

PEETER JOOT PEETERJOOT@PM.ME

QUANTUM FIELD THEORY I

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PEETER JOOT PEETERJOOT@PM.ME

Notes and problems from UofT PHY2403 2018

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Dedicated to:
Aurora and Lance, my awesome kids, and
Sofia, who not only tolerates and encourages my studies, but is also awesome enough to think
that math is sexy.

PREFACE

This is a set of course notes from the Fall 2018, University of Toronto Quantum Field Theory (PHY2403), taught by Prof. Erich Poppitz.

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This document contains:

- Lecture notes.
- Personal notes exploring auxiliary details.
- Worked practice problems.

My thanks go to Professor Poppitz for teaching this course, and to Emily Tyhurst and Stefan Divic who kindly provided me their notes for lecture 22.

Peeter Joot peeterjoot@pm.me

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Part I

LECTURE NOTES

FIELDS, UNITS, AND SCALES.

1.1 WHAT IS A FIELD?

A field is a map from space(time) to some set of numbers. These set of numbers may be organized some how, possibly scalars, or vectors, ...

One example is the familiar spacetime vector, where $\mathbf{x} \in \mathbb{R}^d$

$$(\mathbf{x}, t) \rightarrow \mathbb{R}^{(d,1)} \quad (1.1)$$

Examples of fields:

1. 0 + 1 dimensional “QFT”, where the spatial dimension is zero dimensional and we have one time dimension. Fields in this case are just functions of time $x(t)$. That is, particle mechanics is a 0 + 1 dimensional classical field theory. We know that classical mechanics is described by the action

$$S = \frac{m}{2} \int dt \dot{x}^2. \quad (1.2)$$

This is non-relativistic. We can make this relativistic by saying this is the first order term in the Taylor expansion

$$S = -mc^2 \int dt \sqrt{1 - \dot{x}^2/c^2}. \quad (1.3)$$

Classical field theory (of $x(t)$). The “QFT” of $x(t)$. i.e. QM. All of you know quantum mechanics. If you don’t just leave. Not this way (pointing to the window), but this way (pointing to the door). The solution of a quantum mechanical state is

$$\langle x | e^{-iHt/\hbar} | x' \rangle, \quad (1.4)$$

which can be found by evaluating the “Feynman path integral”

$$\sum_{\text{all paths } x} e^{iS[x]/\hbar} \quad (1.5)$$

This will be particularly useful for QFT, despite the fact that such a sum is really hard to evaluate (try it for the Hydrogen atom for example).

2. 3 + 0 dimensional field theory, where we have 3 spatial dimensions and 0 time dimensions. Classical equilibrium static systems. The field may have a structure like

$$\mathbf{x} \rightarrow \mathbf{M}(\mathbf{x}), \quad (1.6)$$

for example, magnetization. We can write the solution to such a system using the partition function

$$Z \sim \sum_{\text{all } \mathbf{M}(\mathbf{x})} e^{-E[\mathbf{M}]/k_B T}. \quad (1.7)$$

For such a system the energy function may be like

$$E[\mathbf{M}] = \int d^3 \mathbf{x} \left(a \mathbf{M}^2(\mathbf{x}) + b \mathbf{M}^4(\mathbf{x}) + c \sum_{i=1}^3 \left(\frac{\partial}{\partial x_i} \mathbf{M} \right) \cdot \left(\frac{\partial}{\partial x_i} \mathbf{M} \right) \right). \quad (1.8)$$

There is an analogy between the partition function and the Feynman path integral, as both are summing over all possible energy states in both cases. This will be probably be the last time that we mention the partition function and condensed matter physics in this term for this class.

3. 3 + 1 dimensional field theories, with 3 spatial dimensions and 1 time dimension. Example, electromagnetism with $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$ or better use $\mathbf{A}(\mathbf{x}, t)$, $\phi(\mathbf{x}, t)$. The action is

$$S = -\frac{1}{16\pi c} \int d^3 \mathbf{x} dt (\mathbf{E}^2 - \mathbf{B}^2). \quad (1.9)$$

This is our first example of a relativistic field theory in 3 + 1 dimensions. It will take us a while to get there.

These are examples of classical field theories, such as fluid dynamics and general relativity. We want to consider electromagnetism because this is the place that we everything starts to fall apart (i.e. blackbody radiation, relating to the equilibrium states of radiating matter). Part of the resolution of this was the quantization of the energy states, where we studied the normal modes of electromagnetic radiation in a box. These modes can be considered an infinite number of radiating oscillators (the ultraviolet catastrophe). This was resolved by Planck by requiring those energy states to be quantized (an excellent discussion of this can be found in [1]). In that sense you have already seen quantum field theory.

For electromagnetism the classical description is not always good. Examples:

1. blackbody radiation.

2. electron energy e^2/r_e of a point charge diverges as $r_e \rightarrow 0$. We can define the classical radius of the electron by

$$\frac{e^2}{r_e^{\text{cl}}} \sim m_e c^2, \quad (1.10)$$

or

$$r_e^{\text{cl}} \sim \frac{m_e c^2}{e^2} \sim 10^{-15} \text{m} \quad (1.11)$$

Don't treat this very seriously, but it becomes useful at frequencies $\omega \sim c/r_e$, where r_e/c is approximately the time for light to cross a distance r_e . At frequencies like this, we should not believe the solutions that are obtained by classical electrodynamics. In particular, self-accelerating solutions appear at these frequencies in classical EM. This is approximately $\omega_* \sim 10^{23} \text{Hz}$, or

$$\begin{aligned} \hbar\omega_* &\sim (10^{-21} \text{MeVs})(10^{23} \text{1/s}) \\ &\sim 100 \text{MeV}. \end{aligned} \quad (1.12)$$

At such frequencies particle creation becomes possible.

1.2 SCALES.

A (dimensionless) value that is very useful in determining scale is

$$\alpha = \frac{e^2}{4\pi\hbar c} \sim \frac{1}{137}, \quad (1.13)$$

called the fine scale constant, which relates three important scales relevant to quantum mechanics, as sketched in fig. 1.1.

- The Bohr radius (large end of the scale).
- The Compton wavelength of the electron.
- The classical radius of the electron.

1.2.1 Bohr radius.

A quick motivation for the Bohr radius was mentioned in passing in class while discussing scale, following the high school method of deriving the Balmer series ([5]).

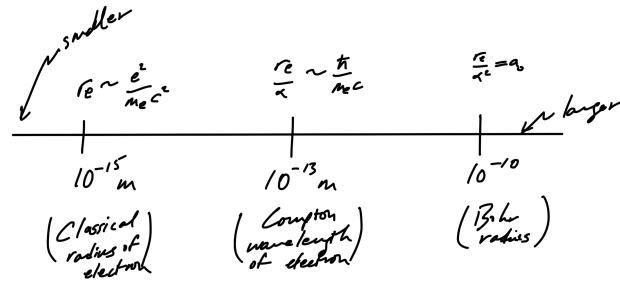


Figure 1.1: Interesting scales in quantum mechanics.

That method assumes a circular electron trajectory ($i = \mathbf{e}_1 \mathbf{e}_2$)

$$\begin{aligned}\mathbf{r} &= r \mathbf{e}_1 e^{i\omega t} \\ \mathbf{v} &= \omega r \mathbf{e}_2 e^{i\omega t} \\ \mathbf{a} &= -\omega^2 r \mathbf{e}_1 e^{i\omega t}\end{aligned}\tag{1.14}$$

The Coulomb force (in cgs units) on the electron is

$$\begin{aligned}\mathbf{F} &= m\mathbf{a} \\ &= -m\omega^2 r \mathbf{e}_1 e^{i\omega t} \\ &= \frac{-e(e)}{r^2} \mathbf{e}_1 e^{i\omega t},\end{aligned}\tag{1.15}$$

or

$$m \left(\frac{v}{r} \right)^2 r = \frac{e^2}{r^2},\tag{1.16}$$

giving

$$mv^2 = \frac{e^2}{r}.\tag{1.17}$$

The energy of the system, including both Kinetic and potential (from an infinite reference point) is

$$\begin{aligned}
 E &= \frac{1}{2}mv^2 - \frac{e^2}{r} \\
 &= -\frac{1}{2}mv^2 \\
 &\sim \hbar\omega \\
 &= \hbar\frac{v}{r},
 \end{aligned} \tag{1.18}$$

or

$$mvr \sim \hbar. \tag{1.19}$$

Eliminating v using eq. (1.17), assuming a ground state radius $r = a_0$ gives

$$a_0 \sim \frac{\hbar^2}{me^2}. \tag{1.20}$$

The Bohr radius is of the order 10^{-10}m .

1.2.2 Compton wavelength.

When particle momentum starts approaching the speed of light, by the uncertainty relation ($\Delta x \Delta p \sim \hbar$) the variation in position must be of the order

$$\lambda_c \sim \frac{\hbar}{m_e c}, \tag{1.21}$$

called the Compton wavelength. Similarly, when the length scales are reduced to the Compton wavelength, the momentum increases to relativistic levels. Because of the relativistic velocities at the Compton wavelength, particle creation and annihilation occurs and any theory has to account for multiple particle states.

1.2.3 Relations.

Scaling the Bohr radius once by the fine structure constant, we obtain the Compton wavelength (after dropping factors of 4π)

$$\begin{aligned}
 a_0 \alpha &= \frac{\hbar^2}{me^2} \frac{e^2}{4\pi \hbar c} \\
 &= \frac{\hbar}{4\pi mc} \\
 &\sim \frac{\hbar}{mc} \\
 &= \lambda_c.
 \end{aligned} \tag{1.22}$$

Scaling once more, we obtain (after dropping another 4π) the classical electron radius

$$\begin{aligned}\lambda_c \alpha &= \frac{e^2}{4\pi m c^2} \\ &\sim \frac{e^2}{m c^2}.\end{aligned}\tag{1.23}$$

1.3 NATURAL UNITS.

$$\begin{aligned}[\hbar] &= [\text{action}] = M \frac{L^2}{T^2} T = \frac{ML^2}{T} \\ [c] &= [\text{velocity}] = \frac{L}{T} \\ [\text{energy}] &= M \frac{L^2}{T^2}.\end{aligned}\tag{1.24}$$

Setting $c = 1$ means

$$\frac{L}{T} = 1\tag{1.25}$$

and setting $\hbar = 1$ means

$$\begin{aligned}[\hbar] &= [\text{action}] \\ &= ML \cancel{\frac{L}{T}} \\ &= ML\end{aligned}\tag{1.26}$$

therefore

$$[L] = \frac{1}{\text{mass}}\tag{1.27}$$

and

$$\begin{aligned}[\text{energy}] &= M \cancel{\frac{L^2}{T^2}} \\ &= \text{mass eV}\end{aligned}\tag{1.28}$$

Summary

- energy $\sim \text{eV}$

- distance $\sim \frac{1}{M}$
- time $\sim \frac{1}{M}$

From:

$$\alpha = \frac{e^2}{4\pi\hbar c} \quad (1.29)$$

which is dimensionless (1/137), so electric charge is dimensionless.

Some useful numbers in natural units

$$\begin{aligned} m_e &\sim 10^{-27} \text{g} \sim 0.5 \text{MeV} \\ m_p &\sim 2000 m_e \sim 1 \text{GeV} \\ m_\pi &\sim 140 \text{MeV} \\ m_\mu &\sim 105 \text{MeV} \\ \hbar c &\sim 200 \text{MeV fm} = 1 \end{aligned} \quad (1.30)$$

1.4 GRAVITY.

Interaction energy of two particles

$$G_N \frac{m_1 m_2}{r} \quad (1.31)$$

$$[\text{energy}] \sim [G_N] \frac{M^2}{L} \quad (1.32)$$

$$[G_N] \sim [\text{energy}] \frac{L}{M^2} \quad (1.33)$$

but energy x distance is dimensionless (action) in our units

$$[G_N] \sim \text{dimensionless} M^2 \quad (1.34)$$

$$\begin{aligned} \frac{G_N}{\hbar c} &\sim \frac{1}{M^2} \\ &\sim \frac{1}{10^{20} \text{GeV}} \end{aligned} \quad (1.35)$$

Planck mass

$$\begin{aligned}
M_{\text{Planck}} &\sim \sqrt{\frac{\hbar c}{G_N}} \\
&\sim 10^{-4} g \\
&\sim \frac{1}{(10^{20} \text{GeV})^2}
\end{aligned}
\tag{1.36}$$

We can revisit the scale diagram from last lecture in terms of MeV mass/energy values, as sketched in fig. 1.2.

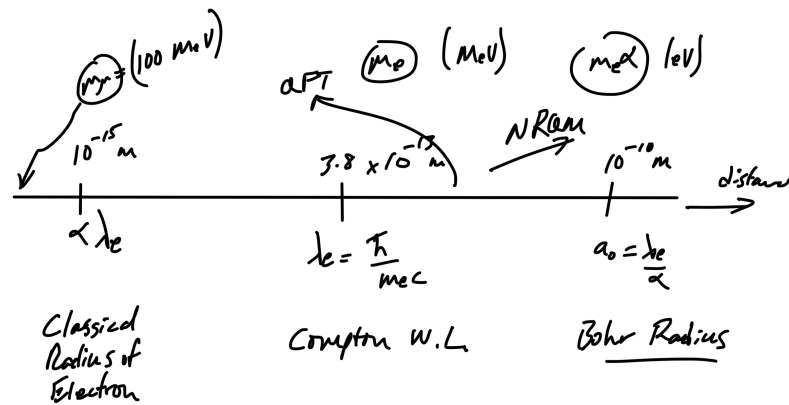


Figure 1.2: Scales, take II.

At the classical electron radius scale, we consider phenomena such as back reaction of radiation, the self energy of electrons. At the Compton wavelength we have to allow for production of multiple particle pairs. At Bohr radius scales we must start using QM instead of classical mechanics.

1.5 CROSS SECTION.

(Verbal discussion of cross section, not captured in these notes). Roughly, the cross section sounds like the number of events per unit time, related to the flux of some source through an area.

We'll compute the cross section of a number of different systems in this course. The cross section is relevant in scattering such as the electron-electron scattering sketched in fig. 1.3.

We assume that QED is highly relativistic. In natural units, our scale factor is basically the square of the electric charge

$$\alpha \sim e^2, \tag{1.37}$$

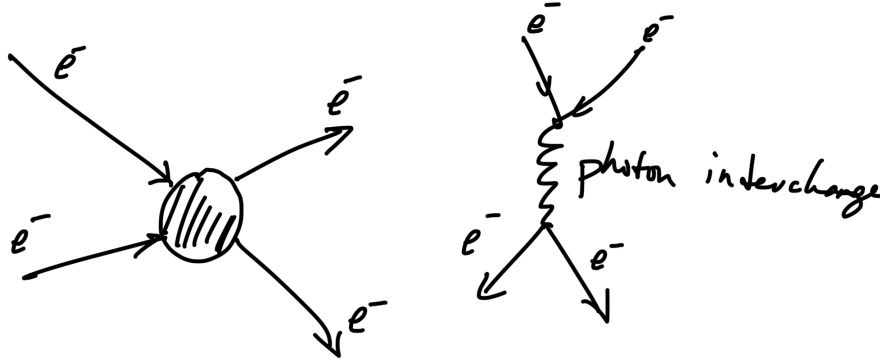


Figure 1.3: Electron electron scattering.

so the cross section has the form

$$\sigma \sim \frac{\alpha^2}{E^2} (1 + O(\alpha) + O(\alpha^2) + \dots) \quad (1.38)$$

In gravity we could consider scattering of electrons, where G_N takes the place of α . However, G_N has dimensions.

For electron-electron scattering due to gravitons

$$\sigma \sim \frac{G_N^2 E^2}{1 + G_N E^2 + \dots} \quad (1.39)$$

Now the cross section grows with energy. This will cause some problems (violating unitarity: probabilities greater than 1!) when $O(G_N E^2) = 1$.

In any quantum field theories when the coupling constant is not-dimensionless we have the same sort of problems at some scale.

The point is that we can get far considering just dimensional analysis.

If the coupling constant has a dimension $(1/\text{mass})^N$, $N > 0$, then unitarity will be violated at high energy. One such theory is the Fermi theory of beta decay (electro-weak theory), which had a coupling constant with dimensions inverse-mass-squared. The relevant scale for beta decay was 4 Fermi, or $G_F \sim (1/100\text{GeV})^2$. This was the motivation for introducing the Higgs theory, which was motivated by restoring unitarity.

Exercise 1.1 **Dimensional analysis. (2015 ps1.4)**

Even though we have set $\hbar = c = 1$, we can still do dimensional analysis because we still have one unit left, mass (or 1/length). In d space-time dimensions (1 time and $d - 1$ space), what is the dimension in mass units of a canonical free scalar field, ϕ ? (Work it out from the equal-time commutation relations.) Still in d dimensions, the Lagrange density for a scalar field with self-interactions might be of the form

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \sum_{n \geq 2} a_n \phi^n. \quad (1.40)$$

- What is the dimension (again in mass units) of the Lagrange density?
- The action?
- The coefficients a_n ? (as a check, whatever the value of d , a_2 had better have the dimensions of mass^2).

Answer for Exercise 1.1

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[REDACTED]

END-REDACTION

Exercise 1.2 **Zero point energy, and unit conversion. (2018 HW1.III)**

In class, we showed that the zero-point energy of the quantized massless scalar field (we are taking this case, because in the physically relevant case of electrodynamics, the number of degrees of freedom and the associated vacuum energy is the same as that of two massless scalar fields) can be written as:

$$E_{\text{vac}} = V_3 \int \frac{d^3 k}{(2\pi)^3} \frac{\omega_k}{2}. \quad (1.41)$$

where V_3 is the (large, i.e., almost infinite) volume of space. This expression diverges, because we assume that electromagnetic fields and photons of arbitrarily large momenta exist. There's no justification to this, as particle physicists have only probed the Standard Model up to energies

of order a few TeV. Assume, then, that the integral above is cut off at some maximum value of the momentum Λ (called the “UV cutoff”), say of order 10 TeV.

- What is the value of the vacuum energy density ρ_{vac} , in units of g/cm^3 .
- What value should Λ have in order that ρ_{vac} matches the observed value of the “dark energy”, of order $\rho_{\text{dark}} \sim 10^{-29} \text{ g}/\text{cm}^3$. Express Λ both as a high-energy scale cutoff and as a short-distance cutoff.
- What is the ratio of ρ_{vac} for $\Lambda \sim M_{\text{Planck}}$ to ρ_{dark} ?
- Note that the zero-point energies of phonons – the zero point energies of the quantized collective sound oscillations of nuclei in a crystal – are given, up to simple numerical factors counting the numbers of polarizations (which we won’t worry about here) by an expression similar to the above. This is because phonons are massless scalar fields propagating with the speed of sound instead of speed of light. Notice that this difference is irrelevant as c appears in E_{vac} simply: k is a wavevector and $\omega_k = ck$ – a frequency (secretly multiplied by \hbar , of course). In the case of phonons, however, we are well aware that a cutoff scale exists and we understand well its nature: it is given by the interatomic separation, as the notion of phonons does not make sense for shorter wavelengths. Now take $k_{\text{max}} = \Lambda \sim 1/a_0$, with a_0 of order the Bohr radius and estimate the energy density of the zero point fluctuations in a crystal. Compare your result to the typical rest energy (i.e. mass) density of crystals.

The results from the first three items above lead to a puzzle commonly referred to as the “cosmological constant problem”. There are various proposals for its solution, ranging from cancellations between the contributions of high and low momentum oscillators, anthropic principle (multiverse) considerations, modifications of gravity at long distances, to name a few. The issue awaits your input!

Answer for Exercise 1.2

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LORENTZ TRANSFORMATIONS.

2.1 LORENTZ TRANSFORMATIONS.

The goal, perhaps not for today, is to study the simplest (relativistic) scalar field theory. First studied classically, and then consider such a quantum field theory. How is relativity implemented when we write the Lagrangian and action?

Our first step must be to consider Lorentz transformations and the Lorentz group.

Spacetime (Minkowski space) is $\mathbb{R}^{3,1}$ (or $\mathbb{R}^{d-1,1}$). Our coordinates are

$$(ct, x^1, x^2, x^3) = (ct, \mathbf{r}). \quad (2.1)$$

Here, we've scaled the time scale by c so that we measure time and space in the same dimensions. We write this as

$$x^\mu = (x^0, x^1, x^2, x^3), \quad (2.2)$$

where $\mu = 0, 1, 2, 3$, and call this a “4-vector”. These are called the space-time coordinates of an event, which tell us where and when an event occurs.

For two events whose spacetime coordinates differ by dx^0, dx^1, dx^2, dx^3 we introduce the notion of a space time interval

$$\begin{aligned} ds^2 &= c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ &= \sum_{\mu, \nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu \end{aligned} \quad (2.3)$$

Here $g_{\mu\nu}$ is the Minkowski space metric, an object with two indexes that run from 0-3. i.e. this is a diagonal matrix

$$g_{\mu\nu} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (2.4)$$

i.e.

$$\begin{aligned} g_{00} &= 1 \\ g_{11} &= -1 \\ g_{22} &= -1 \\ g_{33} &= -1 \end{aligned} \tag{2.5}$$

We will use the Einstein summation convention, where any repeated upper and lower indexes are considered summed over. That is eq. (2.3) is written with an implied sum

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.6}$$

Explicit expansion:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{00} dx^0 dx^0 + g_{11} dx^1 dx^1 + g_{22} dx^2 dx^2 + g_{33} dx^3 dx^3 \\ &= (1) dx^0 dx^0 + (-1) dx^1 dx^1 + (-1) dx^2 dx^2 + (-1) dx^3 dx^3. \end{aligned} \tag{2.7}$$

Recall that rotations (with orthogonal matrix representations) are transformations that leave the dot product unchanged, that is

$$\begin{aligned} (R\mathbf{x}) \cdot (R\mathbf{y}) &= \mathbf{x}^T R^T R \mathbf{y} \\ &= \mathbf{x}^T \mathbf{y} \\ &= \mathbf{x} \cdot \mathbf{y}, \end{aligned} \tag{2.8}$$

where R is a rotation orthogonal 3x3 matrix. The set of such transformations that leave the dot product unchanged have orthonormal matrix representations $R^T R = 1$. We call the set of such transformations that have unit determinant the SO(3) group.

