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Frequency domain time averaged Poynting theorem

The time domain Poynting relationship was found to be

$$0 = \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) + \frac{\epsilon}{2} \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{\mu}{2} \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} + \mathbf{H} \cdot \mathbf{M}_i + \mathbf{E} \cdot \mathbf{J}_i + \sigma \mathbf{E} \cdot \mathbf{E}. \tag{1.1}$$

Let's derive the equivalent relationship for the time averaged portion of the time-harmonic Poynting vector. The time domain representation of the Poynting vector in terms of the time-harmonic (phasor) vectors is

$$\begin{aligned} \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}} &= \frac{1}{4} \left(\mathbf{E} e^{j\omega t} + \mathbf{E}^* e^{-j\omega t} \right) \times \left(\mathbf{H} e^{j\omega t} + \mathbf{H}^* e^{-j\omega t} \right) \\ &= \frac{1}{2} \operatorname{Re} \left(\mathbf{E} \times \mathbf{H}^* + \mathbf{E} \times \mathbf{H} e^{2j\omega t} \right), \end{aligned} \tag{1.2}$$

so if we are looking for the relationships that effect only the time averaged Poynting vector, over integral multiples of the period, we are interested in evaluating the divergence of

$$\frac{1}{2} \mathbf{E} \times \mathbf{H}^*. \tag{1.3}$$

The time-harmonic Maxwell's equations are

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} &= -j\omega\mu \mathbf{H} - \mathbf{M}_i \\ \boldsymbol{\nabla} \times \mathbf{H} &= j\omega\epsilon \mathbf{E} + \mathbf{J}_i + \sigma \mathbf{E} \end{aligned} \tag{1.4}$$

The latter after conjugation is

$$\boldsymbol{\nabla} \times \mathbf{H}^* = -j\omega\epsilon^* \mathbf{E}^* + \mathbf{J}_i^* + \sigma^* \mathbf{E}^*. \tag{1.5}$$

For the divergence we have

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}^*) &= \mathbf{H}^* \cdot (\boldsymbol{\nabla} \cdot \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \cdot \mathbf{H}^*) \\ &= \mathbf{H}^* \cdot \left(-j\omega\mu \mathbf{H} - \mathbf{M}_i \right) - \mathbf{E} \cdot \left(-j\omega\epsilon^* \mathbf{E}^* + \mathbf{J}_i^* + \sigma^* \mathbf{E}^* \right), \end{aligned} \tag{1.6}$$

or

$$0 = \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}^*) + \mathbf{H}^* \cdot \left(j\omega\mu \mathbf{H} + \mathbf{M}_i \right) + \mathbf{E} \cdot \left(-j\omega\epsilon^* \mathbf{E}^* + \mathbf{J}_i^* + \sigma^* \mathbf{E}^* \right), \tag{1.7}$$

so

$$\boxed{0 = \boldsymbol{\nabla} \cdot \frac{1}{2} (\mathbf{E} \times \mathbf{H}^*) + \frac{1}{2} (\mathbf{H}^* \cdot \mathbf{M}_i + \mathbf{E} \cdot \mathbf{J}_i^*) + \frac{1}{2} j\omega \left(\mu \left|\mathbf{H}\right|^2 - \epsilon^* \left|\mathbf{E}\right|^2 \right) + \frac{1}{2} \sigma^* \left|\mathbf{E}\right|^2.} \tag{1.8}$$