflection coefficient can be approximated to first order by summing the reflections at each stage (without considering there may be other internal reflections of transmitted field components). Algebraically that is

$$\Gamma(\Theta) \approx \Gamma_0 + \Gamma_1 e^{-2j\Theta} + \Gamma_2 e^{-4j\Theta} + \dots + \Gamma_N e^{-2Nj\Theta}, \quad (3.1)$$

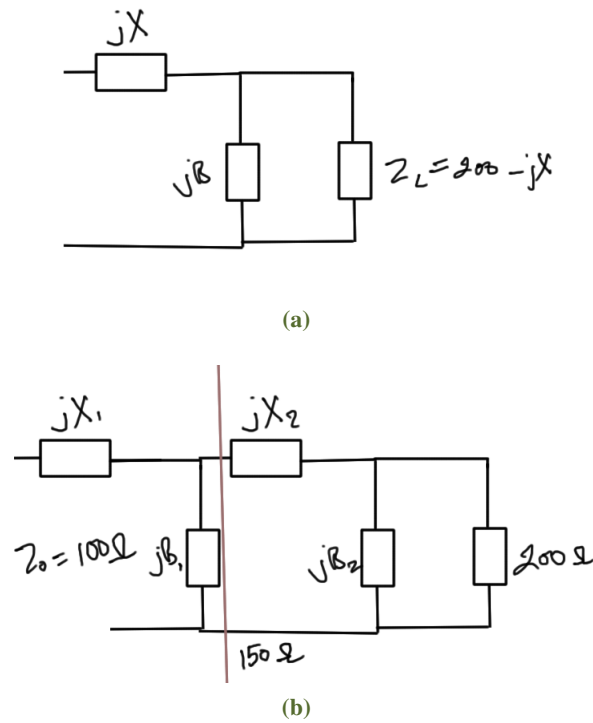


Figure 3.2: Single and multiple stage impedance matching.

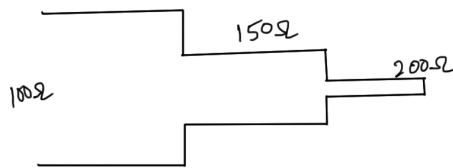


Figure 3.3: Stripline implementation of staged impedance matching.

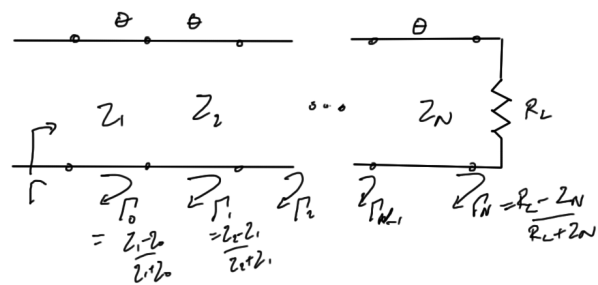


Figure 3.4: Multiple $\lambda/4$ transformers.

where

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \quad (3.2)$$

Why? Consider reflections at the Z_1, Z_2 interface as sketched in fig. 3.5.

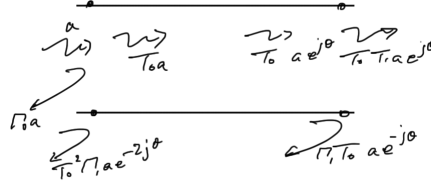


Figure 3.5: Single stage of multiple $\lambda/4$ transformers.

Assuming small reflections, where $|\Gamma| \ll 1$ then $T = 1 + \Gamma \approx 1$. Here

$$\begin{aligned} \Theta &= \beta l \\ &= \frac{2\pi}{\lambda} \frac{\lambda}{4} \\ &= \frac{\pi}{2}. \end{aligned} \quad (3.3)$$

at the design frequency ω_0 . We assume that Z_n are either monotonically increasing if $R_L > Z_0$, or decreasing if $R_L < Z_0$.

Binomial multisection transformers Let

$$\Gamma(\Theta) = A \left(1 + e^{-2j\Theta} \right)^N \quad (3.4)$$

This type of a response is maximally flat, and is plotted in fig. 3.6.