We call a Lorentz transformation, if it is a linear transformation acting on 4 vectors that leaves the spacetime interval (i.e. the inner product of 4 vectors) invariant. That is, a transformation that leaves

$$x^\mu y^\nu g_{\mu\nu} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \tag{2.9}$$

unchanged.

Suppose that transformation has a 4x4 matrix form

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.10}$$

For an example of a possible Λ , consider the transformation sketched in fig. 2.1. We know

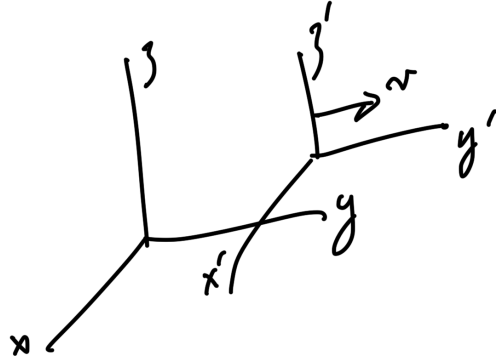


Figure 2.1: Boost transformation.

that boost has the form

$$\begin{aligned}
 x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\
 y &= y' \\
 z &= z' \\
 t &= \frac{t' + (v/c^2)x'}{\sqrt{1 - v^2/c^2}}
 \end{aligned} \tag{2.11}$$

(this is a boost along the x-axis, not y as I'd drawn), or

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1 - v^2/c^2}} & \frac{v/c}{\sqrt{1 - v^2/c^2}} & 0 & 0 \\ \frac{v/c}{\sqrt{1 - v^2/c^2}} & \frac{1}{\sqrt{1 - v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} \tag{2.12}$$

Other examples include rotations ($\lambda^0_0 = 1$ zeros in λ^0_k, λ^k_0 , and a rotation matrix in the remainder.)

Back to Lorentz transformations ($\text{SO}(1, 3)^+$), let

$$\begin{aligned}
 x'^\mu &= \Lambda^\mu_\nu x^\nu \\
 y'^\kappa &= \Lambda^\kappa_\rho y^\rho
 \end{aligned} \tag{2.13}$$

The dot product

$$\begin{aligned} g_{\mu\kappa}x'^{\mu}y'^{\kappa} &= g_{\mu\kappa}\Lambda^{\mu}_{\nu}\Lambda^{\kappa}_{\rho}x^{\nu}y^{\rho} \\ &= g_{\nu\rho}x^{\nu}y^{\rho}, \end{aligned} \quad (2.14)$$

where the last step introduces the invariance requirement of the transformation. That is

$$g_{\nu\rho} = g_{\mu\kappa}\Lambda^{\mu}_{\nu}\Lambda^{\kappa}_{\rho}. \quad (2.15)$$

Upper and lower indexes We've defined

$$x^{\mu} = (t, x^1, x^2, x^3) \quad (2.16)$$

We could also define a four vector with lower indexes

$$\begin{aligned} x_{\nu} &= g_{\nu\mu}x^{\mu} \\ &= (t, -x^1, -x^2, -x^3). \end{aligned} \quad (2.17)$$

That is

$$\begin{aligned} x_0 &= x^0 \\ x_1 &= -x^1 \\ x_2 &= -x^2 \\ x_3 &= -x^3. \end{aligned} \quad (2.18)$$

which allows us to write the dot product as simply $x^{\mu}y_{\mu}$.

We can also define a metric tensor with upper indexes

$$g^{\mu\nu} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (2.19)$$

This is the inverse matrix of $g_{\mu\nu}$, and it satisfies

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho} \quad (2.20)$$

Exercise: Check:

$$\begin{aligned} g_{\mu\nu}x^{\mu}y^{\nu} &= x_{\nu}y^{\nu} \\ &= x^{\nu}y_{\nu} \\ &= g^{\mu\nu}x_{\mu}y_{\nu} \\ &= \delta^{\mu}_{\nu}x_{\mu}y^{\nu} \end{aligned} \quad (2.21)$$

Class ended around this point, but it appeared that we were heading this direction:

Returning to the Lorentz invariant and multiplying both sides of eq. (2.15) with an inverse Lorentz transformation Λ^{-1} , we find

$$\begin{aligned} g_{\nu\rho}(\Lambda^{-1})^\rho_\alpha &= g_{\mu\kappa}\Lambda^\mu_\nu\Lambda^\kappa_\rho(\Lambda^{-1})^\rho_\alpha \\ &= g_{\mu\kappa}\Lambda^\mu_\nu\delta^\kappa_\alpha \\ &= g_{\mu\alpha}\Lambda^\mu_\nu, \end{aligned} \tag{2.22}$$

or

$$(\Lambda^{-1})_{\nu\alpha} = \Lambda_{\alpha\nu}. \tag{2.23}$$

This is clearly analogous to $R^T = R^{-1}$, although the index notation obscures things considerably. Prof. Poppitz said that next week this would all lead to showing that the determinant of any Lorentz transformation was ± 1 .

For what it's worth, it seems to me that this index notation makes life a lot harder than it needs to be, at least for a matrix related question (i.e. determinant of the transformation). In matrix/column-(4)-vector notation, let $x' = \Lambda x, y' = \Lambda y$ be two four vector transformations, then

$$\begin{aligned} x' \cdot y' &= x'^T G y' \\ &= (\Lambda x)^T G \Lambda y \\ &= x^T (\Lambda^T G \Lambda) y \\ &= x^T G y. \end{aligned} \tag{2.24}$$

so

$$\boxed{\Lambda^T G \Lambda = G.} \tag{2.25}$$

Taking determinants of both sides gives $-(\det(\Lambda))^2 = -1$, and thus $\det(\Lambda) = \pm 1$.

2.2 DETERMINANT OF LORENTZ TRANSFORMATIONS.

We require that Lorentz transformations leave the dot product invariant, that is $x \cdot y = x' \cdot y'$, or

$$x^\mu g_{\mu\nu} y^\nu = x'^\mu g_{\mu\nu} y'^\nu. \tag{2.26}$$

Explicitly, with coordinate transformations

$$\begin{aligned} x'^\mu &= \Lambda^\mu_\rho x^\rho \\ y'^\mu &= \Lambda^\mu_\rho y^\rho \end{aligned} \tag{2.27}$$

such a requirement is equivalent to demanding that

$$\begin{aligned} x^\mu g_{\mu\nu} y^\nu &= \Lambda^\mu{}_\rho x^\rho g_{\mu\nu} \Lambda^\nu{}_\kappa y^\kappa \\ &= x^\mu \Lambda^\alpha{}_\mu g_{\alpha\beta} \Lambda^\beta{}_\nu y^\nu, \end{aligned} \quad (2.28)$$

or

$$g_{\mu\nu} = \Lambda^\alpha{}_\mu g_{\alpha\beta} \Lambda^\beta{}_\nu \quad (2.29)$$

multiplying by the inverse we find

$$\begin{aligned} g_{\mu\nu} (\Lambda^{-1})^\nu{}_\lambda &= \Lambda^\alpha{}_\mu g_{\alpha\beta} \Lambda^\beta{}_\nu (\Lambda^{-1})^\nu{}_\lambda \\ &= \Lambda^\alpha{}_\mu g_{\alpha\lambda} \\ &= g_{\lambda\alpha} \Lambda^\alpha{}_\mu. \end{aligned} \quad (2.30)$$

This is now amenable to expressing in matrix form

$$\begin{aligned} (G\Lambda^{-1})_{\mu\lambda} &= (G\Lambda)_{\lambda\mu} \\ &= ((G\Lambda)^T)_{\mu\lambda} \\ &= (\Lambda^T G)_{\mu\lambda}, \end{aligned} \quad (2.31)$$

or

$$G\Lambda^{-1} = (G\Lambda)^T. \quad (2.32)$$

Taking determinants (using the normal identities for products of determinants, determinants of transposes and inverses), we find

$$\cancel{\det(G)} \det(\Lambda^{-1}) = \cancel{\det(G)} \det(\Lambda), \quad (2.33)$$

or

$$\det(\Lambda)^2 = 1, \quad (2.34)$$

or $\det(\Lambda)^2 = \pm 1$. We will generally ignore the case of reflections in spacetime that have a negative determinant.

Smart-alec Peeter pointed out after class last time that we can do the same thing easier in matrix notation

$$\begin{aligned} x' &= \Lambda x \\ y' &= \Lambda y \end{aligned} \quad (2.35)$$

where

$$\begin{aligned} x' \cdot y' &= (x')^T G y' \\ &= x^T \Lambda^T G \Lambda y, \end{aligned} \quad (2.36)$$

which we require to be $x \cdot y = x^T G y$ for all four vectors x, y , that is

$$\Lambda^T G \Lambda = G. \quad (2.37)$$

We can find the result eq. (2.34) immediately without having to first translate from index notation to matrices.

2.3 PROBLEMS.

Exercise 2.1 Lorentz transformation. (2015 ps1.1)

A Lorentz transformation $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is such that it preserves the Minkowski metric $\eta_{\mu\nu}$ meaning that $\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$ for all x .

- a. Show that this implies that

$$\eta_{\mu\nu} = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu. \quad (2.38)$$

- b. Use this result to show that an infinitesimal transformation of the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \quad (2.39)$$

is a Lorentz transformation when $\omega^{\mu\nu}$ is antisymmetric i.e. $\omega^{\mu\nu} = -\omega^{\nu\mu}$. (Note that there an antisymmetric 4×4 matrix has six parameters, as does a Lorentz transformation - 3 rotations and 3 boosts - so the counting works out).

- c. Write down the matrix form for $\omega^\mu{}_\nu$ that corresponds to a rotation through an infinitesimal angle θ about the x^3 -axis.
- d. Do the same for a boost along the x^1 -axis by an infinitesimal velocity v .

Answer for Exercise 2.1

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3

CLASSICAL FIELD THEORY.

3.1 FIELD THEORY.

The electrostatic potential is an example of a scalar field $\phi(\mathbf{x})$ unchanged by SO(3) rotations

$$\mathbf{x} \rightarrow \mathbf{x}' = O\mathbf{x}, \quad (3.1)$$

that is

$$\phi'(\mathbf{x}') = \phi(\mathbf{x}). \quad (3.2)$$

Here $\phi'(\mathbf{x}')$ is the value of the (electrostatic) scalar potential in a primed frame.

However, the electrostatic field is not invariant under Lorentz transformation. We postulate that there is some scalar field

$$\phi'(x') = \phi(x), \quad (3.3)$$

where $x' = \Lambda x$ is an SO(1,3) transformation. There are actually no stable particles (fields that persist at long distances) described by Lorentz scalar fields, although there are some unstable scalar fields such as the Higgs, Pions, and Kaons. However, much of our homework and discussion will be focused on scalar fields, since they are the easiest to start with.

We need to first understand how derivatives $\partial_\mu \phi(x)$ transform. Using the chain rule

$$\begin{aligned} \frac{\partial \phi(x)}{\partial x^\mu} &= \frac{\partial \phi'(x')}{\partial x'^\mu} \\ &= \frac{\partial \phi'(x')}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\mu} \\ &= \frac{\partial \phi'(x')}{\partial x'^\nu} \partial_\mu (\Lambda^\nu{}_\rho x^\rho) \\ &= \frac{\partial \phi'(x')}{\partial x'^\nu} \Lambda^\nu{}_\mu \\ &= \frac{\partial \phi(x)}{\partial x'^\nu} \Lambda^\nu{}_\mu. \end{aligned} \quad (3.4)$$

Multiplying by the inverse $(\Lambda^{-1})^\mu{}_\kappa$ we get

$$\frac{\partial}{\partial x'^\kappa} = (\Lambda^{-1})^\mu{}_\kappa \frac{\partial}{\partial x^\mu} \quad (3.5)$$

This should be familiar to you, and is an analogue of the transformation of the

$$d\mathbf{r} \cdot \nabla_{\mathbf{r}} = d\mathbf{r}' \cdot \nabla_{\mathbf{r}'}. \quad (3.6)$$

3.2 ACTIONS.

We will start with a classical action, and quantize to determine a QFT. In mechanics we have the particle position $q(t)$, which is a classical field in 1+0 time and space dimensions. Our action is

$$\begin{aligned} S &= \int dt \mathcal{L}(t) \\ &= \int dt \left(\frac{1}{2} \dot{q}^2 - V(q) \right). \end{aligned} \tag{3.7}$$

This action depends on the position of the particle that is local in time. You could imagine that we have a more complex action where the action depends on future or past times

$$S = \int dt' q(t') K(t' - t), \tag{3.8}$$

but we don't seem to find such actions in classical mechanics.

3.3 PRINCIPLES DETERMINING THE FORM OF THE ACTION.

- relativity (action is invariant under Lorentz transformation)
- locality (action depends on fields and the derivatives at given (t, \mathbf{x})).
- Gauge principle (the action should be invariant under gauge transformation). We won't discuss this in detail right now since we will start with studying scalar fields. Recall that for Maxwell's equations a gauge transformation has the form

$$\begin{aligned} \phi &\rightarrow \phi + \dot{\chi}, \mathbf{A} \\ &\rightarrow \mathbf{A} - \nabla \chi. \end{aligned} \tag{3.9}$$

Suppose we have a real scalar field $\phi(x)$ where $x \in \mathbb{R}^{1,d-1}$. We will be integrating over space and time $\int dt d^{d-1}x$ which we will write as $\int d^d x$. Our action is

$$S = \int d^d x (\text{Some action density to be determined}) \tag{3.10}$$

The analogue of \dot{q}^2 is

$$\begin{aligned} \left(\frac{\partial \phi}{\partial x^\mu} \right) \left(\frac{\partial \phi}{\partial x^\nu} \right) g^{\mu\nu} &= (\partial_\mu \phi)(\partial_\nu \phi) g^{\mu\nu} \\ &= \partial^\mu \phi \partial_\mu \phi. \end{aligned} \tag{3.11}$$

This has both time and spatial components, that is

$$\partial^\mu \phi \partial_\mu \phi = \dot{\phi}^2 - (\nabla \phi)^2, \quad (3.12)$$

so the desired simplest scalar action is

$$S = \int d^d x (\dot{\phi}^2 - (\nabla \phi)^2). \quad (3.13)$$

The measure transforms using a Jacobian, which we have seen is the Lorentz transform matrix, and has unit determinant

$$d^d x' = d^d x |det(\Lambda^{-1})| = d^d x. \quad (3.14)$$

3.4 PRINCIPLES (CONT.)

- Lorentz (Poincaré : Lorentz and spacetime translations)
- locality
- dimensional analysis
- gauge invariance

These are the requirements for an action. We postulated an action that had the form

$$\int d^d x \partial_\mu \phi \partial^\mu \phi, \quad (3.15)$$

called the “Kinetic term”, which mimics $\int dt \dot{q}^2$ that we’d see in quantum or classical mechanics. In principle there exists an infinite number of local Poincaré invariant terms that we can write. Examples:

- $\partial_\mu \phi \partial^\mu \phi$
- $\partial_\mu \phi \partial_\nu \partial^\nu \partial^\mu \phi$
- $(\partial_\mu \phi \partial^\mu \phi)^2$
- $f(\phi) \partial_\mu \phi \partial^\mu \phi$
- $f(\phi, \partial_\mu \phi \partial^\mu \phi)$
- $V(\phi)$

It turns out that nature (i.e. three spatial dimensions and one time dimension) is described by a finite number of terms. We will now utilize dimensional analysis to determine some of the allowed forms of the action for scalar field theories in $d = 2, 3, 4, 5$ dimensions. Even though the real world is only $d = 4$, some of the $d < 4$ theories are relevant in condensed matter studies, and $d = 5$ is just for fun (but also applies to string theories.)

With $[x] \sim \frac{1}{M}$ in natural units, we must define $[\phi]$ such that the kinetic term is dimensionless in d spacetime dimensions

$$\begin{aligned} [d^d x] &\sim \frac{1}{M^d} \\ [\partial_\mu] &\sim M \end{aligned} \tag{3.16}$$

so it must be that

$$[\phi] = M^{(d-2)/2} \tag{3.17}$$

It will be easier to characterize the dimensionality of any given term by the power of the mass units, that is

$$\begin{aligned} [\text{mass}] &= 1 \\ [d^d x] &= -d \\ [\partial_\mu] &= 1 \\ [\phi] &= (d-2)/2 \\ [S] &= 0. \end{aligned} \tag{3.18}$$

Since the action is

$$S = \int d^d x (\mathcal{L}(\phi, \partial_\mu \phi)), \tag{3.19}$$

and because action had dimensions of \hbar , so in natural units, it must be dimensionless, the Lagrangian density dimensions must be $[d]$. We will abuse language in QFT and call the Lagrangian density the Lagrangian.

3.4.1 $d = 2$.

Because $[\partial_\mu \phi \partial^\mu \phi] = 2$, the scalar field must be dimension zero, or in symbols

$$[\phi] = 0. \tag{3.20}$$

This means that introducing any function $f(\phi) = 1 + a\phi + b\phi^2 + c\phi^3 + \dots$ is also dimensionless, and

$$[f(\phi) \partial_\mu \phi \partial^\mu \phi] = 2, \tag{3.21}$$

for any $f(\phi)$. Another implication of this is that the a potential term in the Lagrangian $[V(\phi)] = 0$ needs a coupling constant of dimension 2. Letting μ have mass dimensions, our Lagrangian must have the form

$$f(\phi)\partial_\mu\phi\partial^\mu\phi + \mu^2 V(\phi). \quad (3.22)$$

An infinite number of coupling constants of positive mass dimensions for $V(\phi)$ are also allowed. If we have higher order derivative terms, then we need to compensate for the negative mass dimensions. Example (still for $d = 2$).

$$\mathcal{L} = f(\phi)\partial_\mu\phi\partial^\mu\phi + \mu^2 V(\phi) + \frac{1}{\mu'^2}\partial_\mu\phi\partial_\nu\partial^\nu\partial^\mu\phi + \left(\partial_\mu\phi\partial^\mu\phi\right)^2 \frac{1}{\tilde{\mu}^2}. \quad (3.23)$$

The last two terms, called couplings (i.e. any non-kinetic term), are examples of terms with negative mass dimension. There is an infinite number of those in any theory in any dimension.

Definitions

- Couplings that are dimensionless are called (classically) marginal.
- Couplings that have positive mass dimension are called (classically) relevant.
- Couplings that have negative mass dimension are called (classically) irrelevant.

In QFT we are generally interested in the couplings that are measurable at long distances for some given energy. Classically irrelevant theories are generally not interesting in $d > 2$, so we are very lucky that we don't live in three dimensional space. This means that we can get away with a finite number of classically marginal and relevant couplings in 3 or 4 dimensions. This was mentioned in the Wilczek's article referenced in the class forum [19]¹

Long distance physics in any dimension is described by the marginal and relevant couplings. The irrelevant couplings die off at low energy. In two dimensions, a priori, an infinite number of marginal and relevant couplings are possible. 2D is a bad place to live!

3.4.2 $d = 3$.

Now we have

$$[\phi] = \frac{1}{2} \quad (3.24)$$

so that

$$[\partial_\mu\phi\partial^\mu\phi] = 3. \quad (3.25)$$

¹ There's currently more in that article that I don't understand than I do, so it is hard to find it terribly illuminating.

A 3D Lagrangian could have local terms such as

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \mu^{3/2} \phi^3 + \mu' \phi^4 + (\mu'') 1/2 \phi^5 + \lambda \phi^6. \quad (3.26)$$

where m, μ, μ'' all have mass dimensions, and λ is dimensionless. i.e. m, μ, μ'' are relevant, and λ marginal. We stop at the sixth power, since any power after that will be irrelevant.

3.4.3 $d = 4$.

Now we have

$$[\phi] = 1 \quad (3.27)$$

so that

$$[\partial_\mu \phi \partial^\mu \phi] = 4. \quad (3.28)$$

In this number of dimensions $\phi^k \partial_\mu \phi \partial^\mu \phi$ is an irrelevant coupling.

A 4D Lagrangian could have local terms such as

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \mu \phi^3 + \lambda \phi^4. \quad (3.29)$$

where m, μ have mass dimensions, and λ is dimensionless. i.e. m, μ are relevant, and λ is marginal.

3.4.4 $d = 5$.

Now we have

$$[\phi] = \frac{3}{2}, \quad (3.30)$$

so that

$$[\partial_\mu \phi \partial^\mu \phi] = 5. \quad (3.31)$$

A 5D Lagrangian could have local terms such as

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \sqrt{\mu} \phi^3 + \frac{1}{\mu'} \phi^4. \quad (3.32)$$

where m, μ, μ' all have mass dimensions. In 5D there are no marginal couplings. Dimension 4 is the last dimension where marginal couplings exist. In condensed matter physics 4D is called the “upper critical dimension”.

From the point of view of particle physics, all the terms in the Lagrangian must be the ones that are relevant at long distances.

3.5 LEAST ACTION PRINCIPLE.

Now we want to study 4D scalar theories. We have some action

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \quad (3.33)$$

Let's keep an example such as the following in mind

Kinetic term

$$\mathcal{L} = \boxed{\frac{1}{2} \partial_\mu \phi \partial^\mu \phi} - \boxed{m^2 \phi - \lambda \phi^4}. \quad (3.34)$$

all relevant and marginal couplings

The even powers can be justified by assuming there is some symmetry that kills the odd powered terms.

