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Energy-momentum tensor for a scalar field

Exercise 1.1 Energy-momentum tensor for a scalar field

It is claimed in [1] (3.2.1) that the momentum components of the energy-momentum tensor was found to be

$$\mathbf{e}_n \int d^3 x T^{0n} = \int d^3 k \mathbf{k} a_k^\dagger a_k. \tag{1.1}$$

Calculate this and the other energy-momentum components.

Answer for Exercise 1.1

First, from the Noether current for the scalar field Lagrangian in question, what is the energy-momentum tensor explicitly?

$$\begin{aligned} T^{\mu\nu} &= \Pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \\ &= \Pi^\mu \partial^\nu \phi - g^{\mu\nu} \frac{1}{2} \left(\partial_\alpha \phi \partial^\alpha \phi - \mu^2 \phi^2 \right) \\ &= \Pi^\mu \Pi^\nu - g^{\mu\nu} \frac{1}{2} \left(\Pi_\alpha \Pi^\alpha - \mu^2 \phi^2 \right) \\ &= \Pi^\mu \Pi^\nu - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} \Pi^\beta \Pi^\alpha + \frac{1}{2} g^{\mu\nu} \mu^2 \phi^2. \end{aligned} \tag{1.2}$$

Consider some special cases for the indexes. For $\mu = \nu = 0$, the result is the Hamiltonian density

$$\begin{aligned} T^{00} &= \Pi^0 \Pi^0 - \frac{1}{2} g^{00} \Pi_\alpha \Pi^\alpha + \frac{1}{2} g^{00} \mu^2 \phi^2 \\ &= \Pi^0 \Pi^0 - \frac{1}{2} \Pi_\alpha \Pi^\alpha + \frac{1}{2} \mu^2 \phi^2 \\ &= \frac{1}{2} \Pi^0 \Pi^0 - \frac{1}{2} \Pi_n \Pi^n + \frac{1}{2} \mu^2 \phi^2 \\ &= \frac{1}{2} \Pi^2 + \frac{1}{2} (\boldsymbol{\nabla} \phi)^2 + \frac{1}{2} \mu^2 \phi^2, \end{aligned} \tag{1.3}$$

where $\Pi^2 = (\partial_0 \phi)^2 \neq \partial^2 \phi$. For any $\mu \neq \nu$ the off diagonal metric elements are zero, leaving just

$$T^{\mu\nu} = \Pi^\mu \Pi^\nu. \tag{1.4}$$

Finally, when $n \neq 0$, the remaining diagonal terms are

$$
\begin{aligned}
T^{nn} &= \Pi^n \Pi^n - \frac{1}{2} g^{nn} \Pi_\alpha \Pi^\alpha + \frac{1}{2} g^{nn} n^2 \phi^2 \\
&= \Pi^n \Pi^n + \frac{1}{2} \Pi_\alpha \Pi^\alpha - \frac{1}{2} \mu^2 \phi^2 \\
&= \frac{1}{2} \Pi^2 + \Pi^n \Pi^n - \frac{1}{2} \Pi^m \Pi^m - \frac{1}{2} \mu^2 \phi^2 \\
&= \frac{1}{2} \Pi^2 + \frac{1}{2} \Pi^n \Pi^n - \frac{1}{2} \sum_{m \neq n, 0} \Pi^m \Pi^m - \frac{1}{2} \mu^2 \phi^2 \\
&= \frac{1}{2} \sum_{m = n, 0} \Pi^m \Pi^m - \frac{1}{2} \sum_{m \neq n, 0} \Pi^m \Pi^m - \frac{1}{2} \mu^2 \phi^2 .
\end{aligned}
\tag{1.5}
$$

The canonical momenta are

$$
\Pi^\mu = \partial^\mu \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right) , \tag{1.6}
$$

but

$$
\begin{aligned}
\partial^\mu e^{ik\cdot x} &= \partial^\mu \exp\left(i k^\alpha x_\alpha \right) \\
&= i k^\mu \exp\left(ik \cdot x \right) ,
\end{aligned}
\tag{1.7}
$$

so

$$
\begin{aligned}
\Pi^\mu &= i \int \frac{d^3 k k^\mu}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left(-a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right) \\
&= i \int \frac{d^3 k k^\mu}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left(-a_k e^{-i\omega_k t + \mathbf{k}\cdot\mathbf{x}} + a_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}} \right) .
\end{aligned}
\tag{1.8}
$$

This gives

$$
\begin{aligned}
\int d^3 x \Pi^\mu \Pi^\nu = &-\frac{1}{2} \int d^3 x \frac{1}{(2\pi)^3} \int d^3 k d^3 j \frac{k^\mu j^\nu}{\sqrt{\omega_k \omega_j}} \Big(\\
& -a_k e^{-i\omega_k t + \mathbf{k}\cdot\mathbf{x}} + a_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}} \Big) \left(-a_j e^{-i\omega_j t + \mathbf{j}\cdot\mathbf{x}} + a_j^\dagger e^{i\omega_j t - i\mathbf{j}\cdot\mathbf{x}} \right) \\
= &-\frac{1}{2} \int d^3 x \frac{1}{(2\pi)^3} \int d^3 k d^3 j \frac{k^\mu j^\nu}{\sqrt{\omega_k \omega_j}} \Big(a_k a_j e^{-i(\omega_j + \omega_k)t + (\mathbf{j}+\mathbf{k})\cdot\mathbf{x}} \\
& - a_k a_j^\dagger e^{i(\omega_j - \omega_k)t - i(\mathbf{j}-\mathbf{k})\cdot\mathbf{x}} - a_k^\dagger a_j e^{-i(\omega_j - \omega_k)t - (\mathbf{k}-\mathbf{j})\cdot\mathbf{x}} + a_k^\dagger a_j^\dagger e^{i(\omega_j + \omega_k)t - i(\mathbf{j}+\mathbf{k})\cdot\mathbf{x}} \Big) \\
= &-\frac{1}{2} \int d^3 k d^3 j \frac{k^\mu j^\nu}{\sqrt{\omega_k \omega_j}} \Big(a_k a_j e^{-i(\omega_j + \omega_k)t} \delta^3(\mathbf{j}+\mathbf{k}) \\
& - a_k a_j^\dagger e^{i(\omega_j - \omega_k)t} \delta^3(\mathbf{j}-\mathbf{k}) - a_k^\dagger a_j e^{-i(\omega_j - \omega_k)t} \delta^3(\mathbf{k}-\mathbf{j}) + a_k^\dagger a_j^\dagger e^{i(\omega_j + \omega_k)t} \delta^3(\mathbf{j}+\mathbf{k}) \Big) .
\end{aligned}
\tag{1.9}
$$