The absolute value of the reflection coefficient is

$$\begin{aligned} |\Gamma(\Theta)| &= |A| \left(e^{j\Theta} + e^{-j\Theta} \right)^N \\ &= 2^N |A| \cos^N \Theta. \end{aligned} \quad (3.5)$$

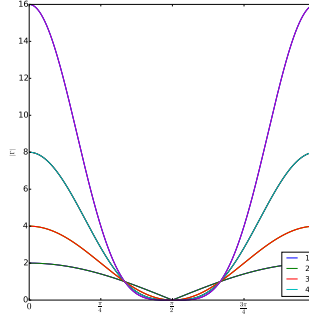


Figure 3.6: Binomial transformer.

When $\Theta = \pi/2$ this is clearly zero. It's derivatives are

$$\begin{aligned}\frac{d|\Gamma|}{d\Theta} &= -N \cos^{N-1} \Theta \sin \Theta \\ \frac{d^2|\Gamma|}{d\Theta^2} &= -N \cos^{N-1} \Theta \cos \Theta N(N-1) \cos^{N-2} \Theta \sin \Theta \\ \frac{d^3|\Gamma|}{d\Theta^3} &= \dots\end{aligned}\tag{3.6}$$

There is a \cos^{N-k} term for all derivatives $d^k/d\Theta^k$ where $k \leq N-1$, so for an N-section transformer

$$\frac{d^n}{d\Theta^n} |\Gamma(\Theta)|_{\omega_0} = 0,\tag{3.7}$$

for $n = 1, 2, \dots, N-1$. The constant A is determined by the limit $\Theta \rightarrow 0$, so

$$\Gamma(0) = 2^N A = \frac{Z_L - Z_0}{Z_L + Z_0},\tag{3.8}$$

because the various Θ sections become DC wires when the segment length goes to zero. This gives

$$\boxed{A = 2^{-N} \frac{Z_L - Z_0}{Z_L + Z_0}}.\tag{3.9}$$

The reflection coefficient can now be expanded using the binomial theorem

$$\begin{aligned}\Gamma(\Theta) &= A \left(1 + e^{2jk\Theta}\right)^N \\ &= \sum_{k=0}^N \binom{N}{k} e^{-2jk\Theta}\end{aligned}\tag{3.10}$$

Recall that

$$\binom{N}{k} = \frac{N!}{k! (N-k)!},\tag{3.11}$$

providing a symmetric set of values

$$\begin{aligned}\binom{N}{1} &= \binom{N}{N} = 1 \\ \binom{N}{1} &= \binom{N}{N-1} = N \\ \binom{N}{k} &= \binom{N}{N-k}.\end{aligned}\tag{3.12}$$

Equating eq. (3.10) with eq. (3.1) we have

$$\boxed{\Gamma_k = A \binom{N}{k}}.\tag{3.13}$$

Approximation for Z_k From [1] (4.6.4), a log series expansion valid for all z is

$$\ln z = \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{z-1}{z+1} \right)^{2k+1},\tag{3.14}$$

so for x near unity a first order approximation of a logarithm is

$$\ln x \approx 2 \frac{x-1}{x+1}.\tag{3.15}$$

Assuming that Z_{k+1}/Z_k is near unity we have

$$\begin{aligned}
\frac{1}{2} \ln \frac{Z_{k+1}}{Z_k} &\approx \frac{\frac{Z_{k+1}}{Z_k} - 1}{\frac{Z_{k+1}}{Z_k} + 1} \\
&= \frac{Z_{k+1} - Z_k}{Z_{k+1} + Z_k} \\
&= \Gamma_k.
\end{aligned} \tag{3.16}$$

Using this approximation, we get

$$\begin{aligned}
\ln \frac{Z_{k+1}}{Z_k} &\approx 2\Gamma_k \\
&= 2A \binom{N}{k} \\
&= 2 \left(2^{-N}\right) \binom{N}{k} \frac{Z_L - Z_0}{Z_L + Z_0} \\
&\approx 2^{-N} \binom{N}{k} \ln \frac{Z_L}{Z_0},
\end{aligned} \tag{3.17}$$

