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Calculating the magnetostatic field from the moment

The vector potential, to first order, for a magnetostatic localized current distribution was found to be

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3}. \tag{1.1}$$

Initially, I tried to calculate the magnetic field from this, but ran into trouble. Here's a new try.

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \times \left(\mathbf{m} \times \frac{\mathbf{x}}{r^3} \right) \\
&= -\frac{\mu_0}{4\pi} \boldsymbol{\nabla} \cdot \left(\mathbf{m} \wedge \frac{\mathbf{x}}{r^3} \right) \\
&= -\frac{\mu_0}{4\pi} \left((\mathbf{m} \cdot \boldsymbol{\nabla}) \frac{\mathbf{x}}{r^3} - \mathbf{m} \boldsymbol{\nabla} \cdot \frac{\mathbf{x}}{r^3} \right) \\
&= \frac{\mu_0}{4\pi} \left(-\frac{(\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{x}}{r^3} - \left(\mathbf{m} \cdot \left(\boldsymbol{\nabla} \frac{1}{r^3} \right) \right) \mathbf{x} + \mathbf{m} (\boldsymbol{\nabla} \cdot \mathbf{x}) \frac{1}{r^3} + \mathbf{m} \left(\boldsymbol{\nabla} \frac{1}{r^3} \right) \cdot \mathbf{x} \right).
\end{aligned} \tag{1.2}$$

Here I've used $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$, and then expanded that with $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}$. Since one of these vectors is the gradient, care must be taken to have it operate on the appropriate terms in such an expansion.

Since we have $\boldsymbol{\nabla} \cdot \mathbf{x} = 3$, $(\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{x} = \mathbf{m}$, and $\boldsymbol{\nabla} 1/r^n = -n\mathbf{x}/r^{n+2}$, this reduces to

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi} \left(-\frac{\mathbf{m}}{r^3} + 3\frac{(\mathbf{m} \cdot \mathbf{x})\mathbf{x}}{r^5} + 3\mathbf{m}\frac{1}{r^3} - 3\mathbf{m}\frac{\mathbf{x}}{r^5} \cdot \mathbf{x} \right) \\
&= \frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - \mathbf{m}}{r^3},
\end{aligned} \tag{1.3}$$

which is the desired result.