We will be integrating over a space time region such as that depicted in fig. 3.1, where a

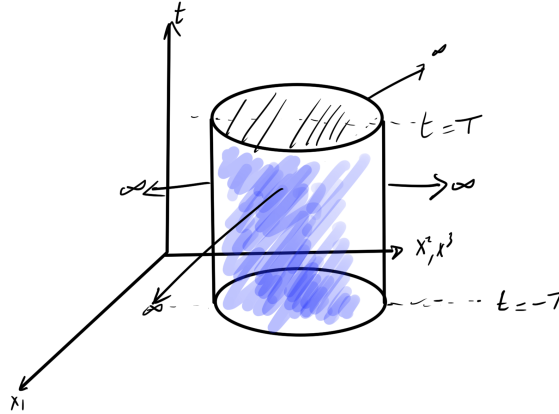


Figure 3.1: Cylindrical spacetime boundary.

cylindrical spatial cross section is depicted that we allow to tend towards infinity. We demand that the field is fixed on the infinite spatial boundaries. The easiest way to demand that the field dies off on the spatial boundaries, that is

$$\lim_{\|\mathbf{x}\| \rightarrow \infty} \phi(\mathbf{x}) \rightarrow 0. \quad (3.35)$$

The functional $\phi(\mathbf{x}, t)$ that obeys the boundary condition as stated extremizes $S[\phi]$.

Extremizing the action means that we seek $\phi(\mathbf{x}, t)$

$$\delta S[\phi] = 0 = S[\phi + \delta\phi] - S[\phi]. \quad (3.36)$$

How do we compute the variation?

$$\begin{aligned} \delta S &= \int d^d x \left(\mathcal{L}(\phi + \delta\phi, \partial_\mu \phi + \partial_\mu \delta\phi) - \mathcal{L}(\phi, \partial_\mu \phi) \right) \\ &= \int d^d x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\mu \delta\phi) \right) \\ &= \int d^d x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right) \\ &= \int d^d x \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \int d^3 \sigma_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \end{aligned} \quad (3.37)$$

If we are explicit about the boundary term, we write it as

$$\begin{aligned} &\int dt d^3 \mathbf{x} \left(\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta\phi \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla \phi)} \delta\phi \right) \right) \\ &= \int d^3 \mathbf{x} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta\phi \Big|_{t=-T}^{t=T} - \int dt d^2 \mathbf{S} \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla \phi)} \delta\phi \right). \end{aligned} \quad (3.38)$$

but $\delta\phi = 0$ at $t = \pm T$ and also at the spatial boundaries of the integration region.

This leaves

$$\begin{aligned} \delta S[\phi] &= \int d^d x \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \\ &= 0 \forall \delta\phi. \end{aligned} \quad (3.39)$$

That is

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.} \quad (3.40)$$

This is the Euler-Lagrange equations for a single scalar field.

Returning to our sample scalar Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4. \quad (3.41)$$

This example is related to the Ising model which has a $\phi \rightarrow -\phi$ symmetry. Applying the Euler-Lagrange equations, we have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi - \lambda \phi^3, \quad (3.42)$$

and

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= \frac{\partial}{\partial(\partial_\mu \phi)} \left(\frac{1}{2} \partial_\nu \phi \partial^\nu \phi \right) \\
&= \frac{1}{2} \partial^\nu \phi \frac{\partial}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \frac{1}{2} \partial_\nu \phi \frac{\partial}{\partial(\partial_\mu \phi)} \partial_\alpha \phi g^{\nu\alpha} \\
&= \frac{1}{2} \partial^\mu \phi + \frac{1}{2} \partial_\nu \phi g^{\nu\mu} \\
&= \partial^\mu \phi
\end{aligned} \tag{3.43}$$

so we have

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \\
&= -m^2 \phi - \lambda \phi^3 - \partial_\mu \partial^\mu \phi.
\end{aligned} \tag{3.44}$$

For $\lambda = 0$, the free field theory limit, this is just

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{3.45}$$

Written out from the observer frame, this is

$$\partial_{tt} \phi - \nabla^2 \phi + m^2 \phi = 0. \tag{3.46}$$

With a non-zero mass term

$$(\partial_{tt} - \nabla^2 + m^2) \phi = 0, \tag{3.47}$$

is called the Klein-Gordon equation.

If we also had $m = 0$ we'd have

$$(\partial_{tt} - \nabla^2) \phi = 0, \tag{3.48}$$

which is the wave equation (for a massless free field). This is also called the D'Alembert equation, which is familiar from electromagnetism where we have

$$\begin{aligned}
(\partial_{tt} - \nabla^2) \mathbf{E} &= 0 \\
(\partial_{tt} - \nabla^2) \mathbf{B} &= 0,
\end{aligned} \tag{3.49}$$

in a source free region.

3.6 PROBLEMS.

Exercise 3.1 Four vector form of the Maxwell gauge transformation.

Show that the transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \quad (3.50)$$

is the desired four-vector form of the gauge transformation eq. (3.9), that is

$$j^\nu = \partial_\mu F'^{\mu\nu} = \partial_\mu F^{\mu\nu}. \quad (3.51)$$

Also relate this four-vector gauge transformation to the spacetime split.

Answer for Exercise 3.1

$$\begin{aligned} \partial_\mu F'^{\mu\nu} &= \partial_\mu (\partial^\mu A'^\nu - \partial_\nu A'^\mu) \\ &= \partial_\mu (\partial^\mu (A^\nu + \partial^\nu \chi) - \partial_\nu (A^\mu + \partial^\mu \chi)) \\ &= \partial_\mu F^{\mu\nu} + \partial_\mu \partial^\mu \partial^\nu \chi - \partial_\mu \partial^\nu \partial^\mu \chi \\ &= \partial_\mu F^{\mu\nu}, \end{aligned} \quad (3.52)$$

by equality of mixed partials. Expanding eq. (3.50) explicitly we find

$$A'^\mu = A^\mu + \partial^\mu \chi, \quad (3.53)$$

which is

$$\begin{aligned} \phi' &= A'^0 = A^0 + \partial^0 \chi = \phi + \dot{\chi} \\ \mathbf{A}' \cdot \mathbf{e}_k &= A'^k = A^k + \partial^k \chi = (\mathbf{A} - \nabla \chi) \cdot \mathbf{e}_k. \end{aligned} \quad (3.54)$$

The last of which can be written in vector notation as $\mathbf{A}' = \mathbf{A} - \nabla \chi$.

Exercise 3.2 One dimensional string (2015 ps1.3)

A string of length a , mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of the transverse displacement $y(x, t)$ is

$$L = \int_0^a dx \left(\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right) \quad (3.55)$$

where x identifies position along the string from one end point.

CLASSICAL FIELD THEORY.

- b. Consider the limiting case of a massless field, $\mu \rightarrow 0$, and identify the field A^μ with the scalar and vector potentials of electrodynamics: $A^\mu = (\phi, \mathbf{A})$, where

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \quad (3.61)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (3.62)$$

Show that the field equations reproduce two of Maxwell's equations, and that the other two hold as identities given the definitions of \mathbf{E} and \mathbf{B} in terms of ϕ and \mathbf{A} .

Answer for Exercise 3.3

PROBLEM SET RELATED MATERIAL REDACTED IN THIS DOCUMENT.PLEASE
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[illegible]

Exercise 3.4 Back to classics: relativistic electrodynamics and variational principle (2018 HW1.D)

Given the action In terms of the four-vector potential A , the Lagrangian density of the electromagnetic field, interacting with a charged particle of mass m can be written as follows:

$$S = \int_{\text{all spacetime}} d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu \right) - m \int_{\text{worldline}} ds. \quad (3.63)$$

Here, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ field strength tensor. The current j^μ is the current corresponding to the particle which can be written as:

$$j^\mu(x) = e \int_{\text{worldline}} dX^\mu(\tau) \delta^{(4)}(x - X(\tau)), \quad (3.64)$$

where $\delta^{(4)}(x)$ is a four-dimensional delta function. All indices are raised and lowered by means of the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

The last term in eq. (3.63) is the relativistic kinetic energy of the particle and the integral is over the particle's worldline, $X^\mu(\tau)$. Note that τ is a parameter used to describe the particle's

location along the worldline. One can take this parameter be equal to x^0 , so that $X^\mu(\tau)$ means ($X^0 = x^0$, $X^i = X^i(x^0)$), where $\mathbf{X}(x^0)$ is simply the trajectory of the particle (such a choice of parametrization can be useful, but is not required). Notice also that the term involving the current in eq. (3.63), after substitution of eq. (3.64) simply becomes

$$-e \int_{\text{worldline}} dX^\mu(\tau) A_\mu(X(\tau)) , \quad (3.65)$$

which is the usual coupling of a charged particle to the electromagnetic field (choose the $\tau = x^0$ parameterization of the worldline to see this). Whether you use this form of the one of eq. (3.63) depends on the problem you're solving (this is a hint).

The dynamical degrees of freedom in the action eq. (3.63) are the four-vector potential A_μ and the particle position $X^\mu(\tau)$.

- Use the identification $A^0 = \phi$, the scalar potential, and $(A^1, A^2, A^3) = \mathbf{A}$, the vector potential, to convince yourself that $F_{01} = E_x, F_{02} = E_y, F_{03} = E_z$, and that $F_{12} = -B_z, F_{31} = -B_y, F_{23} = -B_x$.
- Prove the identity

$$\epsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = 0, \quad (3.66)$$

and use this to show that the source free Maxwell's equations can be recovered directly from the definition of F_{ij} .

- Write the Euler-Lagrange equations obtained when varying eq. (3.63) with respect to A_μ . Show that they can be cast in terms of the field strength tensor F and j . Note that when varying with respect to A_μ , the current is kept fixed. Using the \mathbf{E} and \mathbf{B} fields as the appropriate components of F , show that the Euler-Lagrange equations for A_μ from eq. (3.63) reduce to the Maxwell equations familiar to you from electrodynamics.
- Finally, write the Euler-Lagrange equation varying with respect to the worldline of the particle. Show that they give $mdU^\mu/ds = eF^{\mu\nu}U_\nu$, where $U^\mu = dX^\mu/ds$ is the four velocity of the particle and F is, of course, taken at the particle's position. Convince yourself that this is the relativistic Lorentz force equation.

The point of this problem is to make sure you remember/learn how the action principle works in electrodynamics. The two coupled equations, obtained by varying w.r.t. A_μ and X^μ complete the equations of classical electrodynamics. Feel free to use [7], or [15] while solving this problem.

Answer for Exercise 3.4

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4

CANONICAL QUANTIZATION, KLEIN-GORDON EQUATION, SHOS, MOMENTUM SPACE REPRESENTATION, RAISING AND LOWERING OPERATORS.

4.1 CANONICAL QUANTIZATION.

The harmonic oscillator described by

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2, \quad (4.1)$$

which has solution $\ddot{q} = -\omega^2 q$. With

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q}, \quad (4.2)$$

the Hamiltonian is given by

$$\begin{aligned} H(p, q) &= p\dot{q} - \mathcal{L}|_{\dot{q}(p,q)} \\ &= pp - \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2 \\ &= \frac{p^2}{2} + \frac{\omega^2}{2}q^2. \end{aligned} \quad (4.3)$$

In QM we quantize by mapping Poisson brackets to commutators.

$$[\hat{p}, \hat{q}] = -i \quad (4.4)$$

One way to represent is to say that states are $\Psi(\hat{q})$, a wave function, \hat{q} acts by q

$$\hat{q}\Psi = q\Psi(q) \quad (4.5)$$

With

$$\hat{p} = -i\frac{\partial}{\partial q}, \quad (4.6)$$

so

$$\left[-i\frac{\partial}{\partial q}, q \right] = -i \quad (4.7)$$

Returning to the field Lagrangian. Let's introduce an explicit space time split. We'll write

$$L = \int d^3x \left(\frac{1}{2} (\partial_0 \phi(\mathbf{x}, t))^2 - \frac{1}{2} (\nabla \phi(\mathbf{x}, t))^2 - \frac{m^2}{2} \phi^2 \right), \quad (4.8)$$

so that the action is

$$S = \int dt L. \quad (4.9)$$

The dynamical variables are $\phi(\mathbf{x})$. We define

$$\begin{aligned} \pi(\mathbf{x}, t) &= \frac{\delta L}{\delta(\partial_0 \phi(\mathbf{x}, t))} \\ &= \partial_0 \phi(\mathbf{x}, t) \\ &= \dot{\phi}(\mathbf{x}, t), \end{aligned} \quad (4.10)$$

called the canonical momentum, or the momentum conjugate to $\phi(\mathbf{x}, t)$. Why δ ? Has to do with an implicit Dirac function to eliminate the integral?

$$\begin{aligned} H &= \int d^3x \left(\pi(\bar{\mathbf{x}}, t) \dot{\phi}(\bar{\mathbf{x}}, t) - L \right) \Big|_{\dot{\phi}(\bar{\mathbf{x}}, t) = \pi(\mathbf{x}, t)} \\ &= \int d^3x \left((\pi(\mathbf{x}, t))^2 - \frac{1}{2} (\pi(\mathbf{x}, t))^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right), \end{aligned} \quad (4.11)$$

or

$$H = \int d^3x \left(\frac{1}{2} (\pi(\mathbf{x}, t))^2 + \frac{1}{2} (\nabla \phi(\mathbf{x}, t))^2 + \frac{m^2}{2} (\phi(\mathbf{x}, t))^2 \right) \quad (4.12)$$

In analogy to the momentum, position commutator in QM

$$[\hat{p}_i, \hat{q}_j] = -i\delta_{ij}, \quad (4.13)$$

we “quantize” the scalar field theory by promoting π, ϕ to operators and insisting that they also obey a commutator relationship

$$[\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (4.14)$$

Note that in this commutator, the fields are evaluated at different spatial points, but at the same time.

4.2 CANONICAL QUANTIZATION (CONT.)

Last time we introduced a Lagrangian density associated with the Klein-Gordon equation (with a quadratic potential coupling)

$$L = \int d^3x \left(\frac{1}{2} (\partial_0 \phi)^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4 \right). \quad (4.15)$$

This Lagrangian density was related to the action by

$$S = \int dt L = \int dt d^3x \mathcal{L}, \quad (4.16)$$

with momentum canonically conjugate to the field ϕ defined as

$$\pi(\mathbf{x}, t) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}(\mathbf{x}, t)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\mathbf{x}, t)} \quad (4.17)$$

The Hamiltonian defined as

$$H = \int d^3x \left(\pi(\mathbf{x}, t) \dot{\phi}(\mathbf{x}, t) - \mathcal{L} \right), \quad (4.18)$$

led to

$$H = \int d^3x \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right). \quad (4.19)$$

Like the Lagrangian density, we may introduce a Hamiltonian density \mathcal{H} as

$$H = \int d^3x \mathcal{H}(\mathbf{x}, t). \quad (4.20)$$

For our Klein-Gordon system, this is

$$\mathcal{H}(\mathbf{x}, t) = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4. \quad (4.21)$$

Canonical Commutation Relations (CCR) :

We quantize the system by promoting our fields to Heisenberg-Picture (HP) operators, and imposing commutation relations

$$[\hat{\pi}(\mathbf{x}, t), \hat{\phi}(\mathbf{y}, t)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (4.22)$$

which is analogous to

$$[\hat{p}_i, \hat{q}_j] = -i\delta_{ij}. \quad (4.23)$$

To choose a representation, we may map the Ψ of QM \rightarrow to a wave functional $\Psi[\phi]$

$$\hat{\phi}(\mathbf{y}, t)\Psi[\phi] = \phi(\mathbf{y}, t)\Psi[\phi] \quad (4.24)$$

This is similar to the QM wave functions

$$\begin{aligned} \hat{q}_i\Psi(\{q\}) &= q_i\Psi(q) \\ \hat{p}_i\Psi(\{q\}) &= -i\frac{\partial}{\partial q_i}\Psi(p) \end{aligned} \quad (4.25)$$

Our momentum operator is quantized by expressing it in terms of a variational derivative

$$\hat{\pi}(\mathbf{x}, t) = -i\frac{\delta}{\delta\phi(\mathbf{x}, t)}. \quad (4.26)$$

(Fixme: I'm not really sure exactly what is meant by using the variation derivative δ notation here), and to quantize the Hamiltonian we just add hats, assuming that our fields are all now HP operators

$$\hat{\mathcal{H}}(\mathbf{x}, t) = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4}\hat{\phi}^4. \quad (4.27)$$

QM SHO review Recall the QM SHO had a Hamiltonian

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2, \quad (4.28)$$

where

$$[\hat{p}, \hat{q}] = -i, \quad (4.29)$$

and that HP time evolution operators O satisfied

$$\frac{d\hat{O}}{dt} = i[\hat{H}, \hat{O}]. \quad (4.30)$$

In particular

$$\begin{aligned} \frac{d\hat{p}}{dt} &= i[\hat{H}, \hat{p}] \\ &= i\frac{\omega^2}{2}[\hat{q}^2, \hat{p}] \\ &= i\frac{\omega^2}{2}(2i\hat{q}) \\ &= -i\omega^2\hat{q}, \end{aligned} \quad (4.31)$$

and

$$\begin{aligned}
\frac{d\hat{q}}{dt} &= i [\hat{H}, \hat{q}] \\
&= i \frac{1}{2} [\hat{p}^2, \hat{q}] \\
&= \frac{i}{2} (-2i\hat{p}) \\
&= \hat{p}.
\end{aligned} \tag{4.32}$$

Applying the time evolution operator twice, we find

$$\frac{d^2}{dt^2} \hat{q} = \frac{d\hat{p}}{dt} = -\omega^2 \hat{q}. \tag{4.33}$$

We see that the Heisenberg operators obey the classical equations of motion.

Now we want to try this with the quantized QFT fields we've promoted to operators

$$\begin{aligned}
\frac{d\hat{\pi}}{dt}(\mathbf{x}, t) &= i [\hat{H}, \hat{\pi}(\mathbf{x}, t)] \\
&= i \int d^3y \frac{1}{2} [(\nabla \hat{\phi}(\mathbf{y}))^2, \hat{\pi}(\mathbf{x})] + i \int d^3y \frac{m^2}{2} [\hat{\phi}(\mathbf{y})^2, \hat{\pi}(\mathbf{x})] + i \frac{\lambda}{4} \int d^3y [\hat{\phi}(\mathbf{y})^4, \hat{\pi}(\mathbf{x})].
\end{aligned} \tag{4.34}$$

Starting with the non-gradient commutators, and utilizing the HP field analogues of the relations $[\hat{q}^n, \hat{p}] = ni\hat{q}^{n-1}$, we find

$$\int d^3y [(\hat{\phi}(\mathbf{y}))^2, \hat{\pi}(\mathbf{x})] = \int d^3y 2i\hat{\phi}(\mathbf{y})\delta^{(3)}(\mathbf{x} - \mathbf{y}) = 2i\hat{\phi}(\mathbf{x}). \tag{4.35}$$

$$\int d^3y [(\hat{\phi}(\mathbf{y}))^4, \hat{\pi}(\mathbf{x})] = \int d^3y 4i\hat{\phi}(\mathbf{y})^3\delta^{(3)}(\mathbf{x} - \mathbf{y}) = 4i\hat{\phi}(\mathbf{x})^3. \tag{4.36}$$

For the gradient commutators, we have more work. Prof Poppitz blitzed through that, just calling it integration by parts. I had trouble seeing what he was doing, so here's a more explicit dumb expansion required to calculate the commutator

$$\begin{aligned}
\int d^3y (\nabla \hat{\phi}(\mathbf{y}))^2 \hat{\pi}(\mathbf{x}) &= \int d^3y (\nabla \hat{\phi}(\mathbf{y}) \cdot \nabla \hat{\phi}(\mathbf{y})) \hat{\pi}(\mathbf{x}) \\
&= \int d^3y \nabla \hat{\phi}(\mathbf{y}) \cdot (\nabla (\hat{\phi}(\mathbf{y}) \hat{\pi}(\mathbf{x}))) \\
&= \int d^3y \nabla \hat{\phi}(\mathbf{y}) \cdot (\nabla (\hat{\pi}(\mathbf{x}) \hat{\phi}(\mathbf{y}) + i\delta^{(3)}(\mathbf{x} - \mathbf{y}))) \\
&= \int d^3y \left(\nabla (\hat{\phi}(\mathbf{y}) \hat{\pi}(\mathbf{x})) \cdot \nabla \hat{\phi}(\mathbf{y}) + i \nabla \hat{\phi}(\mathbf{y}) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{y}) \right) \\
&= \int d^3y \left(\nabla (\hat{\pi}(\mathbf{x}) \hat{\phi}(\mathbf{y}) + i\delta^{(3)}(\mathbf{x} - \mathbf{y})) \cdot \nabla \hat{\phi}(\mathbf{y}) + i \nabla \hat{\phi}(\mathbf{y}) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{y}) \right) \\
&= \int d^3y \hat{\pi}(\mathbf{x}) (\nabla \hat{\phi}(\mathbf{y}) \cdot \nabla \hat{\phi}(\mathbf{y})) + 2i \int d^3y \nabla \hat{\phi}(\mathbf{y}) \cdot \nabla \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\
&= \int d^3y \hat{\pi}(\mathbf{x}) \nabla^2 \hat{\phi}(\mathbf{y}) + 2i \int d^3y \nabla \cdot (\delta^{(3)}(\mathbf{x} - \mathbf{y}) \nabla \hat{\phi}(\mathbf{y})) - 2i \int d^3y \delta^{(3)}(\mathbf{x} - \mathbf{y}) \nabla^2 \hat{\phi}(\mathbf{y}) \\
&= \int d^3y \hat{\pi}(\mathbf{x}) \nabla^2 \hat{\phi}(\mathbf{y}) + 2i \int_{\partial} d^2y \delta^{(3)}(\mathbf{x} - \mathbf{y}) \hat{\mathbf{n}} \cdot \nabla \hat{\phi}(\mathbf{y}) - 2i \nabla^2 \hat{\phi}(\mathbf{x}).
\end{aligned} \tag{4.37}$$

Here we take advantage of the fact that the derivative operators $\nabla = \nabla_{\mathbf{y}}$ commute with $\hat{\pi}(\mathbf{x})$, and use the identity $\nabla \cdot (a \nabla b) = (\nabla a) \cdot (\nabla b) + a \nabla^2 b$, so the commutator is

$$\begin{aligned}
\int d^3y [(\nabla \hat{\phi}(\mathbf{y}))^2, \hat{\pi}(\mathbf{x})] &= 2i \int_{\partial} d^2y \delta^{(3)}(\mathbf{x} - \mathbf{y}) \hat{\mathbf{n}} \cdot \nabla \hat{\phi}(\mathbf{y}) - 2i \nabla^2 \hat{\phi}(\mathbf{x}) \\
&= -2i \nabla^2 \hat{\phi}(\mathbf{x}),
\end{aligned} \tag{4.38}$$

where the boundary integral is presumed to be zero (without enough justification.) All the pieces can now be put back together

$$\frac{d}{dt} \hat{\pi}(\mathbf{x}, t) = \nabla^2 \hat{\phi}(\mathbf{x}, t) - m^2 \hat{\phi}(\mathbf{x}, t) - \lambda \hat{\phi}^3(\mathbf{x}, t). \tag{4.39}$$

Now, for the $\hat{\phi}$ time evolution, which is much easier

$$\begin{aligned}
 \frac{d\hat{\phi}}{dt}(\mathbf{x}, t) &= i [\hat{H}, \hat{\phi}(\mathbf{x}, t)] \\
 &= i \frac{1}{2} \int d^3y [\hat{\pi}^2(\mathbf{y}), \hat{\phi}(\mathbf{x})] \\
 &= i \frac{1}{2} \int d^3y (-2i) \hat{\pi}(\mathbf{y}, t) \delta^{(3)}(\mathbf{x} - \mathbf{y}) \\
 &= \hat{\pi}(\mathbf{x}, t)
 \end{aligned} \tag{4.40}$$

$$\frac{d^2}{dt^2} \hat{\phi}(\mathbf{x}, t) = \nabla^2 \hat{\phi} - m^2 \hat{\phi} - \lambda \hat{\phi}^3. \tag{4.41}$$

That is

$$\ddot{\hat{\phi}} - \nabla^2 \hat{\phi} + m^2 \hat{\phi} + \lambda \hat{\phi}^3 = 0, \tag{4.42}$$

which is the classical Euler-Lagrange equation, also obeyed by the Heisenberg operator $\phi(\mathbf{x}, t)$. When $\lambda = 0$ this is the Klein-Gordon equation.