I asked what business do we have in assuming that Z_L/Z_0 is near unity? The answer was that it isn't but surprisingly it works out well enough despite that. As an example, consider $Z_0 = 100\Omega$ and $R_L = 50\Omega$. The exact expression

$$\begin{aligned}
\frac{Z_L - Z_0}{Z_L + Z_0} &= \frac{100 - 50}{100 + 50} \\
&= -0.333,
\end{aligned} \tag{3.18}$$

whereas

$$\frac{1}{2} \ln \frac{Z_L}{Z_0} = -0.3466, \tag{3.19}$$

which is pretty close after all.

Regardless of whether or not that last approximation is used, one can proceed iteratively to Z_{k+1} starting with $k = 0$.

Bandwidth To evaluate the bandwidth, let Γ_m be the maximum tolerable reflection coefficient over the passband, as sketched in fig. 3.7.

That is

$$\begin{aligned}
\Gamma_m &= 2^N |A| |\cos \Theta_m|^N \\
&= 2^N |A| \cos^N \Theta_m,
\end{aligned} \tag{3.20}$$

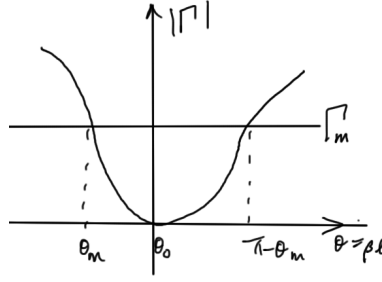


Figure 3.7: Max tolerable reflection.

for $\Theta_m < \pi/2$. Then

$$\Theta_m = \arccos\left(\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right) \quad (3.21)$$

The relative width of the interval is

$$\begin{aligned} \frac{\Delta f_{\max}}{f_0} &= \frac{\Delta \Theta_{\max}}{\Theta_0} \\ &= \frac{2(\Theta_0 - \Theta_{\max})}{\Theta_0} \\ &= 2 - \frac{2\Theta_{\max}}{\Theta_0} \\ &= 2 - \frac{4\Theta_{\max}}{\pi} \\ &= 2 - \frac{4}{\pi} \arccos\left(\frac{1}{2}\left(\frac{\Gamma_{\max}}{|A|}\right)^{1/N}\right). \end{aligned} \quad (3.22)$$

Example 3.1: Three section binomial transformer.

Design a 3-section binomial transformer to match $R_L = 50\Omega$ to a line $Z_0 = 100\Omega$. Calculate the BW based on a maximum $\Gamma_m = 0.05$.

Solution The scaling factor

$$A = 2^{-N} \frac{Z_L - Z_0}{Z_L + Z_0} \approx \frac{1}{2^{N+1}} \ln \frac{Z_L}{Z_0} = -0.0433 \quad (3.23)$$

Now use

$$\ln \frac{Z_{n+1}}{Z_n} \approx 2^{-N} \binom{N}{n} \ln \frac{R_L}{Z_0}, \quad (3.24)$$

starting from

- $n = 0$.

$$\ln \frac{Z_1}{Z_0} \approx 2^{-3} \binom{3}{0} \ln \frac{R_L}{Z_0}, \quad (3.25)$$

or

$$\begin{aligned} \ln Z_1 &= \ln Z_0 + 2^{-3} \binom{3}{0} \ln \frac{R_L}{Z_0} \\ &= \ln 100 + 2^{-3} (1) \ln 0.5 \\ &= 4.518, \end{aligned} \quad (3.26)$$

so

$$Z_1 = 91.7\Omega \quad (3.27)$$

- $n = 1$

$$\begin{aligned} \ln Z_2 &= \ln Z_1 + 2^{-3} \binom{3}{1} \ln \frac{50}{100} \\ &= 4.26 \end{aligned} \quad (3.28)$$

