Peeter Joot
peeterjoot@pm.me

Momentum of scalar field

1.1 Expansion of the field momentum.

Way back in lecture 8, it was claimed that

$$P^k = \int d^3x \hat{\pi} \partial^k \hat{\phi} = \int \frac{d^3p}{(2\pi)^3} p^k a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{1.1}$$

If I compute this, I get a normal ordered variation of this operator, but also get some time dependent terms. Here's the computation (dropping hats)

$$\begin{aligned} P^k &= \int d^3x \hat{\pi} \partial^k \phi \\ &= \int d^3x \partial_0 \phi \partial^k \phi \\ &= \int d^3x \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_p 2\omega_q}} \partial_0 \left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right) \partial^k \left(a_{\mathbf{q}} e^{-iq\cdot x} + a_{\mathbf{q}}^\dagger e^{iq\cdot x} \right). \end{aligned} \tag{1.2}$$

The exponential derivatives are

$$\begin{aligned} \partial_0 e^{\pm ip\cdot x} &= \partial_0 e^{\pm ip_\mu x^\mu} \\ &= \pm i p_0 \partial_0 e^{\pm ip\cdot x}, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \partial^k e^{\pm ip\cdot x} &= \partial^k e^{\pm ip^\mu x_\mu} \\ &= \pm i p^k e^{\pm ip\cdot x}, \end{aligned} \tag{1.4}$$

so

$$
\begin{aligned}
P^k &= -\int d^3x \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_p 2\omega_q}} p_0 q^k \left(-a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x}\right)\left(-a_{\mathbf{q}} e^{-iq\cdot x} + a^\dagger_{\mathbf{q}} e^{iq\cdot x}\right) \\
&= -\frac{1}{2}\int d^3x \frac{d^3p d^3q}{(2\pi)^6} \sqrt{\frac{\omega_p}{\omega_q}} q^k \left(a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)\cdot x} + a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} e^{i(p+q)\cdot x} - a_{\mathbf{p}} a^\dagger_{\mathbf{q}} e^{i(q-p)\cdot x} - a^\dagger_{\mathbf{p}} a_{\mathbf{q}} e^{i(p-q)\cdot x}\right) \\
&= \frac{1}{2}\int \frac{d^3p d^3q}{(2\pi)^3} \sqrt{\frac{\omega_p}{\omega_q}} q^k \Big(-a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(\omega_{\mathbf{p}}+\omega_{\mathbf{q}})t} \delta^3(\mathbf{p}+\mathbf{q}) - a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} e^{i(\omega_{\mathbf{p}}+\omega_{\mathbf{q}})t} \delta^3(-\mathbf{p}-\mathbf{q}) \\
&\qquad + a_{\mathbf{p}} a^\dagger_{\mathbf{q}} e^{i(\omega_{\mathbf{q}}-\omega_{\mathbf{p}})t} \delta^3(\mathbf{p}-\mathbf{q}) + a^\dagger_{\mathbf{p}} a_{\mathbf{q}} e^{i(\omega_{\mathbf{p}}-\omega_{\mathbf{q}})t} \delta^3(\mathbf{q}-\mathbf{p})\Big) \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} p^k \left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\mathbf{p}} - a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}}t} - a^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} e^{2i\omega_{\mathbf{p}}t}\right).
\end{aligned}
\tag{1.5}
$$

What is the rationale for ignoring those time dependent terms? Does normal ordering also implicitly drop any non-paired creation/annihilation operators? If so, why?

1.2 Conservation of the field momentum.

This follows up on unanswered questions related to the apparent time dependent terms in the previous expansion of P^i for a scalar field.

It turns out that examining the reasons that we can say that the field momentum is conserved also sheds some light on the question. P^i is not an a-priori conserved quantity, but we may use the charge conservation argument to justify this despite it not having a four-vector nature (i.e. with zero four divergence.)

The momentum P^i that we have defined is related to the conserved quantity $T^{0\mu}$, the energy-momentum tensor, which satisfies $0 = \partial_\mu T^{0\mu}$ by Noether's theorem (this was the conserved quantity associated with a spacetime translation.)

That tensor was

$$
T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}, \tag{1.6}
$$

and can be used to define the momenta

$$
\begin{aligned}
\int d^3x T^{0k} &= \int d^3x \partial^0 \phi \partial^k \phi \\
&= \int d^3x \pi \partial^k \phi.
\end{aligned}
\tag{1.7}
$$

Charge $Q^i = \int d^3x j^0$ was conserved with respect to a limiting surface argument, and we can make a similar "beer can integral" argument for P^i, integrating over a large time interval $t \in [-T, T]$ as sketched in fig. 1.1. That is

$$
\begin{aligned}
0 &= \partial_\mu \int d^4x T^{0\mu} \\
&= \partial_0 \int d^4x T^{00} + \partial_k \int d^4x T^{0k} \\
&= \partial_0 \int_{-T}^{T} dt \int d^3x T^{00} + \partial_k \int_{-T}^{T} dt \int d^3x T^{0k} \\
&= \partial_0 \int_{-T}^{T} dt \int d^3x T^{00} + \partial_k \int_{-T}^{T} dt \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger - a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}} t} - a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2i\omega_{\mathbf{p}} t} \right) \\
&= \int d^3x \, T^{00} \Big|_{-T}^{T} + T\partial_k \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \right) - \frac{1}{2}\partial_k \int_{-T}^{T} dt \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}} t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2i\omega_{\mathbf{p}} t} \right).
\end{aligned}
\tag{1.8}
$$