4.3 MOMENTUM SPACE REPRESENTATION.

Dropping hats, we now consider the momentum space representation of our operators, as determined by Fourier transform pairs

$$\begin{aligned}
 \phi(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{\phi}(\mathbf{p}, t) \\
 \tilde{\phi}(\mathbf{p}, t) &= \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{x}, t).
 \end{aligned} \tag{4.43}$$

We can discover a representation of the delta function by applying these both in turn

$$\tilde{\phi}(\mathbf{p}, t) = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \tilde{\phi}(\mathbf{q}, t) \tag{4.44}$$

so

$$\int d^3x e^{i\mathbf{A}\cdot\mathbf{x}} = (2\pi)^3 \delta^{(3)}(\mathbf{A}) \tag{4.45}$$

Also observe that $\phi^*(\mathbf{x}, t) = \phi(\mathbf{x}, t)$ iff $\tilde{\phi}(\mathbf{p}, t) = \tilde{\phi}^*(-\mathbf{p}, t)$.

We want the equations of motion for $\tilde{\phi}(\mathbf{p}, t)$ where the operator obeys the Klein-Gordon equation

$$(\partial_t^2 - \nabla^2 + m^2) \phi(\mathbf{x}, t) = 0 \tag{4.46}$$

Inserting the transform relation eq. (4.43) we get

$$\int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \left(\ddot{\tilde{\phi}}(\mathbf{p}, t) + (\mathbf{p}^2 + m^2) \tilde{\phi}(\mathbf{p}, t) \right) = 0, \quad (4.47)$$

or

$$\boxed{\ddot{\tilde{\phi}}(\mathbf{p}, t) = -\omega_{\mathbf{p}}^2 \tilde{\phi}(\mathbf{p}, t),} \quad (4.48)$$

where

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \quad (4.49)$$

The Fourier components of the HP operators are SHOs!

As we have SHO's and know how to deal with these in QM, we use the same strategy, introducing raising and lowering operators

$$\tilde{\phi}(\mathbf{p}, t) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^\dagger \right) \quad (4.50)$$

Observe that

$$\begin{aligned} \tilde{\phi}^\dagger(-\mathbf{p}, t) &= \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^\dagger + e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} \right) \\ &= \tilde{\phi}(\mathbf{p}, t), \end{aligned} \quad (4.51)$$

or

$$\tilde{\phi}^\dagger(\mathbf{p}, t) = \tilde{\phi}(-\mathbf{p}, t), \quad (4.52)$$

so $\phi(\mathbf{p}, t)$ has a real representation in terms of $a_{\mathbf{p}}$.

We will find (Wednesday) that

$$[a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad (4.53)$$

which are equivalent to

$$[\tilde{\pi}(\mathbf{p}, t), \tilde{\phi}(\mathbf{q}, t)] = -i\delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (4.54)$$

4.4 QUANTIZATION OF FIELD THEORY.

We are engaging in the “canonical” or Hamiltonian method of quantization. It is also possible to quantize using path integrals, but it is hard to prove that operators are unitary doing so. In fact, the mechanism used to show unitarity from path integrals is often to find the Lagrangian and show that there is a Hilbert space (i.e. using canonical quantization). Canonical quantization essentially demands that the fields obey a commutator relation of the following form

$$[\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (4.55)$$

We assumed that the quantized fields obey the Hamiltonian relations

$$\begin{aligned} \frac{d\phi}{dt} &= i[H, \phi] \\ \frac{d\pi}{dt} &= i[H, \pi]. \end{aligned} \quad (4.56)$$

We were working with the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}(\pi(\mathbf{x}, t))^2 + \frac{1}{2}(\nabla\phi(\mathbf{x}, t))^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4, \quad (4.57)$$

which included a mass term m and a potential term (λ). We will expand all quantities in Taylor series in λ assuming they have a structure such as

$$f(\lambda) = c_0\lambda^0 + c_1\lambda^1 + c_2\lambda^2 + c_3\lambda^3 + \dots \quad (4.58)$$

We will stop this perturbation theory approach at $O(\lambda^2)$, and will ignore functions such as $e^{-1/\lambda}$.

Within perturbation theory, to leading order, set $\lambda = 0$, so that ϕ obeys the Klein-Gordon equation (if $m = 0$ we have just a d’Alembertian (wave equation)).

We can write our field as a Fourier transform

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \tilde{\phi}(\mathbf{p}, t), \quad (4.59)$$

and due to a Hermitian assumption (i.e. real field) this implies

$$\tilde{\phi}^*(\mathbf{p}, t) = \tilde{\phi}(-\mathbf{p}, t). \quad (4.60)$$

We found that the Klein-Gordon equation implied that the momentum space representation obey Harmonic oscillator equations

$$\ddot{\tilde{\phi}}(\mathbf{p}, t) = -\omega_{\mathbf{p}}^2 \tilde{\phi}(\mathbf{p}, t), \quad (4.61)$$

where $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. The solution of eq. (4.61) may be represented as

$$\tilde{\phi}(\mathbf{q}, t) = \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} \left(e^{-i\omega_{\mathbf{q}}t} a_{\mathbf{q}} + e^{i\omega_{\mathbf{q}}t} b_{\mathbf{q}}^{\dagger} \right). \quad (4.62)$$

This is a general solution, but imposing $a_{\mathbf{q}} = b_{-\mathbf{q}}$ ensures eq. (4.60) is satisfied. This leaves us with

$$\tilde{\phi}(\mathbf{q}, t) = \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} \left(e^{-i\omega_{\mathbf{q}}t} a_{\mathbf{q}} + e^{i\omega_{\mathbf{q}}t} a_{-\mathbf{q}}^{\dagger} \right). \quad (4.63)$$

We want to show that iff

$$[a_{\mathbf{q}}, a_{\mathbf{p}}^{\dagger}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad (4.64)$$

then

$$[\pi(\mathbf{y}, t), \phi(\mathbf{x}, t)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (4.65)$$

where everything else commutes (i.e. $[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^{\dagger}, a_{\mathbf{q}}^{\dagger}] = 0$). We will only show one direction, but you can get the other way too.

$$\phi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p} \cdot \mathbf{x}} \left(e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^{\dagger} \right) \quad (4.66)$$

$$\begin{aligned} \pi(\mathbf{x}, t) &= \dot{\phi} \\ &= i \int \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} \omega_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{x}} \left(-e^{-i\omega_{\mathbf{q}}t} a_{\mathbf{q}} + e^{i\omega_{\mathbf{q}}t} a_{-\mathbf{q}}^{\dagger} \right). \end{aligned} \quad (4.67)$$

The commutator is

$$\begin{aligned} &[\pi(\mathbf{y}, t), \phi(\mathbf{x}, t)] \\ &= i \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} \omega_{\mathbf{q}} e^{i\mathbf{p} \cdot \mathbf{y} + i\mathbf{q} \cdot \mathbf{x}} \left[-e^{-i\omega_{\mathbf{q}}t} a_{\mathbf{q}} + e^{i\omega_{\mathbf{q}}t} a_{-\mathbf{q}}^{\dagger}, e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^{\dagger} \right] \\ &= i \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} \omega_{\mathbf{q}} e^{i\mathbf{p} \cdot \mathbf{y} + i\mathbf{q} \cdot \mathbf{x}} \left(-e^{i(\omega_{\mathbf{p}} - \omega_{\mathbf{q}})t} [a_{\mathbf{q}}, a_{-\mathbf{p}}^{\dagger}] + e^{i(\omega_{\mathbf{q}} - \omega_{\mathbf{p}})t} [a_{-\mathbf{q}}^{\dagger}, a_{\mathbf{p}}] \right) \\ &= i \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} \omega_{\mathbf{q}} (2\pi)^3 e^{i\mathbf{p} \cdot \mathbf{y} + i\mathbf{q} \cdot \mathbf{x}} \left(-e^{i(\omega_{\mathbf{p}} - \omega_{\mathbf{q}})t} \delta^{(3)}(\mathbf{q} + \mathbf{p}) - e^{i(\omega_{\mathbf{q}} - \omega_{\mathbf{p}})t} \delta^{(3)}(-\mathbf{q} - \mathbf{p}) \right) \\ &= -2i \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} \omega_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{y} - \mathbf{x})} \\ &= -i\delta^{(3)}(\mathbf{y} - \mathbf{x}), \end{aligned} \quad (4.68)$$

which is what we wanted to prove.

4.5 FREE HAMILTONIAN.

We call the $\lambda = 0$ case the “free” Hamiltonian. Plugging in the creation and annihilation operator representation we have

$$\begin{aligned}
 H &= \int d^3x \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right) \\
 &= \frac{1}{2} \int d^3x \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}}{\sqrt{2\omega_{\mathbf{p}}} \sqrt{2\omega_{\mathbf{q}}}} \left(\right. \\
 &\quad - (\omega_{\mathbf{p}})(\omega_{\mathbf{q}}) \left(-e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^\dagger \right) \left(-e^{-i\omega_{\mathbf{q}}t} a_{\mathbf{q}} + e^{i\omega_{\mathbf{q}}t} a_{-\mathbf{q}}^\dagger \right) \\
 &\quad - \mathbf{p} \cdot \mathbf{q} \left(e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^\dagger \right) \left(e^{-i\omega_{\mathbf{q}}t} a_{\mathbf{q}} + e^{i\omega_{\mathbf{q}}t} a_{-\mathbf{q}}^\dagger \right) \\
 &\quad \left. + m^2 \left(e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^\dagger \right) \left(e^{-i\omega_{\mathbf{q}}t} a_{\mathbf{q}} + e^{i\omega_{\mathbf{q}}t} a_{-\mathbf{q}}^\dagger \right) \right). \tag{4.69}
 \end{aligned}$$

An immediate simplification is possible by identifying a delta function factor $\int d^3x e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} / (2\pi)^3 = \delta^{(3)}(\mathbf{p} + \mathbf{q})$, so

$$\begin{aligned}
 H &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left(-(\omega_{\mathbf{p}})^2 \left(-e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^\dagger \right) \left(-e^{-i\omega_{\mathbf{p}}t} a_{-\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{\mathbf{p}}^\dagger \right) \right. \\
 &\quad \left. + (\mathbf{p}^2 + m^2) \left(e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^\dagger \right) \left(e^{-i\omega_{\mathbf{p}}t} a_{-\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{\mathbf{p}}^\dagger \right) \right) \\
 &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left(a_{\mathbf{p}} a_{-\mathbf{p}} \left(-\omega_{\mathbf{p}}^2 e^{-2i\omega_{\mathbf{p}}t} + \omega_{\mathbf{p}}^2 e^{-2i\omega_{\mathbf{p}}t} \right) + a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger \left(-\omega_{\mathbf{p}}^2 e^{2i\omega_{\mathbf{p}}t} + \omega_{\mathbf{p}}^2 e^{2i\omega_{\mathbf{p}}t} \right) \right. \\
 &\quad \left. + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \omega_{\mathbf{p}}^2 (1 + 1) + a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} \omega_{\mathbf{p}}^2 (1 + 1) \right) \tag{4.70}
 \end{aligned}$$

When all is said and done we are left with

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\mathbf{p}}}{2} \left(a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \right). \tag{4.71}$$

A final $\mathbf{p} \rightarrow -\mathbf{p}$ transformation¹ in the first integral, puts the free Hamiltonian ($\lambda = 0$) into a nice symmetric form

$$\boxed{H_0 = \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\mathbf{p}}}{2} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \right).} \tag{4.72}$$

¹ $\iiint_{-R}^R d^3p \rightarrow (-1)^3 \iiint_R^{-R} d^3p' = (-1)^6 \iiint_{-R}^R d^3p'.$

Vacuum energy density. From the commutator relationship eq. (4.64) we can write

$$a_{\mathbf{p}}a_{\mathbf{q}}^{\dagger} = a_{\mathbf{q}}^{\dagger}a_{\mathbf{p}} + (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad (4.73)$$

so

$$H_0 = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0) \right). \quad (4.74)$$

The delta function term can be interpreted using

$$(2\pi)^3\delta^{(3)}(\mathbf{q}) = \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}}, \quad (4.75)$$

so when $\mathbf{q} = 0$

$$(2\pi)^3\delta^{(3)}(0) = \int d^3x = V. \quad (4.76)$$

We can write the Hamiltonian now in terms of the volume

$$H_0 = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + V_3 \int \frac{d^3p}{(2\pi)^3} \frac{\omega_{\mathbf{p}}}{2} \times 1. \quad (4.77)$$

4.6 QM SHO REVIEW.

In units with $m = 1$ the non-relativistic QM SHO has the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2. \quad (4.78)$$

If we define a position operator with a time-domain Fourier representation given by

$$q = \frac{1}{\sqrt{2\omega}} \left(a e^{-i\omega t} + a^{\dagger} e^{i\omega t} \right), \quad (4.79)$$

where the Fourier coefficients a, a^{\dagger} are operator valued, then the momentum operator is

$$p = \dot{q} = \frac{i\omega}{\sqrt{2\omega}} \left(-a e^{-i\omega t} + a^{\dagger} e^{i\omega t} \right), \quad (4.80)$$

or inverting for a, a^{\dagger}

$$\begin{aligned} a &= \sqrt{\frac{\omega}{2}} \left(q - \frac{1}{i\omega} p \right) e^{-i\omega t} \\ a^{\dagger} &= \sqrt{\frac{\omega}{2}} \left(q + \frac{1}{i\omega} p \right) e^{i\omega t}. \end{aligned} \quad (4.81)$$

By inspection it is apparent that the product $a^\dagger a$ will be related to the Hamiltonian (i.e. a difference of squares). That product is

$$\begin{aligned} a^\dagger a &= \frac{\omega}{2} \left(q + \frac{1}{i\omega} p \right) \left(q - \frac{1}{i\omega} p \right) \\ &= \frac{\omega}{2} \left(q^2 + \frac{1}{\omega^2} p^2 - \frac{1}{i\omega} [q, p] \right) \\ &= \frac{1}{2\omega} (p^2 + \omega^2 q^2 - \omega), \end{aligned} \quad (4.82)$$

or

$$H = \omega \left(a^\dagger a + \frac{1}{2} \right). \quad (4.83)$$

We can glean some of the properties of a, a^\dagger by computing the commutator of p, q , since that has a well known value

$$\begin{aligned} i &= [q, p] \\ &= \frac{i\omega}{2\omega} [ae^{-i\omega t} + a^\dagger e^{i\omega t}, -ae^{-i\omega t} + a^\dagger e^{i\omega t}] \\ &= \frac{i}{2} ([a, a^\dagger] - [a^\dagger, a]) \\ &= i[a, a^\dagger], \end{aligned} \quad (4.84)$$

so

$$[a, a^\dagger] = 1. \quad (4.85)$$

The operator $a^\dagger a$ is the workhorse of the Hamiltonian and worth studying independently. In particular, assume that we have a set of states $|n\rangle$ that are eigenstates of $a^\dagger a$ with eigenvalues λ_n , that is

$$a^\dagger a |n\rangle = \lambda_n |n\rangle. \quad (4.86)$$

The action of $a^\dagger a$ on $a^\dagger |n\rangle$ is easy to compute

$$\begin{aligned} a^\dagger a a^\dagger |n\rangle &= a^\dagger (a^\dagger a + 1) |n\rangle \\ &= (\lambda_n + 1) a^\dagger |n\rangle, \end{aligned} \quad (4.87)$$

so $\lambda_n + 1$ is an eigenvalue of $a^\dagger |n\rangle$. The state $a^\dagger |n\rangle$ has an energy eigenstate that is one unit of energy larger than $|n\rangle$. For this reason we called a^\dagger the raising (or creation) operator. Similarly,

$$\begin{aligned} a^\dagger a a |n\rangle &= (a a^\dagger - 1) a |n\rangle \\ &= (\lambda_n - 1) a |n\rangle, \end{aligned} \quad (4.88)$$

so $\lambda_n - 1$ is the energy eigenvalue of $a|n\rangle$, having one less unit of energy than $|n\rangle$. We call a the annihilation (or lowering) operator. If we argue that there is a lowest energy state, perhaps designated as $|0\rangle$ then we must have

$$a|0\rangle = 0, \quad (4.89)$$

by the assumption that there are no energy eigenstates with less energy than $|0\rangle$. We can think of higher order states being constructed from the ground state from using the raising operator a^\dagger

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle. \quad (4.90)$$

4.7 DISCUSSION.

We've diagonalized in the Fourier representation for the momentum space fields. For every value of momentum \mathbf{p} we have a quantum SHO.

For our field space we call our space the Fock vacuum and

$$a_{\mathbf{p}}|0\rangle = 0, \quad (4.91)$$

and call $a_{\mathbf{p}}$ the “annihilation operator”, and call $a_{\mathbf{p}}^\dagger$ the “creation operator”. We say that $a_{\mathbf{p}}^\dagger|0\rangle$ is the creation of a state of a single particle of momentum \mathbf{p} by $a_{\mathbf{p}}^\dagger$.

We are discarding the volume term, a procedure called “normal ordering”. We define

$$: \frac{a^\dagger a + a a^\dagger}{2} : \equiv a^\dagger a. \quad (4.92)$$

We are essentially forgetting the vacuum energy as some sort of unobservable quantity, leaving us with the free Hamiltonian of

$$H_0 = \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \quad (4.93)$$

Consider

$$\begin{aligned} H_0 a_{\mathbf{q}}^\dagger |0\rangle &= \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}}^\dagger |0\rangle \\ &= \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})) |0\rangle \\ &= \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger (a_{\mathbf{q}}^\dagger a_{\mathbf{p}} |0\rangle + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle) \\ &= \omega_{\mathbf{q}} a_{\mathbf{q}}^\dagger |0\rangle. \end{aligned} \quad (4.94)$$

Question: Is it possible to modify the Lagrangian or Hamiltonian that we start with so that this vacuum ground state is eliminated? Answer: Only by imposing super-symmetric constraints (that pairs this (bosonic) Hamiltonian to a fermionic system in a way that there is exact cancellation).

We will see that the momentum operator has the form

$$\mathcal{P} = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \quad (4.95)$$

We say that $a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger |0\rangle$ a two particle space with energy $\omega_{\mathbf{p}} + \omega_{\mathbf{q}}$, and $(a_{\mathbf{p}}^\dagger)^m (a_{\mathbf{q}}^\dagger)^n |0\rangle \equiv (a_{\mathbf{p}}^\dagger)^m |0\rangle \otimes (a_{\mathbf{q}}^\dagger)^n |0\rangle$, a $m + n$ particle space.

There is a connection to statistical mechanics that is of interest

$$\begin{aligned} \langle E \rangle &= \frac{1}{Z} \sum_n E_n e^{-E_n/k_B T} \\ &= \frac{1}{Z} \sum_n \langle n | e^{-\hat{H}/k_B T} \hat{H} | n \rangle, \end{aligned} \quad (4.96)$$

so for a SHO Hamiltonian system

$$\begin{aligned} \langle E \rangle &= \frac{1}{Z} \sum_n e^{-E_n/k_B T} \langle n | \hat{H} | n \rangle \\ &= \frac{1}{Z} \sum_n e^{-E_n/k_B T} \langle n | \omega a^\dagger a | n \rangle \\ &= \frac{\omega}{e^{\omega/k_B T} - 1} \\ &= \langle \omega a^\dagger a \rangle_{k_B T}, \end{aligned} \quad (4.97)$$

which is the $k_B T$ ensemble average energy for a SHO system. Note that this sum was evaluated by noting that $\langle n | a^\dagger a | n \rangle = n$ which leaves sums of the form

$$\begin{aligned} \frac{\sum_{n=0}^{\infty} n a^n}{\sum_{n=0}^{\infty} a^n} &= a \frac{\sum_{n=1}^{\infty} n a^{n-1}}{\sum_{n=0}^{\infty} a^n} \\ &= a(1-a) \frac{d}{da} \left(\frac{1}{1-a} \right) \\ &= \frac{a}{1-a}. \end{aligned} \quad (4.98)$$

If we consider a real scalar field of mass m we have $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, but for a Maxwell field \mathbf{E}, \mathbf{B} where $m = 0$, our dispersion relation is $\omega_{\mathbf{p}} = \|\mathbf{p}\|$.

We will see that for a free Maxwell field (no charges or currents) the Hamiltonian is

$$H_{\text{Maxwell}} = \sum_{i=1}^2 \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^{i\dagger} a_{\mathbf{p}}^i, \quad (4.99)$$

where i is a polarization index.