so

$$Z_2 = 70.7\Omega \quad (3.29)$$

- $n = 2$

$$\begin{aligned} \ln Z_3 &= \ln Z_2 + 2^{-3} \binom{3}{2} \ln \frac{50}{100} \\ &= 4.0, \end{aligned} \quad (3.30)$$

so

$$Z_3 = 54.5\Omega. \quad (3.31)$$

With the fractional BW for $\Gamma_m = 0.05$, where $10 \log_{10} |\Gamma_m|^2 = -26\text{dB}$

$$\begin{aligned} \frac{\Delta f}{f_0} &\approx 2 - \frac{4}{\pi} \arccos \left(\frac{1}{2} \left(\frac{\Gamma_m}{|A|} \right)^{1/N} \right) \\ &= 2 - \frac{4}{\pi} \arccos \left(\frac{1}{2} \left(\frac{0.05}{0.0433} \right)^{1/3} \right) \\ &= 0.7 \end{aligned} \quad (3.32)$$

At 2GHz, $BW = 0.7(70\%) = 1.4\text{GHz}$, or $[2.3, 2.7]\text{GHz}$, whereas a single $\lambda/4$ transformer $Z_T = \sqrt{(100)(50)} = 70.7\Omega$ yields a BW of just 0.36GHz (18%).

3.2 CONTINUUM TRANSFORMER

A non-discrete impedance matching transformation, as sketched in fig. 3.8, is also possible.

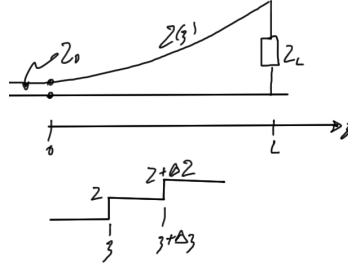


Figure 3.8: Tapered impedance matching.

$$\Delta\Gamma = \frac{(Z + \Delta Z) - Z}{(Z + \Delta Z) + Z} = \frac{\Delta Z}{2Z} \quad (3.33)$$

$$\Delta Z \rightarrow 0 \quad (3.34)$$

$$\begin{aligned}
d\Gamma &= \frac{dZ}{2Z} \\
&= \frac{1}{2} \frac{d(\ln Z)}{dz} \\
&= \frac{Z_0}{Z} \frac{d(Z/Z_0)}{dz} \\
&= \frac{1}{Z} \frac{dZ}{dz}.
\end{aligned} \tag{3.35}$$

Hence as we did for multisection transformers, associate $\Delta\Gamma$ with $e^{-2j\beta z}$ as sketched in fig. 3.9.

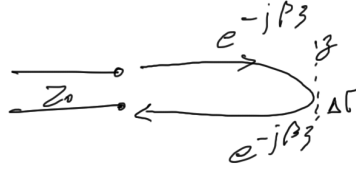


Figure 3.9: Reflection coefficient over an interval

assuming small reflections (i.e. $Z(z)$ is a slowly varying (adiabatic)). Then

$$\begin{aligned}
\Gamma(\omega) &= \int_0^L e^{-2j\beta z} d\Gamma \\
&= \frac{1}{2} \int_0^L e^{-2j\beta z} \frac{d(\ln Z)}{dz} dz
\end{aligned} \tag{3.36}$$

This supplies the means to calculate the reflection coefficient for any impedance curve. As with the step impedance matching process, it is assumed that only first order reflections are of interest.

3.3 EXPONENTIAL TAPER

Let

$$Z(z) = Z_0 e^{az}, \quad 0 < z < L \tag{3.37}$$

subject to

$$\begin{aligned}
Z(0) &= Z_0 \\
Z(L) &= Z_0 e^{aL} = Z_L,
\end{aligned} \tag{3.38}$$