There are two cases here to consider. The first is $\nu = 0$, for which

$$
\int d^3 x \Pi^\mu \Pi^0 = -\frac{1}{2} \int d^3 k k^\mu \left(a_k a_{-k} e^{-2i\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} \right) . \tag{1.10}
$$

For $\nu \neq 0$

$$\begin{aligned}\int d^3x \Pi^\mu \Pi^\nu &= -\frac{1}{2}\int d^3k \frac{k^\mu k^\nu}{\omega_k}\left(-a_k a_{-k} e^{-2i\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k - a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right) \\ &= \frac{1}{2}\int d^3k \frac{k^\mu k^\nu}{\omega_k}\left(a_k a_{-k} e^{-2i\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right).\end{aligned} \tag{1.11}$$

Here's a summary of these products

$$\int d^3x \Pi^0 \Pi^0 = -\frac{1}{2}\int d^3k \omega_k \left(a_k a_{-k} e^{-2i\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right), \tag{1.12a}$$

$$\begin{aligned}\int d^3x \Pi^n \Pi^0 &= \int d^3x \Pi^0 \Pi^n \\ &= -\frac{1}{2}\int d^3k k^n \left(a_k a_{-k} e^{-2i\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right),\end{aligned} \tag{1.12b}$$

$$\int d^3x \Pi^m \Pi^n = \frac{1}{2}\int d^3k \frac{k^m k^n}{\omega_k}\left(a_k a_{-k} e^{-2i\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right). \tag{1.12c}$$

For the mass term it was previously found that

$$\frac{1}{2}\int d^3x \mu^2 \phi^2 = \frac{\mu^2}{4}\int d^3k \frac{1}{\omega_k}\left(a_{-k} a_k e^{-2i\omega_k t} + a_{-k}^\dagger a_k^\dagger e^{2i\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k\right). \tag{1.13}$$

The Hamiltonian component has been previously calculated, and resolves to

$$\int d^3x T^{00} = \frac{1}{2}\int d^3k \omega_k \left(a_k a_k^\dagger + a_k^\dagger a_k\right). \tag{1.14}$$

The other diagonal components, for $r \neq s \neq t$ are

$$\begin{aligned}\int d^3x T^{rr} &= \int d^3x \left(\frac{1}{2}\sum_{m=r,0} \Pi^m \Pi^m - \frac{1}{2}\sum_{m=s,t} \Pi^m \Pi^m - \frac{1}{2}\mu^2\phi^2\right) \\ &= \frac{1}{4}\int d^3k \frac{(k^r)^2 - (k^s)^2 - (k^t)^2 - \mu^2}{\omega_k}\left(a_k a_{-k} e^{-2i\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k + \right. \\ &\left. a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right) - \frac{1}{4}\int d^3k \omega_k \left(a_k a_{-k} e^{-2i\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right) \\ &= \frac{1}{4}\int d^3k \frac{(k^r)^2 - (k^s)^2 - (k^t)^2 - \mu^2 - \omega_k^2}{\omega_k}\left(a_k a_{-k} e^{-2i\omega_k t}\right. \\ &\left. + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right) + \frac{1}{4}\int d^3k \frac{(k^r)^2 - (k^s)^2 - (k^t)^2 - \mu^2 + \omega_k^2}{\omega_k}\left(a_k a_k^\dagger + a_k^\dagger a_k\right) \\ &= \frac{1}{2}\int d^3k \frac{(k^r)^2 - \omega_k^2}{\omega_k}\left(a_k a_{-k} e^{-2i\omega_k t} + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t}\right) + \frac{1}{2}\int d^3k \frac{(k^r)^2}{\omega_k}\left(a_k a_k^\dagger + a_k^\dagger a_k\right).\end{aligned} \tag{1.15}$$

This doesn't have the nice cancelation that killed the time dependent terms in the Hamiltonian. Such cancellation also doesn't appear in the off diagonal energy-momentum tensor components, which are

$$\begin{aligned} \int d^3 x T^{n0} &= \int d^3 x T^{n0} \\ &= -\frac{1}{2} \int d^3 k k^n \left(a_k a_{-k} e^{-2i\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} \right), \end{aligned} \tag{1.16}$$

and for $m \neq n \neq 0$

$$\int d^3 x T^{mn} = \frac{1}{2} \int d^3 k \frac{k^m k^n}{\omega_k} \left(a_k a_{-k} e^{-2i\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} \right). \tag{1.17}$$

The eq. (1.16) result has time dependence that the stated result does not (but is linear in **k** as desired)? Did I miss something?

Bibliography

[1] Michael Luke. *PHY2403F Lecture Notes: Quantum Field Theory*, 2015. URL https://piazza.com/utoronto.ca/fall2015/phy2403f/resources. [Online; accessed 02-Jan-2016]. 1.1