We expect that we can evaluate an average such as eq. (4.97) for our field, and operate using the analogy

$$\begin{aligned} aa^\dagger &= a^\dagger a + 1 \\ a_{\mathbf{p}} a_{\mathbf{p}}^\dagger &= a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + V_3. \end{aligned} \quad (4.100)$$

so if we rescale by $\sqrt{V_3}$

$$a_{\mathbf{p}} = \sqrt{V_3} \tilde{a}_{\mathbf{p}}, \quad (4.101)$$

then we have commutator relations like standard QM

$$\tilde{a} \tilde{a}^\dagger = \tilde{a}^\dagger \tilde{a} + 1. \quad (4.102)$$

So we can immediately evaluate the energy expectation for our quantized fields

$$\begin{aligned} \langle H_0 \rangle &= \left\langle \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right\rangle \\ &= \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} V_3 \langle \tilde{a}_{\mathbf{p}}^\dagger \tilde{a}_{\mathbf{p}} \rangle \\ &= V_3 \int \frac{d^3 p}{(2\pi)^3} \frac{\omega_{\mathbf{p}}}{e^{\omega_{\mathbf{p}}/k_B T} - 1}. \end{aligned} \quad (4.103)$$

Using this with the Maxwell field, we have a factor of two from polarization

$$U^{\text{Maxwell}} = 2V_3 \int \frac{d^3 p}{(2\pi)^3} \frac{\|\mathbf{p}\|}{e^{\omega_{\mathbf{p}}/k_B T} - 1}, \quad (4.104)$$

which is Planck's law describing the blackbody energy spectrum.

4.8 PROBLEMS.

Exercise 4.1 Scalar field creation operator commutator.

In [8] it is stated that the creation operators of eq. 2.78

$$\alpha_k = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left(\phi(x, 0) + \frac{i}{\omega_k} \partial_0 \phi(x, 0) \right) e^{-ik \cdot x} \quad (4.105)$$

associated with field operator ϕ commute. Verify that.

Answer for Exercise 4.1

$$\begin{aligned}
[\alpha_k, \alpha_m] &= \frac{1}{4} \frac{1}{(2\pi)^6} \int d^3x d^3y e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{m}\cdot\mathbf{y}} \left[\phi(x, 0) + \frac{i}{\omega_k} \partial_0 \phi(x, 0), \phi(y, 0) + \frac{i}{\omega_m} \partial_0 \phi(y, 0) \right] \\
&= \frac{i}{4} \frac{1}{(2\pi)^6} \int d^3x d^3y e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{m}\cdot\mathbf{y}} \left(\left[\phi(x, 0), \frac{1}{\omega_m} \partial_0 \phi(y, 0) \right] + \left[\frac{1}{\omega_k} \partial_0 \phi(x, 0), \phi(y, 0) \right] \right) \\
&= \frac{i}{4} \frac{1}{(2\pi)^6} \int d^3x d^3y e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{m}\cdot\mathbf{y}} \left(\frac{i}{\omega_m} \delta^{(3)}(\mathbf{x} - \mathbf{y}) - \frac{i}{\omega_k} \delta^{(3)}(\mathbf{x} - \mathbf{y}) \right) \\
&= -\frac{1}{4} \frac{1}{(2\pi)^6} \int d^3x e^{-i(\mathbf{k}+\mathbf{m})\cdot\mathbf{x}} \left(\frac{1}{\omega_m} - \frac{1}{\omega_k} \right) \\
&= -\frac{1}{4} \frac{1}{(2\pi)^3} \left(\frac{1}{\omega_m} - \frac{1}{\omega_k} \right) \delta^{(3)}(\mathbf{k} + \mathbf{m}) \\
&= -\frac{1}{4} \frac{1}{(2\pi)^3} \left(\frac{1}{\omega_{\|\mathbf{k}+\mathbf{m}\|}} - \frac{1}{\omega_{\|\mathbf{k}\|}} \right) \delta^{(3)}(\mathbf{k} + \mathbf{m}) \\
&= 0.
\end{aligned} \tag{4.106}$$

Exercise 4.2

In [8] it is left as an exercise to expand the scalar field Hamiltonian in terms of the raising and lowering operators. Let's do that.

Answer for Exercise 4.2

The field operator expanded in terms of the raising and lowering operators is

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} (a_{\mathbf{k}} e^{-ik\cdot x} + a_{\mathbf{k}}^\dagger e^{ik\cdot x}) \\
&= \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} (a_{\mathbf{k}} e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}}) \\
&= \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} (a_{\mathbf{k}} e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t + i\mathbf{k}\cdot\mathbf{x}}) \\
&= \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} (a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) e^{i\mathbf{k}\cdot\mathbf{x}}.
\end{aligned} \tag{4.107}$$

Note that x and k here are both four-vectors, so this field is dependent on a spacetime point, but the integration is over a spatial volume. This is discussed in the class notes but also justified nicely in [14] using the structure of the raising and lower operators. The trick of reversing the sign above is also from that text.

The Hamiltonian in terms of the fields was

$$H = \frac{1}{2} \int d^3x \left(\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2 \right). \quad (4.108)$$

The field derivatives are

$$\begin{aligned} \pi &= \partial_0\phi \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left(a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= i \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} \left(-a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t} \right) e^{i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \quad (4.109)$$

and

$$\begin{aligned} \partial_n\phi &= \partial_n \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \left(a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= i \int \frac{d^3k}{(2\pi)^{3/2}} \frac{k^n}{\sqrt{2\omega_k}} \left(a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t} \right) e^{i\mathbf{k}\cdot\mathbf{x}}. \end{aligned} \quad (4.110)$$

Introducing a second set of momentum variables \mathbf{j} , the momentum portion of the Hamiltonian is

$$\begin{aligned} \frac{1}{2} \int d^3x \pi^2 &= -\frac{1}{2} \frac{1}{(2\pi)^3} \int d^3x \int d^3j d^3k \frac{\omega_j \omega_k}{2} \left(-a_{\mathbf{j}} e^{-i\omega_j t} + a_{-\mathbf{j}}^\dagger e^{i\omega_j t} \right) \left(-a_{\mathbf{k}} e^{-i\omega_k t} \right. \\ &\quad \left. + a_{-\mathbf{k}}^\dagger e^{i\omega_k t} \right) e^{i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{j}\cdot\mathbf{x}} \\ &= -\frac{1}{2} \int d^3j d^3k \frac{\omega_j \omega_k}{2} \left(-a_{\mathbf{j}} e^{-i\omega_j t} + a_{-\mathbf{j}}^\dagger e^{i\omega_j t} \right) \left(-a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t} \right) \delta^{(3)}(\mathbf{k} + \mathbf{j}) \\ &= -\frac{1}{2} \int d^3k \frac{\omega_k^2}{2} \left(-a_{-\mathbf{k}} e^{-i\omega_k t} + a_{\mathbf{k}}^\dagger e^{i\omega_k t} \right) \left(-a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t} \right) \\ &= -\frac{1}{4} \int d^3k \omega_k \left(a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} + a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - a_{-\mathbf{k}} a_{-\mathbf{k}}^\dagger \right). \end{aligned} \quad (4.111)$$

For the gradient portion of the Hamiltonian we have

$$\begin{aligned}
\frac{1}{2} \int d^3x (\nabla\phi)^2 &= -\frac{1}{2} \frac{1}{(2\pi)^3} \int d^3x \int d^3j d^3k \frac{1}{\sqrt{4\omega_j\omega_k}} \left(\sum_{n=1}^3 j^n k^n \right) (a_{\mathbf{j}} e^{-i\omega_j t} + a_{-\mathbf{j}}^\dagger e^{i\omega_j t}) (a_{\mathbf{k}} e^{-i\omega_k t} \\
&\quad + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) e^{i\mathbf{j}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= -\frac{1}{2} \int d^3j d^3k \frac{1}{\sqrt{4\omega_j\omega_k}} \mathbf{j} \cdot \mathbf{k} (a_{\mathbf{j}} e^{-i\omega_j t} + a_{-\mathbf{j}}^\dagger e^{i\omega_j t}) (a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) \delta^{(3)}(\mathbf{j} + \mathbf{k}) \\
&= \frac{1}{2} \int d^3k \frac{1}{\sqrt{4\omega_k\omega_k}} \mathbf{k}^2 (a_{-\mathbf{k}} e^{-i\omega_k t} + a_{\mathbf{k}}^\dagger e^{i\omega_k t}) (a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) \\
&= \frac{1}{4} \int d^3k \frac{\mathbf{k}^2}{\omega_k} (a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} + a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}} a_{-\mathbf{k}}^\dagger).
\end{aligned} \tag{4.112}$$

Finally, for the mass term, we have

$$\begin{aligned}
\frac{1}{2} \int d^3x \mu^2 \phi^2 &= \frac{\mu^2}{2} \frac{1}{(2\pi)^3} \int d^3x \int d^3j d^3k \frac{1}{\sqrt{4\omega_j\omega_k}} (a_{\mathbf{j}} e^{-i\omega_j t} + a_{-\mathbf{j}}^\dagger e^{i\omega_j t}) (a_{\mathbf{k}} e^{-i\omega_k t} \\
&\quad + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) e^{i\mathbf{j}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \frac{\mu^2}{2} \int d^3j d^3k \frac{1}{\sqrt{4\omega_j\omega_k}} (a_{\mathbf{j}} e^{-i\omega_j t} + a_{-\mathbf{j}}^\dagger e^{i\omega_j t}) (a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) \delta^{(3)}(\mathbf{j} + \mathbf{k}) \\
&= \frac{\mu^2}{2} \int d^3k \frac{1}{2\omega_k} (a_{-\mathbf{k}} e^{-i\omega_k t} + a_{\mathbf{k}}^\dagger e^{i\omega_k t}) (a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) \\
&= \frac{\mu^2}{4} \int d^3k \frac{1}{\omega_k} (a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} + a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} + a_{-\mathbf{k}} a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}).
\end{aligned} \tag{4.113}$$

Now all the pieces can be put back together again

$$\begin{aligned}
H &= \frac{1}{4} \int d^3k \frac{1}{\omega_k} (\\
&\quad -\omega_k^2 (a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} + a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - a_{-\mathbf{k}} a_{-\mathbf{k}}^\dagger) \\
&\quad + \mathbf{k}^2 (a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} + a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}} a_{-\mathbf{k}}^\dagger) \\
&\quad + \mu^2 (a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} + a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} + a_{-\mathbf{k}} a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}})) \\
&= \frac{1}{4} \int d^3k \frac{1}{\omega_k} (a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} (-\omega_k^2 + \mathbf{k}^2 + \mu^2) \\
&\quad + a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} (-\omega_k^2 + \mathbf{k}^2 + \mu^2) \\
&\quad + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger (\omega_k^2 + \mathbf{k}^2 + \mu^2) \\
&\quad + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} (\omega_k^2 + \mathbf{k}^2 + \mu^2)).
\end{aligned} \tag{4.114}$$

Expression eq. (4.117) gives the zero point energy of the field in a spatial volume V_3 . This energy is, of course, infinite and is usually discarded (as we learned, by applying a “normal ordering” procedure) as unobservable. Nevertheless, there are circumstances under which *changes* in the zero point energy lead to measurable effects. The most celebrated example is the *Casimir effect*², predicted by Casimir in 1948 [2] and discovered experimentally in 1958 (see Lamoreaux’s more recent article linked to in the “Summary of Sept. 25th class”). Another instance where this has been “observed” (in numerical simulations) is the L’uscher term in the confining string in QCD. Casimir energies generally also appear whenever the topology of space(time) is changed and people have speculated that dark energy may have something to do with that...

The Casimir effect can be described very simply (!): the zero point energy of the electromagnetic field between two infinite conducting plates is smaller than it would be in the absence of the plates. This is because the boundary conditions on the plates eliminate some of the modes of the field that would be otherwise present. The vacuum energy in the space between the plates should be proportional to the area A of the plates, as well as to \hbar (as zero point energies are proportional to \hbar). It can also depend on a , the distance between the plates, and the speed of light c . By dimensional analysis, the excess energy (negative) in the volume aA between the plates should be

$$\Delta E_{vac}(a) \sim -aA \frac{\hbar c}{a^4} = -A \frac{\hbar c}{a^3}, \quad (4.118)$$

where the aA factor is the volume, \hbar has dimensions of energy \times time, c/a has dimensions of inverse time, and the extra factor of $1/a^3$ is there to make the dimension of energy right. Thus, to minimize E_{vac} the plates “want to” get closer. In other words, there should be an attractive force per unit area of the plates, called “Casimir pressure”

$$p_{\text{Casimir}} \sim \frac{\hbar c}{a^4}, \quad (4.119)$$

proportional to the inverse fourth power of the distance between the plates. In what follows we shall calculate this force.

We will use our real scalar *massless* field theory as a model for the real thing (the electromagnetic field, that we have not formally learned how to quantize yet). Casimir considered two infinite, conducting plates stretching in the y, z plane and located at $x = 0$ and $x = a$, respectively; furthermore, he used perfect conductor boundary conditions on the plates. These require that the tangential component of the vector potential, $\mathbf{A}_{\text{tang.}}$, vanishes at the plates (in Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, $A^0 = 0$). Our two toy “conducting plates” will be made of a “material” that requires that the scalar field ϕ vanish at the plates.

² Notice that just like for the Planck derivation of blackbody radiation formula, where some people would say that it does not imply that the electromagnetic radiation is quantized, but only its sources (as radiation is emitted by the atoms of the cavity), there are similar claims for the Casimir force (my take is to ignore these, as we know that the radiation is quantized). See article by Lamoreaux that I put a link to online.

- a. Show that the boundary conditions on the plates impose a quantization condition on the allowed values of field momentum perpendicular to the plates, i.e. $k_x = n\pi/a, n = 0, \pm 1, \pm 2, \dots$ [e.g., recall your waveguide physics].
- b. Consider now the contribution to the energy of the vacuum fluctuations of the field in the space between the plates and find the zero point energy per unit area of the plates... Consider now the contribution to the energy of the vacuum fluctuations of the field in the space between the plates and find the zero point energy per unit area of the plates. To do this, replace the integral over k_x in eq. (4.117) by a sum over n , $\int dk_x = (\pi/a) \sum_n$ [Hint: to save work, use the fact that the correct expression should have the property that as the plates are removed, $a \rightarrow \infty$, the energy (per unit volume) should give back eq. (4.117)]. Does the resulting expression for the zero point energy still diverge?
- c. Show now, starting from eq. (4.117), with integral replaced by sum, that the difference between the zero point energies per unit area, in the space between the plates in the presence of the plates and without the plates is:

$$\Delta E_{\text{vac}}(a) = \hbar c \int_0^\infty \frac{dk}{2\pi} k \left(\frac{k}{4} + \frac{1}{2} \sum_{n=1}^\infty \sqrt{k^2 + \frac{n^2 \pi^2}{a^2}} - \frac{1}{2} \int_0^\infty dn \sqrt{k^2 + \frac{n^2 \pi^2}{a^2}} \right).$$

where, obviously, k is radial wave vector in y, z -directions.

- d. The expression eq. (4.120) is still ill-defined, as every single term is infinite. Now, to make progress, we note that the idealization of perfect conducting plates and the corresponding macroscopic boundary conditions do not make sense for wavelengths smaller than the atomic size. In particular, for frequencies above $1/a_0$ (a_0 is of the order of the Bohr radius) the conducting plates are totally invisible for the electromagnetic field. To incorporate this in our calculation, introduce a function $f(k)$ into the integrand in eq. (4.120) such that $f(k) = 1$ for $k < 1/a_0$ and $f(k) = 0$ for $k > 1/a_0$, somehow smoothly interpolating between these two values. The integrals in eq. (4.120) thus become absolutely convergent—all momenta larger than the inverse Bohr size are cut off. Show that eq. (4.120) (with the cutoff $f(k)$ as described in the original problem spec) can be written as:

$$\Delta E_{\text{vac}}(a) = \frac{\hbar c \pi^2}{8a^3} \left(\frac{1}{2} F(0) + \sum_{n=1}^\infty F(n) - \int_0^\infty dn F(n) \right), \quad (4.121)$$

where

$$F(n) = \int_0^\infty du \sqrt{u + n^2} f((\pi/a) \sqrt{u + n^2}). \quad (4.122)$$

- e. To calculate eq. (4.121), use the Euler-Maclaurin formula:³

$$\frac{1}{2}F(0) + F(1) + F(2) + \dots - \int_{n=0}^{\infty} dn F(n) = -\frac{1}{2!}B_2 F'(0) - \frac{1}{4!}B_4 F'''(0) + \dots, \quad (4.123)$$

where $B_2 = 1/6$, $B_4 = -1/30$, etc. are Bernoulli numbers, and primes denote derivatives. Now, $f(0) = 1$ as stated above; furthermore, assume that all derivatives of our smearing function $f(k)$ vanish at zero (it is not difficult to construct examples of such functions). Show that $F'(0) = 0$, $F'''(0) = -4$, and that all higher derivatives of F vanish.

Thus the “cutoff” function f does not enter the final result—or the fact that we assumed a cutoff at scales of order the inverse Bohr radius; it only mattered that $a_0 \ll L$.

- f. Show, now, that the final result for the Casimir energy per unit area of the plates is:

$$\Delta E_{vac.}(a) = \frac{\pi^2}{2a^3} \frac{B_4}{4!} = -\frac{\pi^2}{2 \times 720} \frac{1}{a^3}, \quad (4.124)$$

giving rise to an attractive force between the plates. This force—for the electromagnetic field, where there is an additional factor of two—was measured in 1958, and not only the sign, but also the $\sim a^{-4}$ distance dependence was observed! In fact, measuring the distance dependence is crucial for verifying the nature of this force—at atomic distances the Casimir force competes with Van-der-Vaals forces, which however have a different, $\sim a^{-7}$, dependence on the distance.

- g. To get some idea of what experimentalists have to go through, estimate the force acting on plates of area 1cm^2 a micron apart... Compare with the magnitude of forces whose measurements you are familiar with. Note that the 1990’s Lamoreaux measurements are accurate within 5%.
- h. A final bonus question: what if the scalar field had a mass, m ? Would you expect an effect if $m \gg 1/a$? What if $m \ll 1/a$?

You just saw the first example of extracting a finite and physically meaningful result from seemingly infinite expressions. Infinities result from assuming that quantum field theory makes sense at arbitrarily short distances, or large momenta k in eq. (4.120). The possibility of extracting finite results (e.g., the Casimir force) from quantum field

³ This formula is used to approximate sums with integrals. See, e.g., Wikipedia article for a derivation by induction. Other, fun ways to proceed exist, my favorite is [6].

Most importantly, the result is independent of the method of regularization. “By definition”, this is what we call a physical result in QFT (=cutoff independent). Notice the striking difference with the E_{vac} of eq. (1.41), which inherently depends on the cutoff and can not be made physical sense within QFT ... as you see, many lessons lurk in this “simple” problem!

theory simply means that in many cases (most cases, in fact: the so-called “renormalizable” ones—and even in “non-renormalizable” if one is happy with finite precision—see QFT2) the long-distance physics is independent of the details of the short-distance, most often not understood, physics, when expressed only through quantities observed at long distances.⁴

In this example, this was seen by the independence of the final answer on the cutoff function $f(k)$. This independence really means that field modes with wave vectors $\gg 1/L$ do not contribute to the Casimir effect, i.e., it is an IR (infrared) effect.

Answer for Exercise 4.4

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Exercise 4.5 Playing with the non-relativistic limit (2018 HW2.IV)

Consider a real scalar relativistic field theory of mass m with $\lambda\phi^4$ interaction. Let there be N particles of momenta labeled by p_1, \dots, p_N , whose energies are such that they are insufficient to create any new particles. Nevertheless, the particles can scatter and exchange momenta. In what follows you will study this N -particle nonrelativistic limit in some detail.

⁴ This is already familiar from classical electrodynamics although may not be always stressed. The electrostatic energy of a point charge diverges, as is well known, hence it gives an infinite contribution to the charge's rest energy. However, in the non relativistic limit (to order v^2/c^2 , in fact) the equations describing the motion of charged particles do not depend at all on whatever structure one might ascribe to the electron (it could be a ball, a hollow sphere, or a tiny string). The relative motion of particles in this limit (and, of course, at relative distances larger than the “classical radius of the electron”) is determined by two “relevant” parameters: their mass m and charge e . These are quantities determined by experiment, not calculated from first principles. These experiments are made at the long distance/time scales, where classical electromagnetic theory applies. There is no way to calculate m and e from first principles. The situation in QFT is not that different—its calculational tools are a way to relate measurable quantities to measurable quantities. Its usefulness is in that there are more measurable quantities than the number of measurements required to fix the relevant parameters in the Lagrangian (e.g., the same m and e for QED), so it has predictive power. When QFT is used to relate observables to observables, no infinities appear. There we go. QFT in a nutshell.

- Write down the Hamiltonian of the field theory, including the interaction term, restricted to the N -particle sector of Hilbert space. (Use the creation and annihilation operator representation, i.e. write the result as sums of products of creation and annihilation operators of particles of various momenta.)
- Does the resulting Hamiltonian preserve particle number? Is there an associated symmetry? What is the operator that generates it?
- Consider now the interaction term in your reduced (to the N -particle sector of Hilbert space) Hamiltonian. How does a typical interaction term (for given configurations of momenta) act on an N -particle state? What kinds of scattering processes does it describe?
- What do you think is the potential, in x -space, that allows the various particles to scatter and exchange momentum? How would you describe the resulting nonrelativistic quantum system to friends who never took QFT but are well-versed in quantum mechanics?