which gives

$$\ln \frac{Z_L}{Z_0} = aL, \quad (3.39)$$

or

$$a = \frac{1}{L} \ln \frac{Z_L}{Z_0} \quad (3.40)$$

Also

$$\frac{d}{dz} \ln \frac{Z_L}{Z_0} = \frac{d}{dz}(az) = a, \quad (3.41)$$

Hence

$$\begin{aligned} \Gamma(\omega) &= \frac{1}{2} \int_0^L e^{-2j\beta z} \frac{d}{dz} \ln \frac{Z_L}{Z_0} dz \\ &= \frac{a}{2} \int_0^L e^{-2j\beta z} dz \\ &= \frac{1}{2L} \ln \frac{Z_L}{Z_0} \left. \frac{e^{-2j\beta z}}{-2j\beta} \right|_0^L \\ &= \frac{1}{2L\beta} \ln \frac{Z_L}{Z_0} \frac{1 - e^{-2j\beta L}}{2j} \\ &= \frac{1}{2} \ln \frac{Z_L}{Z_0} e^{-j\beta L} \frac{\sin(\beta L)}{\beta L}, \end{aligned} \quad (3.42)$$

or

$$\Gamma(\omega) = \frac{1}{2} \ln \frac{Z_L}{Z_0} e^{-j\beta L} \text{sinc}(\beta L). \quad (3.43)$$

1. β is constant with Z varying: this is good only for TEM lines.
2. $|\Gamma|$ decreases with increasing length.
3. An electrical length $\beta L > \pi$, is required to minimize low frequency mismatch ($L > \lambda/2$).

This is sketched in fig. 3.10.

Want:

$$\beta L = \pi, \quad (3.44)$$

or

$$\frac{\omega_c}{v_\phi} L = \pi \quad (3.45)$$

where ω_c is the cutoff frequency. This gives

$$\omega_c = \frac{\pi v_\phi}{L}. \quad (3.46)$$

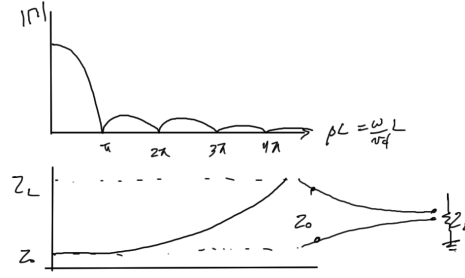


Figure 3.10: Exponential taper reflection coefficient.

Triangular Taper

$$Z(z) = \begin{cases} Z_0 e^{2(z/L)^2 \ln(Z_L/Z_0)} & 0 \leq z \leq L/2 \\ Z_0 e^{(4z/L - 2z^2/L^2) \ln(Z_L/Z_0)} & L/2 \leq z \leq L \end{cases} \quad (3.47)$$

$$\frac{d}{dz} \ln(Z/Z_0) = \begin{cases} (4z/L^2) \ln(Z_L/Z_0) & 0 \leq z \leq L/2 \\ (4/L - 4z/L^2) \ln(Z_L/Z_0) & L/2 \leq z \leq L \end{cases} \quad (3.48)$$

In this case

$$\Gamma(\omega) = \frac{1}{2} e^{-\beta L} \ln \frac{Z_L}{Z_0} e^{-j\beta L} \text{sinc}^2(\beta L/2). \quad (3.49)$$

Compared to the exponential taper $\text{sinc}(\beta L)$ for the $\beta L > 2\pi$ the peaks of $|\Gamma|$ are lower, but the first null occurs at $\beta L = 2\pi$ whereas for the exponential taper it occurs at $\beta L = \pi$. This is sketched in fig. 3.11. The price to pay for this is that the zero is at 2π so we have to make it twice as long to get the ripple down.

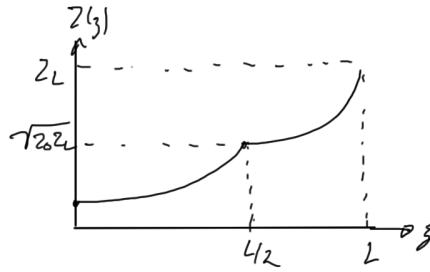


Figure 3.11: Triangular taper impedance curve.

Klopfenstein Taper For a given taper length L , the Klopfenstein taper is optimum in the sense that the reflection coefficient in the passband is minimum. Alternatively, for a given minimum reflection coefficient in the passband, the Klopfenstein taper yields the shortest length L .