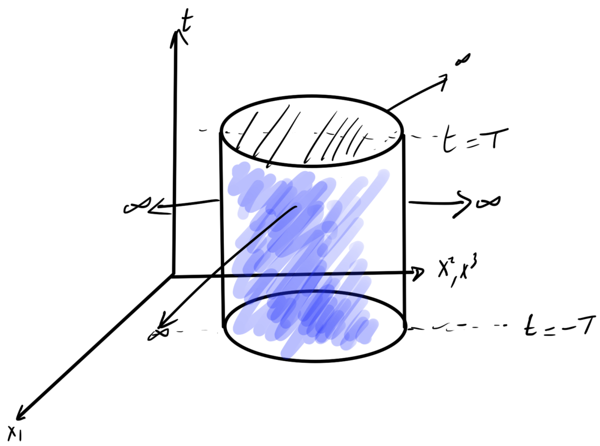

Figure 1.1: Cylindrical spacetime boundary.

The first integral can be said to vanish if the field energy goes to zero at the time boundaries, and the last integral reduces to

$$
\begin{aligned}
&-\frac{1}{2}\partial_k \int_{-T}^{T} dt \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}} t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2i\omega_{\mathbf{p}} t} \right) \\
&= -\int \frac{d^3p}{2(2\pi)^3} p^k \left(a_{\mathbf{p}} a_{-\mathbf{p}} \frac{\sin(-2\omega_{\mathbf{p}} T)}{-2\omega_{\mathbf{p}}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \frac{\sin(2\omega_{\mathbf{p}} T)}{2\omega_{\mathbf{p}}} \right) \\
&= -\int \frac{d^3p}{2(2\pi)^3} p^k \left(a_{\mathbf{p}} a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \right) \frac{\sin(2\omega_{\mathbf{p}} T)}{2\omega_{\mathbf{p}}}.
\end{aligned}
\tag{1.9}
$$

The sin term can be interpretted as a sinc like function of $\omega_{\mathbf{p}}$ which vanishes for large $\mathbf{p}$. It's not entirely sinc like for a massive field as $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, which never hits zero, as shown in fig. 1.2. Vanishing for large $\mathbf{p}$ doesn't help the whole integral vanish, but we can resort to the Riemann-Lebesque

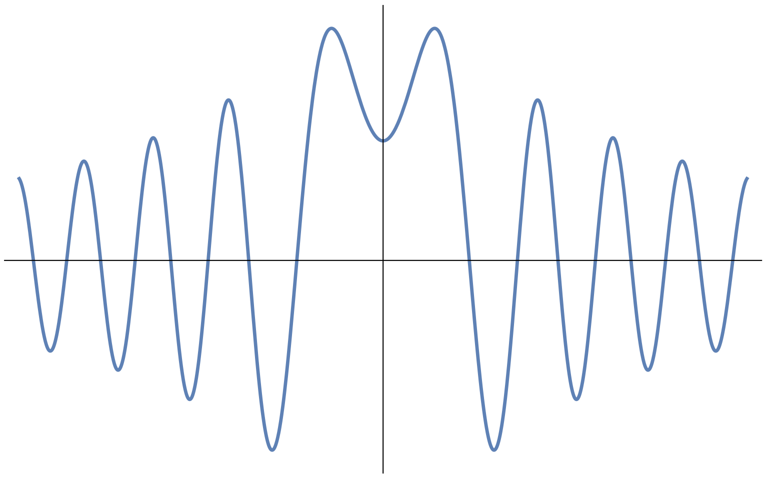

Figure 1.2: $\sin(2\omega_{\mathbf{p}} T)/\omega_{\mathbf{p}}$

lemma [1] instead and interpret this integral as one with a plain old high frequency oscillation that is presumed to vanish (i.e. the rest is well behaved enough that it can be labelled as L_1 integrable.)

We see that only the non-time dependent portion of $\mathbf{P}$ matters from a conserved quantity point of view, and having killed off all the time dependent terms, we are left with a conservation relationship for the momenta $\boldsymbol{\nabla} \cdot \mathbf{P} = 0$, where $\mathbf{P}$ in normal order is just

$$:\mathbf{P}: = \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{1.10}$$

Bibliography

[1] Wikipedia contributors. Riemann-lebesgue lemma — Wikipedia, the free encyclopedia, 2018. URL https://en.wikipedia.org/w/index.php?title=Riemann%E2%80%93Lebesgue_lemma&oldid=856778941. [Online; accessed 29-October-2018]. 1.2