Hint: For ?? d, consider $N = 2$ first. Start with a two particle nonrelativistic quantum mechanics with Hamiltonian:

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1 - x_2) , \quad (4.125)$$

where p_i, x_i are the operators of momentum and position of the i -th particle (three vectors, arrows omitted for brevity). Use as a basis the eigenstates of the free Hamiltonian, i.e. plane waves, $|\vec{p}_1, \vec{p}_2\rangle$, symmetric with respect to interchange of the momenta (even better, use the corresponding wavefunctions $\psi_{p_1, p_2}(x_1, x_2) = \langle x_1, x_2 | p_1, p_2 \rangle$). Compute the matrix elements

$$\langle q_1, q_2 | H | p_1, p_2 \rangle \quad (4.126)$$

in this basis. To compare to the nonrelativistic limit of the scalar field theory, compute the same matrix elements of the Hamiltonian you found in (1.) above, in the basis of states of the restricted ($N = 2$) Hilbert space $|p_1, p_2\rangle$. Are they similar to the matrix elements you found in the quantum mechanics problem for some choice of $V(x_1 - x_2)$? Explain the difference (if any). Then go on to answer (4.) for any N .

Answer for Exercise 4.5

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5

SYMMETRIES.

5.1 SWITCHING GEARS: SYMMETRIES.

The question is how to apply the CCR results to moving frames, which is done using Lorentz transformations. Just like we know that the exponential of the Hamiltonian (times time) represents time translations, we will examine symmetries that relate results in different frames.

Examples. For scalar field(s) with action

$$S = \int d^d x \mathcal{L}(\phi^i, \partial_\mu \phi^i). \quad (5.1)$$

For example, we've been using our massive (boson) real scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - V(\phi). \quad (5.2)$$

Internal symmetry example

$$H = J \sum_{\langle n, n' \rangle} \mathbf{S}_n \cdot \mathbf{S}_{n'}, \quad (5.3)$$

where the sum means the sum over neighbouring indexes n, n' as sketched in fig. 5.1.

Such a Hamiltonian is left invariant by the transformation $\mathbf{S}_n \rightarrow -\mathbf{S}_n$ since the Hamiltonian is quadratic.

Suppose that $\phi \rightarrow -\phi$ is a symmetry (it leaves the Lagrangian unchanged). Example

$$\phi = \begin{bmatrix} \phi^1 \\ \phi^2 \\ \vdots \\ \phi^n \end{bmatrix} \quad (5.4)$$

the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^T \partial^\mu \phi - \frac{m^2}{2} \phi^T \phi - V(\phi^T \phi). \quad (5.5)$$

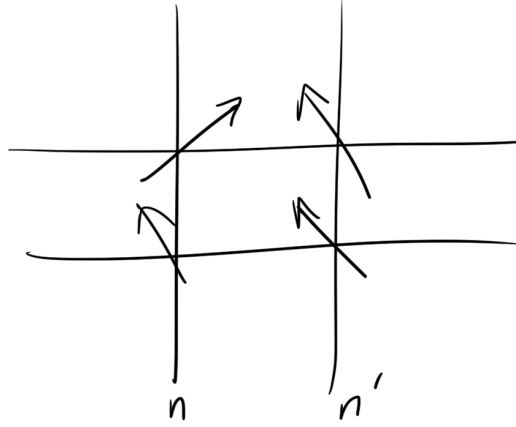


Figure 5.1: Neighbouring spin cells.

If O is any $n \times n$ orthogonal matrix, then it is symmetry since

$$\begin{aligned}\phi^T \phi &\rightarrow \phi^T O^T O \phi \\ &= \phi^T \phi.\end{aligned}\tag{5.6}$$

$O(2)$ model (exercise 4.3). Example for complex ϕ

$$\phi \rightarrow e^{i\phi} \phi,\tag{5.7}$$

$$\phi = \frac{\psi_1 + i\psi_2}{\sqrt{2}}\tag{5.8}$$

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \rightarrow \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}\tag{5.9}$$

5.2 SYMMETRIES.

Given the complexities of the non-linear systems we want to investigate, examination of symmetries gives us simpler problems that we can solve.

- “internal” symmetries. This means that the symmetries do not act on space time (\mathbf{x}, t) . An example is

$$\phi^i = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix} \quad (5.10)$$

If we map $\phi^i \rightarrow O^i_j \phi^j$ where $O^T O = 1$, then we call this an internal symmetry. The corresponding Lagrangian density might be something like

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{m^2}{2} \phi \cdot \phi - V(\phi \cdot \phi) \quad (5.11)$$

- spacetime symmetries: Translations, rotations, boosts, dilatations. We will consider continuous symmetries, which can be defined as a succession of infinitesimal transformations. An example from $O(2)$ is a rotation

$$\begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \rightarrow \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix}, \quad (5.12)$$

or if $\alpha \sim 0$

$$\begin{aligned} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \\ &= \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} + \alpha \begin{bmatrix} \phi^2 \\ -\phi^1 \end{bmatrix} \end{aligned} \quad (5.13)$$

In index notation we write

$$\phi^i \rightarrow \phi^i + \alpha \epsilon^{ij} \phi^j, \quad (5.14)$$

where $\epsilon^{12} = +1, \epsilon^{21} = -1$ is the completely antisymmetric tensor. This can be written in more general form as

$$\phi^i \rightarrow \phi^i + \delta \phi^i, \quad (5.15)$$

where $\delta \phi^i$ is considered to be an infinitesimal transformation.

Definition 5.1: Symmetry

A symmetry means that there is some transformation

$$\phi^i \rightarrow \phi^i + \delta\phi^i,$$

where $\delta\phi^i$ is an infinitesimal transformation, and the equations of motion are invariant under this transformation.

Theorem 5.1: Noether's theorem (1st).

If the equations of motion are invariant under $\phi^\mu \rightarrow \phi^\mu + \delta\phi^\mu$, then there exists a conserved current j^μ such that $\partial_\mu j^\mu = 0$.

Noether's first theorem applies to global symmetries, where the parameters are the same for all (\mathbf{x}, t) . Gauge symmetries are not examples of such global symmetries.

Given a Lagrangian density $\mathcal{L}(\phi(x), \phi_{,\mu}(x))$, where $\phi_{,\mu} \equiv \partial_\mu \phi$. The action is

$$S = \int d^d x \mathcal{L}. \quad (5.16)$$

The equations of motion are invariant if under $\phi(x) \rightarrow \phi'(x) = \phi(x) + \delta_\epsilon \phi(x)$, we have

$$\begin{aligned} \mathcal{L}(\phi) &\rightarrow \mathcal{L}'(\phi') \\ &= \mathcal{L}(\phi) + \partial_\mu J_\epsilon^\mu(\phi) + O(\epsilon^2). \end{aligned} \quad (5.17)$$

Then there exists a conserved current. In QFT we say that the E.O.M's are "on shell". Note that eq. (5.17) is a symmetry since we have added a total derivative to the Lagrangian which leaves the equations of motion unchanged.

In general, the change of action under arbitrary variation of $\delta\phi$ of the fields is

$$\begin{aligned} \delta S &= \int d^d x \delta \mathcal{L}(\phi, \partial_\mu \phi) \\ &= \int d^d x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \partial_\mu \phi \right) \\ &= \int d^d x \left(\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \delta \phi \right) \\ &= \int d^d x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right). \end{aligned} \quad (5.18)$$

However from eq. (5.17)

$$\delta_\epsilon \mathcal{L} = \partial_\mu J_\epsilon^\mu(\phi, \partial_\mu \phi), \quad (5.19)$$

so after equating these variations we find that

$$\begin{aligned} \delta S &= \int d^d x \delta_\epsilon \mathcal{L} \\ &= \int d^d x \partial_\mu J_\epsilon^\mu, \end{aligned} \quad (5.20)$$

or

$$0 = \int d^d x \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta \phi - J_\epsilon^\mu \right), \quad (5.21)$$

or $\partial_\mu j^\mu = 0$ provided

$$\boxed{j^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta_\epsilon \phi - J_\epsilon^\mu.} \quad (5.22)$$

Integrating the divergence of the current over a space time volume, perhaps that of fig. 3.1, is also zero. That is

$$\begin{aligned} 0 &= \int d^4 x \partial_\mu j^\mu \\ &= \int d^3 \mathbf{x} dt \partial_\mu j^\mu \\ &= \int d^3 \mathbf{x} dt \partial_t j^0 - \int d^3 \mathbf{x} dt \nabla \cdot \mathbf{j}, \end{aligned} \quad (5.23)$$

where the spatial divergence is zero assuming there's no current leaving the volume on the infinite boundary (no \mathbf{j} at spatial infinity.)

We write

$$Q = \int d^3 x j^0, \quad (5.24)$$

and call this the on-shell charge associated with the symmetry.

5.3 SPACETIME TRANSLATION.

A spacetime translation has the form

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu, \quad (5.25)$$

where the fields transform as

$$\phi(x) \rightarrow \phi'(x') = \phi(x). \quad (5.26)$$

Contrast this to a Lorentz transformation that had the form $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

If $\phi'(x + a) = \phi(x)$, then

$$\begin{aligned} \phi'(x) + a^\mu \partial_\mu \phi'(x) &= \phi'(x) + a^\mu \partial_\mu \phi(x) \\ &= \phi(x), \end{aligned} \quad (5.27)$$

so

$$\begin{aligned} \phi'(x) &= \phi(x) - a^\mu \partial_\mu \phi'(x) \\ &= \phi(x) + \delta_a \phi(x), \end{aligned} \quad (5.28)$$

or

$$\delta_a \phi(x) = -a^\mu \partial_\mu \phi(x). \quad (5.29)$$

Under $\phi \rightarrow \phi - a^\mu \partial_\mu \phi$, we have

$$\mathcal{L}(\phi) \rightarrow \mathcal{L}(\phi) - a^\mu \partial_\mu \mathcal{L}. \quad (5.30)$$

Let's calculate this with our scalar theory Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - V(\phi). \quad (5.31)$$

The Lagrangian variation¹ is

$$\begin{aligned} \delta \mathcal{L}|_{\phi \rightarrow \phi + \delta \phi, \delta \phi = -a^\mu \partial_\mu \phi} &= (\partial_\mu \phi) \delta(\partial^\mu \phi) - m^2 \phi \delta \phi - \frac{\partial V}{\partial \phi} \delta \phi \\ &= (\partial_\mu \phi) (-a^\nu \partial_\nu \partial^\mu \phi) + m^2 \phi a^\nu \partial_\nu \phi + \frac{\partial V}{\partial \phi} a^\nu \partial_\nu \phi \\ &= -a^\nu \partial_\nu \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - V(\phi) \right) \\ &= -a^\nu \partial_\nu \mathcal{L}, \end{aligned} \quad (5.32)$$

¹ Using: $\partial_\alpha((1/2)\partial_\mu \phi \partial^\mu \phi) = 2(1/2)\partial_\mu \phi (\partial_\alpha \partial^\mu \phi)$.

so the current is

$$\begin{aligned} j^\mu &= (\partial^\mu \phi)(-a^\nu \partial_\nu \phi) + a^\mu \mathcal{L} \\ &= -a^\nu (\partial^\mu \phi \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}). \end{aligned} \quad (5.33)$$

We really have a current for each ν direction and can make that explicit writing

$$\begin{aligned} \delta_\nu \mathcal{L} &= -\partial_\nu \mathcal{L} \\ &= -\partial_\mu (\delta^\mu_\nu \mathcal{L}) \\ &= \partial_\mu j^\mu_\nu \end{aligned} \quad (5.34)$$

we write

$$j^\mu_\nu = \frac{\partial \phi}{\partial x_\mu} \left(-\frac{\partial \phi}{\partial x^\nu} \right) + \delta^\mu_\nu \mathcal{L}, \quad (5.35)$$

where ν are labels which coordinates are translated:

$$\begin{aligned} \partial_\nu \phi &= -\partial_\nu \phi \\ \partial_\nu \mathcal{L} &= -\partial_\nu \mathcal{L}. \end{aligned} \quad (5.36)$$

We call the conserved quantities elements of the energy-momentum tensor, and write it as

$$T^\mu_\nu = -\frac{\partial \phi}{\partial x_\mu} \frac{\partial \phi}{\partial x^\nu} + \delta^\mu_\nu \mathcal{L}. \quad (5.37)$$

Incidentally, we picked a non-standard sign convention for the tensor, as an explicit expansion of T^{00} , the energy density component, shows

$$\begin{aligned} T^0_0 &= -\frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial t} - \frac{1}{2} (\nabla \phi) \cdot (\nabla \phi) - \frac{m^2}{2} \phi^2 - V(\phi) \\ &= -\frac{1}{2} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial t} - \frac{1}{2} (\nabla \phi) \cdot (\nabla \phi) - \frac{m^2}{2} \phi^2 - V(\phi). \end{aligned} \quad (5.38)$$

Had we translated by $-a^\mu$ we'd have a positive definite tensor instead.

5.4 1ST NOETHER THEOREM.

Recall that, given a transformation

$$\phi(x) \rightarrow \phi(x) + \delta \phi(x), \quad (5.39)$$

such that the transformation of the Lagrangian is only changed by a total derivative

$$\mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}(\phi, \partial_\mu \phi) + \partial_\mu J^\mu_\epsilon, \quad (5.40)$$

then there is a conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\epsilon \phi - J_\epsilon^\mu. \quad (5.41)$$

Here ϵ is an x -independent quantity (i.e. a global symmetry). This is in contrast to “gauge symmetries”, which can be more accurately be categorized as a redundancy in the description.

As an example, for $\mathcal{L} = (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)/2$, let

$$\phi(x) \rightarrow \phi(x) - a^\mu \partial_\mu \phi. \quad (5.42)$$

The Lagrangian density transforms as

$$\begin{aligned} \mathcal{L}(\phi, \partial_\mu \phi) &\rightarrow \mathcal{L}(\phi, \partial_\mu \phi) - a^\mu \partial_\mu \mathcal{L} \\ &= \mathcal{L}(\phi, \partial_\mu \phi) + \partial_\mu (-\delta^\mu{}_\nu a^\nu \mathcal{L}). \end{aligned} \quad (5.43)$$

Here $J_\epsilon^\mu = J_\epsilon^\mu|_{\epsilon=a^\nu}$, and the current is

$$J^\mu = (\partial^\mu \phi)(-a^\nu \partial_\nu \phi) + \delta^\mu{}_\nu a^\nu \mathcal{L}. \quad (5.44)$$

In particular, we have one such current for each ν , and we write

$$T^\mu{}_\nu = -(\partial^\mu \phi)(\partial_\nu \phi) + \delta^\mu{}_\nu \mathcal{L}. \quad (5.45)$$

By Noether’s theorem, we must have

$$\partial_\mu T^\mu{}_\nu = 0, \quad \forall \nu. \quad (5.46)$$

Check:

$$\begin{aligned} \partial_\mu T^\mu{}_\nu &= -(\partial_\mu \partial^\mu \phi)(\partial_\nu \phi) - (\partial^\mu \phi)(\partial_\mu \partial_\nu \phi) + \delta^\mu{}_\nu \partial_\mu \left(\frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{m^2}{2} \phi^2 \right) \\ &= -(\partial_\mu \partial^\mu \phi)(\partial_\nu \phi) - (\partial^\mu \phi)(\partial_\mu \partial_\nu \phi) + \frac{1}{2} (\partial_\nu \partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} (\partial_\mu \phi)(\partial_\nu \partial^\mu \phi) - m^2 (\partial_\nu \phi) \phi \\ &= -(\partial_\mu \partial^\mu \phi + m^2 \phi)(\partial_\nu \phi) - (\partial_\mu \phi)(\partial^\mu \partial_\nu \phi) + \frac{1}{2} (\partial_\nu \partial^\mu \phi)(\partial_\mu \phi) + \frac{1}{2} (\partial_\mu \phi)(\partial_\nu \partial^\mu \phi) \\ &= 0. \end{aligned} \quad (5.47)$$

Example: our potential Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \quad (5.48)$$

Written with upper indexes

$$\begin{aligned} T^{\mu\nu} &= -(\partial^\mu\phi)(\partial^\nu\phi) + g^{\mu\nu}\mathcal{L} \\ &= -(\partial^\mu\phi)(\partial^\nu\phi) + g^{\mu\nu}\left(\frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4\right) \end{aligned} \quad (5.49)$$

There are 4 conserved currents $J^{\mu(\nu)} = T^{\mu\nu}$. Observe that this is symmetric ($T^{\mu\nu} = T^{\nu\mu}$).

We have four associated charges

$$Q^\nu = \int d^3x T^{0\nu}. \quad (5.50)$$

We call

$$Q^0 = \int d^3x T^{00}, \quad (5.51)$$

the energy density, and call

$$P^i = \int d^3x T^{0i}, \quad (5.52)$$

($i = 1, 2, 3$) the momentum density.

writing this out explicitly the energy density is

$$\begin{aligned} T^{00} &= -\dot{\phi}^2 + \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4\right) \\ &= -\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\right), \end{aligned} \quad (5.53)$$

and

$$T^{0i} = \partial^0\phi\partial^i\phi, \quad (5.54)$$

$$P^i = - \int d^3x \partial^0\phi\partial^i\phi. \quad (5.55)$$

Since the energy density is negative definite (due to an arbitrary choice of translation sign), let's redefine $T^{\mu\nu}$ to have a positive sign

$$T^{00} \equiv \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4, \quad (5.56)$$

and

$$P^i = \int d^3x \partial^0 \phi \partial^i \phi \quad (5.57)$$

As an operator the charge is

$$\begin{aligned} \hat{Q} &= \int d^3x \hat{T}^{00} \\ &= \int d^3x \left(\frac{1}{2} \hat{\pi}^2 + \frac{1}{2} (\nabla \hat{\phi})^2 + \frac{m^2}{2} \hat{\phi}^2 + \frac{\lambda}{4} \hat{\phi}^4 \right), \end{aligned} \quad (5.58)$$

and the momenta are

$$\hat{P}^i = \int d^3x \hat{\pi} \partial^i \phi. \quad (5.59)$$

We showed that

$$\frac{d\hat{O}}{dt} = i [\hat{H}, \hat{O}]. \quad (5.60)$$

This implied that $\hat{\phi}, \hat{\pi}$ obey the classical equations of motion

$$\frac{d\hat{\phi}}{dt} = i [\hat{H}, \hat{\phi}] = \frac{d\hat{\pi}}{dt} \quad (5.61)$$

$$\frac{d\hat{\pi}}{dt} = i [\hat{H}, \hat{\pi}] = \dots \quad (5.62)$$

In terms of creation and annihilation operators (for the $\lambda = 0$ free field), up to a constant

$$\begin{aligned} \hat{H} &= \int d^3x \hat{T}^{00} \\ &= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}. \end{aligned} \quad (5.63)$$

It can be shown (appendix B) that the operator form of the field momentum is

$$\begin{aligned} \hat{P}^i &= \int d^3x \hat{\pi} \partial^i \phi \\ &= \int \frac{d^3p}{(2\pi)^3} p^i \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}}. \end{aligned} \quad (5.64)$$

Now we see the energy and momentum as conserved quantities associated with spacetime translation.

5.5 UNITARY OPERATORS.

In QM we say that $\hat{\mathbf{P}}$ “generates translations”. With $\hat{\mathbf{P}} \equiv -i\hbar\nabla$ that translation is

$$\hat{U}(\mathbf{a}) = e^{i\mathbf{a}\cdot\hat{\mathbf{P}}} = e^{-\mathbf{a}\cdot\nabla}. \quad (5.65)$$

In particular

$$\begin{aligned} \langle \mathbf{x} | \hat{U}(\mathbf{a}) | \psi \rangle &= \int d^3p \langle \mathbf{x} | \hat{U}(\mathbf{a}) | \mathbf{p} \rangle \langle \mathbf{p} | \psi \rangle \\ &= \int d^3p \langle \mathbf{x} | e^{i\mathbf{a}\cdot\hat{\mathbf{P}}} | \mathbf{p} \rangle \langle \mathbf{p} | \psi \rangle \\ &= \int d^3p e^{i\mathbf{a}\cdot\hat{\mathbf{P}}} \langle \mathbf{x} | \mathbf{p} \rangle \tilde{\psi}(\mathbf{p}) \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{a}\cdot\hat{\mathbf{P}}} e^{i\mathbf{x}\cdot\mathbf{p}} \tilde{\psi}(\mathbf{p}) \\ &= \psi(\mathbf{x} + \mathbf{a}). \end{aligned} \quad (5.66)$$

Implicitly, this shows that the action of the translation operator on just a bra is

$$\langle \mathbf{x} | \hat{U}(\mathbf{a}) = \langle \mathbf{x} + \mathbf{a} |, \quad (5.67)$$

or

$$\begin{aligned} \hat{U}(-\mathbf{a}) | \mathbf{x} \rangle &= \hat{U}^\dagger(\mathbf{a}) | \mathbf{x} \rangle \\ &= | \mathbf{x} + \mathbf{a} \rangle. \end{aligned} \quad (5.68)$$

This is a different sign convention for the translation operator than is found in some other texts².

In one dimension, we can compute

$$\hat{U}(a)\hat{X}\hat{U}^\dagger(a) = e^{ia\hat{P}}\hat{X}e^{-ia\hat{P}} = \hat{X} + a\hat{1}, \quad (5.69)$$

which is a consequence of the Baker-Campbell-Hausdorff theorem.

Theorem 5.2: Baker-Campbell-Hausdorff

$$e^B A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} [B \cdots [B, A]], \quad (5.70)$$

where the n-th commutator is denoted above

- $n = 0 : A$

² In particular [3] uses $D(\mathbf{a}) = e^{-i\mathbf{a}\cdot\hat{\mathbf{P}}/\hbar}$ defined by the property $D(\mathbf{a}) | \mathbf{x} \rangle = | \mathbf{x} + \mathbf{a} \rangle$.