Definition:

$$\ln Z = \frac{1}{2} \ln(Z_0 Z_L) + \frac{\Gamma_0}{\cosh A} A^2 \phi(2z/L - 1, A), \quad 0 \leq z \leq L, \quad (3.50)$$

where

$$\phi(x, A) = \int_0^x \frac{I_1(A \sqrt{1-y^2})}{A \sqrt{1-y^2}} dy, \quad |x| \leq 1. \quad (3.51)$$

Here $I_1(x)$ is the modified Bessel function. Note that

$$\begin{aligned} \phi(0, A) &= 0 \\ \phi(x, 0) &= x/2 \\ \phi(1, A) &= \frac{\cosh A - 1}{A^2} \end{aligned} \quad (3.52)$$

The resulting reflection coefficient is

$$\Gamma(\omega) = \begin{cases} \Gamma_0 e^{-j\beta L} \frac{\cos \sqrt{(\beta L)^2 - A^2}}{\cosh A} & \beta L > A \\ \Gamma_0 e^{-j\beta L} \frac{\cos \sqrt{A^2 - (\beta L)^2}}{\cosh A} & \beta L < A \end{cases}, \quad (3.53)$$

where as usual

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} \approx \frac{1}{2} \ln(Z_L/Z_0). \quad (3.54)$$

The passband is defined by $\beta L \geq A$ and the maximum ripple in the passband is

$$\Gamma_m = \frac{\Gamma_0}{\cosh A}. \quad (3.55)$$

Example 3.2: Triangular taper vs. exponential taper vs. Klopfenstein taper.

Design a triangular taper, an exponential taper, and a Klopfenstein taper (with $\Gamma_m = 0.02$) to match a 50Ω load to a 100Ω line.

- Triangular taper:

$$Z(z) = \begin{cases} Z_0 e^{2(z/L)^2 \ln Z_L/Z_0} & 0 \leq z \leq L/2 \\ Z_0 e^{(4z/L - 2z^2/L^2 - 1) \ln Z_L/Z_0} & L/2 \leq z \leq L \end{cases} \quad (3.56)$$

The resulting Γ is

$$|\Gamma| = \frac{1}{2} \ln(Z_L/Z_0) \operatorname{sinc}^2(\beta L/2). \quad (3.57)$$

- Exponential taper:

$$\begin{aligned} Z(z) &= Z_0 e^{az}, \quad 0 \leq z \leq L \\ a &= \frac{1}{L} \ln(Z_L/Z_0) = \frac{0.693}{L} \\ |\Gamma| &= \frac{1}{2} \ln(Z_L/Z_0) \operatorname{sinc}(\beta L) \end{aligned} \quad (3.58)$$

- Klopfenstein taper:

$$\begin{aligned} Z(z) &= \frac{1}{2} \ln(Z_L/Z_0) = 0.346 \\ A &= \cosh^{-1} \left(\frac{\Gamma_0}{\Gamma_m} \right) = \cosh^{-1} \left(\frac{0.346}{0.02} \right) = 3.543 \\ |\Gamma| &= \Gamma_0 \frac{\cos \sqrt{(\beta L)^2 - A^2}}{\cosh A}, \end{aligned} \quad (3.59)$$

The passband $\beta L > A = 3.543 = 1.13\pi$. The impedance $Z(z)$ must be evaluated numerically.

To illustrate some of the differences, we are referred to fig. 5.21 [5]. It is noted that

1. The exponential taper has the lowest cutoff frequency $\beta L = \pi$. Then is the Klopfenstein taper which is close $\beta L = 1.13\pi$. Last is the triangular with $\beta L = 2\pi$.
2. The Klopfenstein taper has the lowest $|\Gamma|$ in the passband and meets the spec of $\Gamma_m = 0.02$. The worst $|\Gamma|$ in the passband is from the exponential taper and the triangular ripple is between the two others.

Part II

APPENDICES

USEFUL FORMULAS AND REVIEW

Terminology review

$$Z_{\text{in}} = R + jX \quad (\text{A.1})$$

$$Y_{\text{in}} = G + jB \quad (\text{A.2})$$

- Z_{in} : impedance
- R : resistance
- X : reactance
- Y_{in} : admittance
- G : conductance
- B : susceptance

Apparently this notation goes all the way back to Heavyside!

Periodic motion

$$v_{\phi} = \frac{\omega}{\beta} = \frac{\lambda}{T} \quad (\text{A.3})$$

$$\beta = \frac{2\pi}{\lambda} = \frac{\omega}{v_{\phi}} \quad (\text{A.4})$$

Units

$$V = (\text{kgm}^2)/(\text{s}^3\text{A}) \quad (\text{A.5})$$

$$A = C/s \quad (\text{A.6})$$

$$\Omega = V/A \quad (\text{A.7})$$

$$F = As/V = s/\Omega \quad (\text{A.8})$$

$$H = Vs/A = \Omega s \quad (\text{A.9})$$

$$W = Vs \quad (\text{A.10})$$

$$[\epsilon] = F/m \quad (\text{A.11})$$

$$[\mathbf{D}] = C/m^2 \quad (\text{A.12})$$

$$[\mathbf{E}] = V/m \quad (\text{A.13})$$

$$[\mathbf{H}] = A/m \quad (\text{A.14})$$

$$[\mathbf{B}] = \text{Wb}/m \quad (\text{A.15})$$

$$[\mathbf{J}] = A/m^2 \quad (\text{A.16})$$

$$[\rho] = C/m^3 \quad (\text{A.17})$$

$$[\sigma] = S/m = [\omega\epsilon] = F/m/s = A/V/m = \mathcal{U}/m \quad (\text{A.18})$$

$$[\mu] = H/m \quad (\text{A.19})$$

Consistutive relations

$$\mathbf{B} = \mu \mathbf{H} \quad (\text{A.20})$$

$$\mathbf{D} = \epsilon \mathbf{E} \quad (\text{A.21})$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (\text{A.22})$$

constants

$$c = 3 \times 10^8 \text{ m/s} \quad (\text{A.23})$$

$$\epsilon_0 = 8.8 \times 10^{-12} \text{ F/m} \quad (\text{A.24})$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m} \quad (\text{A.25})$$

$$\eta_0 = \mu_0 c \approx 120\pi \Omega \quad (\text{A.26})$$

ODDS AND ENDS

B.1 AVERAGE POWER FOR CIRCUIT ELEMENTS

In [5] §2.2 is a comparison of field energy expressions with their circuit equivalents. It's clearly been too long since I've worked with circuits, because I'd forgotten all the circuit energy expressions:

$$\begin{aligned} W_R &= \frac{R}{2}|I|^2 \\ W_C &= \frac{C}{4}|V|^2 \\ W_L &= \frac{L}{4}|I|^2 \\ W_G &= \frac{G}{2}|V|^2 \end{aligned} \tag{B.1}$$

Here's a recap of where these come from

Energy lost to resistance Given

$$v(t) = Ri(t) \tag{B.2}$$

the average power lost to a resistor is

$$\begin{aligned} p_R &= \frac{1}{T} \int_0^T v(t)i(t)dt \\ &= \frac{1}{T} \int_0^T \operatorname{Re}(Ve^{j\omega t}) \operatorname{Re}(Ie^{j\omega t})dt \\ &= \frac{1}{4T} \int_0^T (Ve^{j\omega t} + V^*e^{-j\omega t})(Ie^{j\omega t} + I^*e^{-j\omega t})dt \\ &= \frac{1}{4T} \int_0^T (V I e^{2j\omega t} + V^* I^* e^{-2j\omega t} + V I^* + V^* I)dt \\ &= \frac{1}{2} \operatorname{Re}(V I^*) \\ &= \frac{1}{2} \operatorname{Re}(I R I^*) \\ &= \frac{R}{2}|I|^2. \end{aligned} \tag{B.3}$$

Here it is assumed that the averaging is done over some integer multiple of the period, which kills off all the exponentials.