- $n = 1 : [B, A]$
- $n = 2 : [B, [B, A]]$
- $n = 3 : [B, [B, [B, A]]]$

Proof:

$$\begin{aligned} f(t) &= e^{tB} A e^{-tB} \\ &= f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \cdots + \frac{t^n}{n!} f^{(n)}(0) \end{aligned} \quad (5.71)$$

$$f(0) = A \quad (5.72)$$

$$\begin{aligned} f'(t) &= e^{tB} B A e^{-tB} + e^{tB} A (-B) e^{-tB} \\ &= e^{tB} [B, A] e^{-tB} \end{aligned} \quad (5.73)$$

$$\begin{aligned} f''(t) &= e^{tB} B [B, A] e^{-tB} + e^{tB} [B, A] (-B) e^{-tB} \\ &= e^{tB} [B, [B, A]] e^{-tB}. \end{aligned} \quad (5.74)$$

From

$$f(1) = f(0) + f'(0) + \frac{1}{2} f''(0) + \cdots + \frac{1}{n!} f^{(n)}(0) \quad (5.75)$$

we have

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2} [B, [B, A]] + \cdots \quad (5.76)$$

Example (as claimed above) :

$$\begin{aligned} e^{ia\hat{P}} \hat{X} e^{-ia\hat{P}} &= \hat{X} + [ia\hat{P}, \hat{X}] + \cdots \\ &= \hat{X} + ia(-i)\hat{1} \\ &= \hat{X} + a\hat{1}. \end{aligned} \quad (5.77)$$

Application:

$$e^{i\text{Hermitian}} = \text{unitary} \quad (5.78)$$

$$e^{i\text{Hermitian}} \times e^{-i\text{Hermitian}} = 1 \quad (5.79)$$

So

$$\hat{U}(\mathbf{a}) = e^{ia^j \hat{p}^j} \quad (5.80)$$

is a unitary operator representing finite translations in a Hilbert space.

In particular, we can apply the BCH theorem to a field operator

$$\begin{aligned} \hat{U}(\mathbf{a})\hat{\phi}(\mathbf{x})\hat{U}^\dagger(\mathbf{a}) &= e^{ia^j \hat{P}^j} \hat{\phi}(\mathbf{x}) e^{-ia^k \hat{P}^k} \\ &= \hat{\phi}(\mathbf{x}) + ia^j [\hat{P}^j, \hat{\phi}(\mathbf{x})] + \frac{-a^{j_1} a^{j_2}}{2} [\hat{P}^{j_1}, [\hat{P}^{j_2}, \hat{\phi}(\mathbf{x})]], \end{aligned} \quad (5.81)$$

where the first order commutator is

$$\begin{aligned} [\hat{P}^j, \hat{\phi}(\mathbf{x})] &= \int d^3y [\hat{\pi}(\mathbf{y}) \partial^j \hat{\phi}(\mathbf{y}), \hat{\phi}(\mathbf{x})] \\ &= \int d^3y [\hat{\pi}(\mathbf{y}), \hat{\phi}(\mathbf{x})] \partial^j \hat{\phi}(\mathbf{y}) \\ &= \int d^3y (-i) \delta^{(3)}(\mathbf{y} - \mathbf{x}) \partial^j \hat{\phi}(\mathbf{y}) \\ &= -i \partial^j \hat{\phi}(\mathbf{x}), \end{aligned} \quad (5.82)$$

and any higher order commutator is zero

$$[\hat{P}^k, [\hat{P}^j, \phi(x)]] = \int d^3y [\pi(y) \partial^k \phi(y), -i \partial^j \phi(x)] = 0. \quad (5.83)$$

This gives

$$\begin{aligned} \hat{U}(\mathbf{a})\hat{\phi}(\mathbf{x})\hat{U}^\dagger(\mathbf{a}) &= \hat{\phi}(\mathbf{x}) + ia^j (-i) \partial^j \hat{\phi}(\mathbf{x}) + \dots \\ &= \hat{\phi}(\mathbf{x}) + a^j \partial^j \hat{\phi}(\mathbf{x}) + \dots \\ &= \hat{\phi}(\mathbf{x}) + a^j \frac{\partial}{\partial x_j} \hat{\phi}(\mathbf{x}) + \dots \\ &= \hat{\phi}(\mathbf{x}) - a^j \frac{\partial}{\partial x^j} \hat{\phi}(\mathbf{x}) + \dots \\ &= \hat{\phi}(\mathbf{x} - \mathbf{a}). \end{aligned} \quad (5.84)$$

5.6 CONTINUOUS SYMMETRIES.

For all infinitesimal transformations, continuous symmetries lead to conserved charges Q . In QFT we map these charges to Hermitian operators $Q \rightarrow \hat{Q}$. We say that these charges are “generators of the corresponding symmetry” through unitary operators

$$\hat{U} = e^{i\text{parameter}\hat{Q}}. \quad (5.85)$$

These represent the action of the symmetry in the Hilbert space.

Example: spatial translation

$$\hat{U}(\mathbf{a}) = e^{i\mathbf{a}\cdot\hat{\mathbf{P}}} \quad (5.86)$$

Example: time translation

$$\hat{U}(t) = e^{it\hat{H}}. \quad (5.87)$$

5.7 CLASSICAL SCALAR THEORY.

For $d > 2$ let's look at

$$S = \int d^d x \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2 - \lambda \phi^{d-2} \right) \quad (5.88)$$

Take $m^2, \lambda \rightarrow 0$, the free massless scalar field. We have a shift symmetry in this case since $\phi(x) \rightarrow \phi(x) + \text{constant}$. The current is just

$$\begin{aligned} j^\mu &= \frac{\partial \phi}{\partial(\partial_\mu \phi)} \delta \phi - \mathcal{J}^\mu \\ &= \text{constant} \times \partial^\mu \phi \\ &= \partial^\mu \phi, \end{aligned} \quad (5.89)$$

where the constant factor has been set to one. This current is clearly conserved since $\partial_\mu J^\mu = \partial_\mu \partial^\mu \phi = 0$ (the equation of motion). These are called “Goldstone bosons”, or “Nambu-Goldstone bosons”.

With $m = \lambda = 0, d = 4$ we have NOTE: We did this in class differently with $d \neq 4, m, \lambda \neq 0$, and then switched to $m = \lambda = 0, d = 4$, which was confusing. I've reworked my notes to $d = 4$ like the supplemental handout that did the same.

$$S = \int d^4x \left(\frac{1}{2} \partial^\mu \phi \partial_\mu \phi \right) \quad (5.90)$$

Here we have a scale or dilatation invariance

$$x \rightarrow x' = e^\lambda x, \quad (5.91)$$

$$\phi(x) \rightarrow \phi'(x') = e^{-\lambda} \phi, \quad (5.92)$$

$$d^4x \rightarrow d^4x' = e^{4\lambda} d^4x. \quad (5.93)$$

The partials transform as

$$\begin{aligned} \partial^\mu &\rightarrow \frac{\partial}{\partial x'_\mu} \\ &= \frac{\partial x_\mu}{\partial x'_\mu} \frac{\partial}{\partial x_\mu} \\ &= e^{-\lambda} \frac{\partial}{\partial x_\mu}, \end{aligned} \quad (5.94)$$

so the partial of the field transforms as

$$\partial^\mu \phi(x) \rightarrow \frac{\partial \phi'(x')}{\partial x'_\mu} = e^{-2\lambda} \partial^\mu \phi(x), \quad (5.95)$$

and finally

$$(\partial_\mu \phi)^2 \rightarrow e^{-4\lambda} (\partial_\mu \phi(x))^2. \quad (5.96)$$

With a -4λ power in the transformed quadratic term, and 4λ in the volume element, we see that the action is invariant. To find Noether current, we need to vary the field and its derivatives

$$\begin{aligned} \delta_\lambda \phi &= \phi'(x) - \phi(x) \\ &= \phi'(e^{-\lambda} x') - \phi(x) \\ &\approx \phi'(x' - \lambda x') - \phi(x) \\ &\approx \phi'(x') - \lambda x'^\alpha \partial_\alpha \phi'(x') - \phi(x) \\ &\approx (1 - \lambda) \phi(x) - \lambda x'^\alpha \partial_\alpha \phi'(x') - \phi(x) \\ &= -\lambda (1 + x^\alpha \partial_\alpha) \phi, \end{aligned} \quad (5.97)$$

where the last step assumes that $x' \rightarrow x, \phi' \rightarrow \phi$, effectively weeding out any terms that are quadratic or higher in λ .

Now we need the variation of the derivatives of ϕ

$$\delta \partial_\mu \phi(x) = \partial'_\mu \phi'(x) - \partial_\mu \phi(x), \quad (5.98)$$

By eq. (5.95)

$$\begin{aligned} \partial'_\mu \phi'(x') &= e^{-2\lambda} \partial_\mu \phi(x) \\ &= e^{-2\lambda} \partial_\mu \phi(e^{-\lambda} x') \\ &\approx e^{-2\lambda} \partial_\mu (\phi(x') - \lambda x'^\alpha \partial_\alpha \phi(x')) \\ &\approx (1 - 2\lambda) \partial_\mu (\phi(x') - \lambda x'^\alpha \partial_\alpha \phi(x')), \end{aligned} \quad (5.99)$$

so

$$\begin{aligned} \delta \partial_\mu \phi &= -\lambda x^\alpha \partial_\alpha \partial_\mu \phi(x) - 2\lambda \partial_\mu \phi(x) + O(\lambda^2) \\ &= -\lambda (x^\alpha \partial_\alpha + 2) \partial_\mu \phi(x). \end{aligned} \quad (5.100)$$

$$\begin{aligned} \delta \mathcal{L} &= (\partial^\mu \phi) \delta(\partial_\mu \phi) \\ &= -\lambda (2\partial_\mu \phi + x^\alpha \partial_\alpha \partial_\mu \phi) \partial^\mu \phi, \end{aligned} \quad (5.101)$$

or

$$\begin{aligned} \frac{\delta \mathcal{L}}{-\lambda} &= 4\mathcal{L} + x^\alpha (\partial_\alpha \partial_\mu \phi) \partial^\mu \phi \\ &= 4\mathcal{L} + x^\alpha \partial_\alpha (\mathcal{L}) \\ &= \cancel{4\mathcal{L}} + \partial_\alpha (x^\alpha \mathcal{L}) - \mathcal{L} \cancel{\partial_\alpha x^\alpha}. \end{aligned} \quad (5.102)$$

The variation in the Lagrangian density is thus

$$\delta \mathcal{L} = \partial_\mu J_\lambda^\mu = \partial_\mu (-\lambda x^\mu \mathcal{L}), \quad (5.103)$$

and the current is

$$J_\lambda^\mu = -\lambda x^\mu \mathcal{L}. \quad (5.104)$$

The Noether current is

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi - J^\mu \\ &= -\partial^\mu \phi (1 + x^\nu \partial_\nu) \phi + \frac{1}{2} x^\mu \partial_\nu \phi \partial^\nu \phi, \end{aligned} \quad (5.105)$$

or after flipping signs

$$\begin{aligned} j_{\text{dil}}^\mu &= \partial^\mu \phi (1 + x^\nu \partial_\nu) \phi - \frac{1}{2} x^\mu \partial_\nu \phi \partial^\nu \phi \\ &= x_\nu \left(\partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\lambda \phi \partial^\lambda \phi \right) + \frac{1}{2} \partial^\mu (\phi^2), \end{aligned} \quad (5.106)$$

$$j_{\text{dil}}^\mu = -x_\nu T^{\nu\mu} + \frac{1}{2} \partial^\mu (\phi^2). \quad (5.107)$$

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \quad (5.108)$$

The current and $T^{\mu\nu}$ can both be redefined $j^{\mu'} = j^\mu + \partial_\nu C^{\nu\mu}$ adding an antisymmetric $C^{\mu\nu} = -C^{\nu\mu}$

$$j_{\text{dil conformal}}^\mu = -x_\nu T_{\text{conformal}}^{\nu\mu} \quad (5.109)$$

$$\partial_\mu j_{\text{dil conformal}}^\mu = -T_{\text{conformal}}^\mu{}_\mu \quad (5.110)$$

consequence: $0 = T^{00} - T^{11} - T^{22} - T^{33}$, which is essentially

$$0 = \rho - 3p = 0. \quad (5.111)$$

5.8 LAST TIME.

We followed a sequence of operations

1. Noether's theorem
2. \rightarrow conserved currents
3. \rightarrow charges (classical)
4. \rightarrow "correspondence principle"
5. $\rightarrow \hat{Q}$
 - Hermitian operators
 - "generators of symmetry"

$$\hat{U}(\alpha) = e^{i\alpha \hat{Q}} \quad (5.112)$$

We found

$$\hat{U}(\alpha) \hat{\phi} \hat{U}^\dagger(\alpha) = \hat{\phi} + i\alpha [\hat{Q}, \hat{\phi}] + \dots \quad (5.113)$$

Example: internal symmetries: (non-spacetime), such as $O(N)$ or $U(1)$.

In QFT internal symmetries can have different “modes of realization”.

I “Wigner mode”. These are also called “unbroken symmetries”.

$$\hat{Q}|0\rangle = 0 \quad (5.114)$$

i.e. $\hat{U}(\alpha)|0\rangle = 0$. Ground state invariant. Formally : \hat{Q} : annihilates $|0\rangle$. $[\hat{Q}, \hat{H}] = 0$ implies that all eigenstates are eigenstates of \hat{Q} in $U(1)$. Example from HW 1

$$\hat{Q} = \text{“charge” under } U(1). \quad (5.115)$$

All states have definite charge, just live in QU.

II “Nambu-Goldstone mode” (Landau-Ginsburg). This is also called a “spontaneously broken symmetry”³. H or L is invariant under symmetry, but ground state is not.

Example:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|), \quad (5.116)$$

where

$$V(|\phi|) = m^2 \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2. \quad (5.117)$$

When $m^2 > 0$ we have a Wigner mode, but when $m^2 < 0$ we have an issue: $\phi = 0$ is not a minimum of potential. When $m^2 < 0$ we write

$$\begin{aligned} V(\phi) &= -m^2 \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 \\ &= \frac{\lambda}{4} \left((\phi^* \phi)^2 - \frac{4}{\lambda} m^2 \right) \\ &= \frac{\lambda}{4} \left(\phi^* \phi - \frac{2}{\lambda} m^2 \right)^2 - \frac{4m^4}{\lambda^2}, \end{aligned} \quad (5.118)$$

or simply

$$V(\phi) = \frac{\lambda}{4} \left(\phi^* \phi - v^2 \right)^2 + \text{const.} \quad (5.119)$$

The potential (called the Mexican hat potential) is illustrated in fig. 5.2 for non-zero v , and in fig. 5.3 for $v = 0$. The following is a Mathematica code listing that can be used to play with this shape

³ First encounter example (HWII, $SU(2) \times SU(2) \rightarrow SU(2)$). Here a $U(1)$ spontaneous broken symmetry.


```

In[1]:= ClearAll[potential]
potential[x_, y_, v_] := (x^2 + y^2 - v^2)^2

Manipulate[
  Plot3D[potential[x, y, v], {x, -5, 5}, {y, -5, 5}, PlotRange
→Full],
  {{v, 4}, 0, 10}
]

```

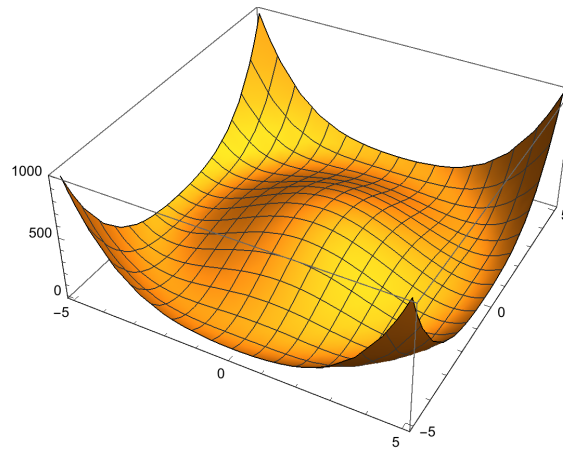


Figure 5.2: Mexican hat potential.

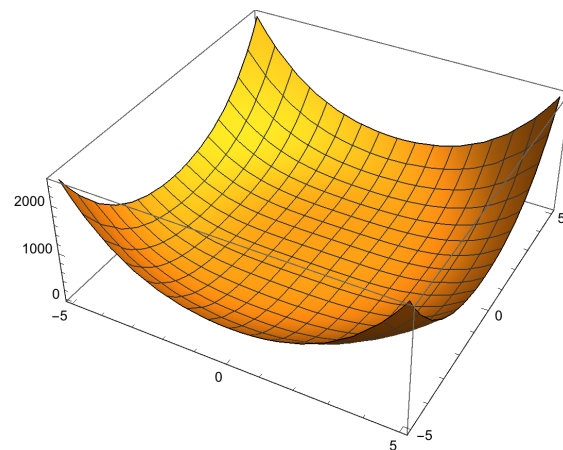


Figure 5.3: Degenerate Mexican hat potential $v = 0$.

We choose to expand around some point on the minimum ring (it doesn't matter which one). When there is no potential, we call the field massless (i.e. if we are in the minimum ring). We expand as

$$\phi(x) = v \left(1 + \frac{\rho(x)}{v} \right) e^{i\alpha(x)/v}, \quad (5.120)$$

so

$$\begin{aligned} \frac{\lambda}{4} (\phi^* \phi - v^2)^2 &= \left(v^2 \left(1 + \frac{\rho(x)}{v} \right)^2 - v^2 \right)^2 \\ &= \frac{\lambda}{4} v^4 \left(\left(1 + \frac{\rho(x)}{v} \right)^2 - 1 \right)^2 \\ &= \frac{\lambda}{4} v^4 \left(\frac{2\rho}{v} + \frac{\rho^2}{v^2} \right)^2, \end{aligned} \quad (5.121)$$

and

$$\partial_\mu \phi = \left(v \left(1 + \frac{\rho(x)}{v} \right) \right) \frac{i}{v} \partial_\mu \alpha + \partial_\mu \rho e^{i\alpha}. \quad (5.122)$$

The Lagrangian takes the form

$$\begin{aligned} \mathcal{L} &= |\partial \phi^*|^2 - \frac{\lambda}{4} (|\phi^*|^2 - v^2)^2 \\ &= \partial_\mu \rho \partial^\mu \rho + \partial_\mu \alpha \partial^\mu \alpha \left(1 + \frac{\rho}{v} \right) - \frac{\lambda v^4}{4} \frac{4\rho^2}{v^2} + O(\rho^3) \\ &= \partial_\mu \rho \partial^\mu \rho - \lambda v^2 \rho^2 + \partial_\mu \alpha \partial^\mu \alpha \left(1 + \frac{\rho}{v} \right). \end{aligned} \quad (5.123)$$

We have two fields, ρ : a massive scalar field, the ‘‘Higgs’’, and a massless field α (the Goldstone boson).

$U(1)$ symmetry acts on $\phi(x) \rightarrow e^{i\omega} \phi(x)$ i.t.o $\alpha(x) \rightarrow \alpha(x) + v\omega$. $U(1)$ global symmetry (broken) acts on the Goldstone field $\alpha(x)$ by a constant shift. ($U(1)$ is still a symmetry of the Lagrangian.)

The current of the $U(1)$ symmetry is:

$$j_\mu = \partial_\mu \alpha (1 + \text{higher dimensional } \rho \text{ terms}). \quad (5.124)$$

When we quantize

$$\alpha(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_p}} e^{i\omega_p t - i\mathbf{p} \cdot \mathbf{x}} \hat{a}_{\mathbf{p}}^\dagger + \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_p}} e^{-i\omega_p t + i\mathbf{p} \cdot \mathbf{x}} \hat{a}_{\mathbf{p}} \quad (5.125)$$

$$\begin{aligned}
j^\mu(x) &= \partial^\mu \alpha(x) \\
&= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_p}} (i\omega_{\mathbf{p}} - i\mathbf{p}) e^{i\omega_p t - i\mathbf{p} \cdot \mathbf{x}} \hat{a}_{\mathbf{p}}^\dagger \\
&\quad + \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_p}} (-i\omega_{\mathbf{p}} + i\mathbf{p}) e^{-i\omega_p t + i\mathbf{p} \cdot \mathbf{x}} \hat{a}_{\mathbf{p}}.
\end{aligned} \tag{5.126}$$

$$j^\mu(x) |0\rangle \neq 0, \tag{5.127}$$

instead it creates a single particle state.

5.9 EXAMPLES OF SYMMETRIES.

In particle physics, examples of Wigner vs Nambu-Goldstone, ignoring gravity the only exact internal symmetry in the standard module is $(B\# - L\#)$, believed to be a $U(1)$ symmetry in Wigner mode.

Here $B\#$ is the Baryon number, and $L\#$ is the Lepton number. Examples:

- $B(p) = 1$, proton.
- $B(q) = 1/3$, quark
- $B(e) = 1$, electron
- $B(n) = 1$, neutron.
- $L(p) = 1$, proton.
- $L(q) = 0$, quark.
- $L(e) = 0$, electron.

The major use of global internal symmetries in the standard model is as “approximate” ones. They become symmetries when one neglects some effect(“terms in \mathcal{L} ”). There are other approximate symmetries (use of group theory to find the Balmer series).

Example from exercise 5.4 (Hw2): QCD in limit

$$m_u = m_d = 0. \tag{5.128}$$

$m_u m_d \ll m_p$ (the products of the up-quark mass and the down-quark mass are much less than a composite one (name?)). $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$

EWSB (Electro-Weak-Symmetry-Breaking) sector When the couplings $g_2, g_1 = 0$. ($g_2 \in SU(2), g_1 \in U(1)$).

5.10 SCALE INVARIANCE.

$$\begin{aligned} x &\rightarrow e^\lambda x \\ \phi &\rightarrow e^{-\lambda} \phi \\ A_\mu &\rightarrow e^{-\lambda} A_\mu \end{aligned} \quad (5.129)$$

Any unitary theory which is scale invariant is also conformal invariant. Conformal invariance means that angles are preserved. The point here is that there is more than scale invariance.

We have classical internal global continuous symmetries. These can be either

1. “unbroken” (Wigner mode)

$$\hat{Q} |0\rangle = 0. \quad (5.130)$$

2. “spontaneously broken”

$$j^\mu(x) |0\rangle \neq 0 \quad (5.131)$$

(creates Goldstone modes).

3. “anomalous”. Classical symmetries are not a symmetry of QFT. Examples:

- Scale symmetry (to be studied in QFT II), although this is not truly internal.
- In QCD again when $\omega_q = 0$, a $U(1)$ symmetry (chiral symmetry) becomes exact, and cannot be preserved in QFT.
- In the standard model (E.W sector), the Baryon number and Lepton numbers are not symmetries, but their difference $B\# - L\#$ is a symmetry.