Energy stored in a capacitor I tried the same sort of analysis for a capacitor in phasor form, but everything canceled out. Referring to [2], the approach used to figure this out is to operate first strictly in the time domain. Specifically, for the capacitor where $i = Cdv/dt$ the power supplied up to a time t is

$$\begin{aligned} p_C(t) &= \int_{-\infty}^t C \frac{dv}{dt} v(t) dt \\ &= \frac{1}{2} C v^2(t). \end{aligned} \tag{B.4}$$

The $v^2(t)$ term can now be expanded in terms of phasors and averaged for

$$\begin{aligned} \bar{p}_C &= \frac{C}{2T} \int_0^T \frac{1}{4} (V e^{j\omega t} + V^* e^{-j\omega t}) (V e^{j\omega t} + V^* e^{-j\omega t}) dt \\ &= \frac{C}{2T} \int_0^T \frac{1}{4} 2|V|^2 dt \\ &= \frac{C}{4} |V|^2. \end{aligned} \tag{B.5}$$

Energy stored in an inductor The inductor energy is found the same way, with

$$\begin{aligned} p_L(t) &= \int_{-\infty}^t L \frac{di}{dt} i(t) dt \\ &= \frac{1}{2} L i^2(t), \end{aligned} \tag{B.6}$$

which leads to

$$\bar{p}_L = \frac{L}{4} |I|^2. \tag{B.7}$$

Energy lost due to conductance Finally, we have conductance. In phasor space that is defined by

$$G = \frac{I}{V} = \frac{1}{R}, \tag{B.8}$$

so power lost due to conductance follows from power lost due to resistance. In the average we have

$$\begin{aligned} p_G &= \frac{1}{2G}|I|^2 \\ &= \frac{1}{2G}|VG|^2 \\ &= \frac{G}{2}|V|^2 \end{aligned} \tag{B.9}$$



MATHEMATICA NOTEBOOKS

These Mathematica notebooks, some just trivial ones used to generate figures, others more elaborate, and perhaps some even polished, can be found in

<https://github.com/peeterjoot/mathematica/tree/master/>.

The free **Wolfram CDF player**, is capable of read-only viewing these notebooks to some extent.

- Feb 5, 2016 [ece1236/transmissionLineUnits.nb](#)

UnitConvert verification that $1/\sqrt{FH}$ is an Ohm, and that $1/\sqrt{(F/m)(H/m)} = m/s$

MATLAB NOTEBOOKS

These Matlab notebooks, some just trivial ones used to generate figures, others more elaborate, and perhaps some even polished, can be found in

<https://github.com/peeterjoot/matlab>.

These notebooks are text files, but a matlab product is required to execute them.

- Feb 4, 2016 [ece1236/smith/run.m](#)
run mathworks plotSmithChart() and save output to a file.

JULIA NOTEBOOKS

These Julia notebooks, can be found in

<https://github.com/peeterjoot/julia>.

These notebooks are text files. The julia program, available freely at www.julialang.org, is required to execute them. Some Julia code can also be evaluated with Matlab.

- Feb 6, 2016 [ece1236/ps1/ps1_3.jl](#)
Numerical substitutions for ps1 p3.
- Feb 6, 2016 [ece1236/ps1/ps1_2.jl](#)
Numerical substitutions for ps1 p2.
- Feb 8, 2016 [ece1236/ps1/ps1_4.jl](#)
Numerical substitutions for ps1 p4.
- Feb 8, 2016 [ece1236/lecturenotes7/multisectionTransformer.jl](#)
This plots $\Gamma(\Theta) = |1 - \exp(2j\theta)|^N$, the multisection binomial quarter wavelength transformer.
- Feb 10, 2016 [ece1236/lecturenotes7/multisectionExample.jl](#)
A quick numerical check of the logarithm approximation.

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- [2] J.D. Irwin. *Basic Engineering Circuit Analysis*. MacMillian, 1993. (Cited on page 98.)
- [3] EETech Media. *Simple Series Resonance*, 2016. URL <http://www.allaboutcircuits.com/textbook/alternating-current/chpt-6/simple-series-resonance/>. [Online; accessed 04-Feb-2016]. (Cited on page 53.)
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