5.11 LORENTZ INVARIANCE.

We’d like to study the action of Lorentz symmetries on quantum states. We are going to “go by the book”, finding symmetries, currents, quantize, find generators, and so forth.

Under a Lorentz transformation

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad (5.132)$$

We are going to consider infinitesimal Lorentz transformations

$$\Lambda^\mu{}_\nu \approx \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad (5.133)$$

where $\omega^\mu{}_\nu$ is small. A Lorentz transformation Λ must satisfy $\Lambda^T G \Lambda = G$, or

$$g_{\mu\nu} = \Lambda^\alpha{}_\mu g_{\alpha\beta} \Lambda^\beta{}_\nu, \quad (5.134)$$

into which we insert the infinitesimal transformation representation

$$\begin{aligned} 0 &= -g_{\mu\nu} + (\delta^\alpha{}_\mu + \omega^\alpha{}_\mu) g_{\alpha\beta} (\delta^\beta{}_\nu + \omega^\beta{}_\nu) \\ &= -g_{\mu\nu} + (g_{\mu\beta} + \omega_{\beta\mu}) (\delta^\beta{}_\nu + \omega^\beta{}_\nu) \\ &= -g_{\mu\nu} + g_{\mu\nu} + \omega_{\nu\mu} + \omega_{\mu\nu} + \omega_{\beta\mu} \omega^\beta{}_\nu. \end{aligned} \quad (5.135)$$

The quadratic term can be ignored, leaving just

$$0 = \omega_{\nu\mu} + \omega_{\mu\nu}, \quad (5.136)$$

or

$$\omega_{\nu\mu} = -\omega_{\mu\nu}. \quad (5.137)$$

Note that ω is a completely antisymmetric tensor, and like $F_{\mu\nu}$ this has only 6 elements. This means that the infinitesimal transformation of the coordinates is

$$x^\mu \rightarrow x'^\mu \approx x^\mu + \omega^{\mu\nu} x_\nu, \quad (5.138)$$

the field transforms as

$$\phi(x) \rightarrow \phi'(x') = \phi(x) \quad (5.139)$$

or

$$\begin{aligned} \phi'(x^\mu + \omega^{\mu\nu} x_\nu) &= \phi'(x) + \omega^{\mu\nu} x_\nu \partial_\mu \phi(x) \\ &= \phi(x), \end{aligned} \quad (5.140)$$

so

$$\begin{aligned} \delta\phi &= \phi'(x) - \phi(x) \\ &= -\omega^{\mu\nu} x_\nu \partial_\mu \phi. \end{aligned} \quad (5.141)$$

Since \mathcal{L} is a scalar

$$\begin{aligned} \delta\mathcal{L} &= -\omega^{\mu\nu} x_\nu \partial_\mu \mathcal{L} \\ &= -\partial_\mu (\omega^{\mu\nu} x_\nu \mathcal{L}) + (\partial_\mu x_\nu) \omega^{\mu\nu} \mathcal{L} \\ &= \partial_\mu (-\omega^{\mu\nu} x_\nu \mathcal{L}), \end{aligned} \quad (5.142)$$

since $\partial_\nu x_\mu = g_{\nu\mu}$ is symmetric, and ω is antisymmetric. Our current is

$$J^\mu{}_\omega = -\omega^{\mu\nu} x_\nu \mathcal{L}. \quad (5.143)$$

FIXME: index mismatch above! .

Our Noether current is

$$\begin{aligned}
 j_{\omega\mu\rho}^\nu &= \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} \delta \phi - J_\omega^\mu \\
 &= \partial^\nu \phi \left(-\omega^{\mu\rho} x_\rho \partial_\mu \phi \right) + \omega^{\nu\rho} x_\rho \mathcal{L} \\
 &= \omega^{\mu\rho} \left(\partial^\nu \phi \left(-x_\rho \partial_\mu \phi \right) + \delta^\nu_\mu x_\rho \mathcal{L} \right) \\
 &= \omega^{\mu\rho} x_\rho \left(-\partial^\nu \phi \partial_\mu \phi + \delta^\nu_\mu \mathcal{L} \right).
 \end{aligned} \tag{5.144}$$

We identify

$$-T^\nu_\mu = -\partial^\nu \phi \partial_\mu \phi + \delta^\nu_\mu \mathcal{L}, \tag{5.145}$$

so the current is

$$j_{\omega\mu\rho}^\nu = -\omega^{\mu\rho} x_\rho T^\nu_\mu = -\omega_{\mu\rho} x^\rho T^{\nu\mu}. \tag{5.146}$$

Define

$$j^{\nu\mu\rho} = \frac{1}{2} (x^\rho T^{\nu\mu} - x^\mu T^{\nu\rho}), \tag{5.147}$$

which retains the antisymmetry in $\mu\rho$ yet still drops the parameter $\omega^{\mu\rho}$. To check that this makes sense, we can contract $j^{\nu\mu\rho}$ with $\omega_{\rho\mu}$

$$\begin{aligned}
 j^{\nu\mu\rho} \omega_{\rho\mu} &= -\frac{1}{2} (x^\rho T^{\nu\mu} - x^\mu T^{\nu\rho}) \omega_{\mu\rho} \\
 &= -\frac{1}{2} x^\rho T^{\nu\mu} \omega_{\mu\rho} - \frac{1}{2} x^\mu T^{\nu\rho} \omega_{\rho\mu} \\
 &= -\frac{1}{2} x^\rho T^{\nu\mu} \omega_{\mu\rho} - \frac{1}{2} x^\rho T^{\nu\mu} \omega_{\mu\rho} \\
 &= -x^\rho T^{\nu\mu} \omega_{\mu\rho},
 \end{aligned} \tag{5.148}$$

which matches eq. (5.146) as desired.

Example. Rotations $\mu\rho = ij$

$$\begin{aligned}
 J^{0ij} \epsilon_{ijk} &= \frac{1}{2} (x^i T^{0j} - x^j T^{0i}) \epsilon_{ijk} \\
 &= x^i T^{0j} \epsilon_{ijk}.
 \end{aligned} \tag{5.149}$$

Observe that this has the structure of $(\mathbf{x} \times \mathbf{p})_k$, where \mathbf{p} is the momentum density of the field. Let

$$L_k \equiv Q_k = \int d^3x J^{0ij} \epsilon_{ijk}. \tag{5.150}$$

We can now quantize and build a generator

$$\begin{aligned}\hat{U}(\alpha) &= e^{i\alpha \cdot \hat{\mathbf{L}}} \\ &= \exp \left(i\alpha_k \int d^3x x^i \hat{T}^{0j} \epsilon_{ijk} \right)\end{aligned}\tag{5.151}$$

From eq. (5.145) we can quantize with $T^{0j} = \partial^0 \phi \partial^j \phi \rightarrow \hat{\pi} (\nabla \hat{\phi})_j$, or

$$\begin{aligned}\hat{U}(\alpha) &= \exp \left(i\alpha_k \int d^3x x^i \hat{\pi} (\nabla \hat{\phi})_j \epsilon_{ijk} \right) \\ &= \exp \left(i\alpha \cdot \int d^3x \hat{\pi} \nabla \hat{\phi} \times \mathbf{x} \right)\end{aligned}\tag{5.152}$$

(up to a sign in the exponent which doesn't matter)

$$\begin{aligned}\hat{\phi}(\mathbf{y}) &\rightarrow \hat{U}(\alpha) \hat{\phi}(\mathbf{y}) \hat{U}^\dagger(\alpha) \\ &\approx \hat{\phi}(\mathbf{y}) + i\alpha \cdot \left[\int d^3x \hat{\pi}(\mathbf{x}) \nabla \hat{\phi}(\mathbf{x}) \times \mathbf{x}, \hat{\phi}(\mathbf{y}) \right] \\ &= \hat{\phi}(\mathbf{y}) + i\alpha \cdot \int d^3x (-i) \delta^{(3)}(\mathbf{x} - \mathbf{y}) \nabla \hat{\phi}(\mathbf{x}) \times \mathbf{x} \\ &= \hat{\phi}(\mathbf{y}) + \alpha \cdot (\nabla \hat{\phi}(\mathbf{y}) \times \mathbf{y}).\end{aligned}\tag{5.153}$$

Explicitly, in coordinates, this is

$$\begin{aligned}\hat{\phi}(\mathbf{y}) &\rightarrow \hat{\phi}(\mathbf{y}) + \alpha^i (\partial^j \hat{\phi}(\mathbf{y}) y^k \epsilon_{jki}) \\ &= \hat{\phi}(\mathbf{y}) - \epsilon_{ikj} \alpha^i y^k \partial^j \hat{\phi} \\ &= \hat{\phi}(y^j - \epsilon^{ikj} \alpha^i y^k).\end{aligned}\tag{5.154}$$

This is a rotation. To illustrate, pick $\alpha = (0, 0, \alpha)$, so $y^j \rightarrow y^j - \epsilon^{ikj} \alpha y^k \delta_{i3} = y^j - \epsilon^{3kj} \alpha y^k$, or

$$\begin{aligned}y^1 &\rightarrow y^1 - \epsilon^{3k1} \alpha y^k = y^1 + \alpha y^2 \\ y^2 &\rightarrow y^2 - \epsilon^{3k2} \alpha y^k = y^2 - \alpha y^1 \\ y^3 &\rightarrow y^3 - \epsilon^{3k3} \alpha y^k = y^3,\end{aligned}\tag{5.155}$$

or in matrix form

$$\begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix}.\tag{5.156}$$

5.12 PROBLEMS.

Exercise 5.1 Energy-momentum tensor for a scalar field

It is claimed in [8] (3.2.1) that the momentum components of the energy-momentum tensor was found to be

$$\mathbf{e}_n \int d^3x T^{0n} = \int d^3k \mathbf{k} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}. \quad (5.157)$$

- Calculate this.
- Calculate the other energy-momentum tensor components for the spacelike components.
- Calculate the other energy-momentum tensor components for the Hamiltonian component.

Answer for Exercise 5.1

First, from the Noether current for the scalar field Lagrangian in question, what is the energy-momentum tensor explicitly?

$$\begin{aligned} T^{\mu\nu} &= \pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \\ &= \pi^\mu \partial^\nu \phi - g^{\mu\nu} \frac{1}{2} (\partial_\alpha \phi \partial^\alpha \phi - \mu^2 \phi^2) \\ &= \pi^\mu \pi^\nu - g^{\mu\nu} \frac{1}{2} (\pi_\alpha \pi^\alpha - \mu^2 \phi^2) \\ &= \pi^\mu \pi^\nu - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} \pi^\beta \pi^\alpha + \frac{1}{2} g^{\mu\nu} \mu^2 \phi^2. \end{aligned} \quad (5.158)$$

Consider some special cases for the indexes. For $\mu = \nu = 0$, the result is the Hamiltonian density

$$\begin{aligned} T^{00} &= \pi^0 \pi^0 - \frac{1}{2} g^{00} \pi_\alpha \pi^\alpha + \frac{1}{2} g^{00} \mu^2 \phi^2 \\ &= \pi^0 \pi^0 - \frac{1}{2} \pi_\alpha \pi^\alpha + \frac{1}{2} \mu^2 \phi^2 \\ &= \frac{1}{2} \pi^0 \pi^0 - \frac{1}{2} \pi_n \pi^n + \frac{1}{2} \mu^2 \phi^2 \\ &= \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2, \end{aligned} \quad (5.159)$$

where $\pi^2 = (\partial_0 \phi)^2 \neq \partial^2 \phi$. For any $\mu \neq \nu$ the off diagonal metric elements are zero, leaving just

$$T^{\mu\nu} = \pi^\mu \pi^\nu. \quad (5.160)$$

Finally, when $n \neq 0$, the remaining diagonal terms are

$$\begin{aligned}
 T^{nn} &= \pi^n \pi^n - \frac{1}{2} g^{nn} \pi_\alpha \pi^\alpha + \frac{1}{2} g^{nn} n^2 \phi^2 \\
 &= \pi^n \pi^n + \frac{1}{2} \pi_\alpha \pi^\alpha - \frac{1}{2} \mu^2 \phi^2 \\
 &= \frac{1}{2} \pi^2 + \pi^n \pi^n - \frac{1}{2} \pi^m \pi^m - \frac{1}{2} \mu^2 \phi^2 \\
 &= \frac{1}{2} \pi^2 + \frac{1}{2} \pi^n \pi^n - \frac{1}{2} \sum_{m \neq n, 0} \pi^m \pi^m - \frac{1}{2} \mu^2 \phi^2 \\
 &= \frac{1}{2} \sum_{m=n, 0} \pi^m \pi^m - \frac{1}{2} \sum_{m \neq n, 0} \pi^m \pi^m - \frac{1}{2} \mu^2 \phi^2.
 \end{aligned} \tag{5.161}$$

The canonical momenta are

$$\pi^\mu = \partial^\mu \int \frac{d^3 k}{(2\pi)^{3/2} \sqrt{2\omega_k}} (a_{\mathbf{k}} e^{-ik \cdot x} + a_{\mathbf{k}}^\dagger e^{ik \cdot x}), \tag{5.162}$$

but

$$\begin{aligned}
 \partial^\mu e^{ik \cdot x} &= \partial^\mu \exp(ik^\alpha x_\alpha) \\
 &= ik^\mu \exp(ik \cdot x),
 \end{aligned} \tag{5.163}$$

so

$$\begin{aligned}
 \pi^\mu &= i \int \frac{d^3 k k^\mu}{(2\pi)^{3/2} \sqrt{2\omega_k}} (-a_{\mathbf{k}} e^{-ik \cdot x} + a_{\mathbf{k}}^\dagger e^{ik \cdot x}) \\
 &= i \int \frac{d^3 k k^\mu}{(2\pi)^{3/2} \sqrt{2\omega_k}} (-a_{\mathbf{k}} e^{-i\omega_k t + \mathbf{k} \cdot \mathbf{x}} + a_{\mathbf{k}}^\dagger e^{i\omega_k t - i\mathbf{k} \cdot \mathbf{x}}) \\
 &= i \int \frac{d^3 k k^\mu}{(2\pi)^{3/2} \sqrt{2\omega_k}} (-a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) e^{i\mathbf{k} \cdot \mathbf{x}}.
 \end{aligned} \tag{5.164}$$

This gives

$$\begin{aligned}
 \int d^3 x \pi^\mu \pi^\nu &= -\frac{1}{2} \int d^3 x \frac{d^3 k d^3 p}{(2\pi)^3} \frac{k^\mu p^\nu}{\sqrt{\omega_k \omega_p}} (-a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) (-a_{\mathbf{p}} e^{-i\omega_p t} + a_{-\mathbf{p}}^\dagger e^{i\omega_p t}) e^{i(\mathbf{p} + \mathbf{k}) \cdot \mathbf{x}} \\
 &= -\frac{1}{2} \int d^3 k d^3 p \frac{k^\mu p^\nu}{\sqrt{\omega_k \omega_p}} (-a_{\mathbf{k}} e^{-i\omega_k t} + a_{-\mathbf{k}}^\dagger e^{i\omega_k t}) (-a_{\mathbf{p}} e^{-i\omega_p t} + a_{-\mathbf{p}}^\dagger e^{i\omega_p t}) \delta^{(3)}(\mathbf{p} + \mathbf{k}) \\
 &= -\frac{1}{2} \int d^3 k d^3 p \frac{k^\mu p^\nu}{\omega_k} (a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}} a_{\mathbf{k}}^\dagger - a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t}) \delta^{(3)}(\mathbf{p} + \mathbf{k}).
 \end{aligned} \tag{5.165}$$

Further reduction of the leading $k^\mu p^\nu$ term has a sign that depends on the values of the indices.

Part a. First consider the momentum case where one of μ , or ν is zero

$$\begin{aligned}\int d^3x \pi^\mu \pi^0 &= \int d^3x \pi^0 \pi^\mu \\ &= -\frac{1}{2} \int d^3k k^\mu \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}} a_{\mathbf{k}}^\dagger - a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right).\end{aligned}\tag{5.166}$$

For $\mu \neq 0$ this can be written as a vector operator

$$\begin{aligned}\mathbf{e}_n \int d^3x T^{0n} &= -\frac{1}{2} \int d^3k k^\mu \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right) \\ &\quad + \frac{1}{2} \int d^3k k^\mu \left(a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right)\end{aligned}\tag{5.167}$$

To get the desired result the time dependent terms have to be made to go away somehow. Consider a spherical parameterization of the momentum space

$$\mathbf{k} = k (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),\tag{5.168}$$

Note that the volume element is

$$d^3k = k^2 \sin \theta dk \wedge d\theta \wedge d\phi,\tag{5.169}$$

where $k \in [0, \infty]$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$. If we map $\mathbf{k} \rightarrow -\mathbf{k}$, the volume element becomes

$$d^3k = (-k)^2 \sin \theta d(-k) \wedge d\theta \wedge d\phi,\tag{5.170}$$

over the same angular intervals, but $k \in [-\infty, 0]$. Flipping the sign of the time dependent operator products gives

$$\begin{aligned}a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} &\rightarrow a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} + a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} \\ &= a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t},\end{aligned}\tag{5.171}$$

which shows that this is an even function in \mathbf{k} . The even characteristics of the volume element and time dependent terms and the odd character of the momentum vector \mathbf{k} can be used to show that these terms integrate out to zero. Let's compute the integral by averaging the momentum operator using both parameterization sign options. First write

$$f(\mathbf{k}) = a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t},\tag{5.172}$$

so

$$\begin{aligned}
\int d^3 k \mathbf{k} f(\mathbf{k}) &= \frac{1}{2} \int d^3 k \mathbf{k} f(\mathbf{k}) + \frac{1}{2} \int d^3 k' \mathbf{k}' f(\mathbf{k}') \\
&= \frac{1}{2} \int_0^\infty k^2 dk \int_0^\pi \sin \theta d\theta \int_0^{2\pi} k \hat{\mathbf{k}}(\theta, \phi) f(\mathbf{k}) \\
&\quad + \frac{1}{2} \int_{-\infty}^0 k^2 d(-k) \int_0^\pi \sin \theta d\theta \int_0^{2\pi} (-k) \hat{\mathbf{k}}(\theta, \phi) f(-\mathbf{k}) \\
&= \frac{1}{2} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \hat{\mathbf{k}} \left(\int_0^\infty k^3 dk f(\mathbf{k}) + \int_{-\infty}^0 k^3 dk f(-\mathbf{k}) \right) \\
&= \frac{1}{2} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \hat{\mathbf{k}} \left(\int_0^\infty k^3 dk f(\mathbf{k}) - \int_0^\infty k^3 dk f(\mathbf{k}) \right) \\
&= 0,
\end{aligned} \tag{5.173}$$

so the momentum is reduced to

$$\begin{aligned}
\mathbf{e}_n \int d^3 x T^{0n} &= \frac{1}{2} \int d^3 k \mathbf{k} (a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) \\
&= \frac{1}{2} \int d^3 k \mathbf{k} (2a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + [a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger]) \\
&= \int d^3 k \mathbf{k} \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} \delta^{(3)}(0) \right).
\end{aligned} \tag{5.174}$$

An argument like that of [14] can be used to dismiss the unphysical infinity associated with the ground state energy level, leaving just

$$\boxed{\mathbf{e}_n \int d^3 x T^{0n} = \int d^3 k \mathbf{k} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}.} \tag{5.175}$$

Part b. For $\mu = m \neq 0$, and $\nu = n \neq 0$, we have

$$\int d^3 x \pi^m \pi^n = \frac{1}{2} \int d^3 k \frac{k^m k^n}{\omega_k} (a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}} a_{\mathbf{k}}^\dagger - a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t}). \tag{5.176}$$

Can the time dependent terms be killed in this case?

Part c. TODO: some stuff is wrong here.

For $\nu \neq 0$

$$\begin{aligned}
\int d^3 x \pi^\mu \pi^\nu &= -\frac{1}{2} \int d^3 k \frac{k^\mu k^\nu}{\omega_k} (-a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}} a_{\mathbf{k}}^\dagger - a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t}) \\
&= \frac{1}{2} \int d^3 k \frac{k^\mu k^\nu}{\omega_k} (a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t}).
\end{aligned} \tag{5.177}$$

Here's a summary of these products

$$\int d^3x \pi^0 \pi^0 = -\frac{1}{2} \int d^3k \omega_k \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}} a_{\mathbf{k}}^\dagger - a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right), \quad (5.178a)$$

$$\begin{aligned} \int d^3x \pi^n \pi^0 &= \int d^3x \pi^0 \pi^n \\ &= -\frac{1}{2} \int d^3k k^n \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}} a_{\mathbf{k}}^\dagger - a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right), \end{aligned} \quad (5.178b)$$

$$\int d^3x \pi^m \pi^n = \frac{1}{2} \int d^3k \frac{k^m k^n}{\omega_k} \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right). \quad (5.178c)$$

For the mass term it was previously found that

$$\frac{1}{2} \int d^3x \mu^2 \phi^2 = \frac{\mu^2}{4} \int d^3k \frac{1}{\omega_k} \left(a_{-\mathbf{k}} a_{\mathbf{k}} e^{-2i\omega_k t} + a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_k t} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right). \quad (5.179)$$

The Hamiltonian component has been previously calculated, and resolves to

$$\int d^3x T^{00} = \frac{1}{2} \int d^3k \omega_k \left(a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right). \quad (5.180)$$

The other diagonal components, for $r \neq s \neq t$ are

$$\begin{aligned} \int d^3x T^{rr} &= \int d^3x \left(\frac{1}{2} \sum_{m=r,0} \pi^m \pi^m - \frac{1}{2} \sum_{m=s,t} \pi^m \pi^m - \frac{1}{2} \mu^2 \phi^2 \right) \\ &= \frac{1}{4} \int d^3k \frac{(k^r)^2 - (k^s)^2 - (k^t)^2 - \mu^2}{\omega_k} \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right) \\ &\quad - \frac{1}{4} \int d^3k \omega_k \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} - a_{\mathbf{k}} a_{\mathbf{k}}^\dagger - a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right) \\ &= \frac{1}{4} \int d^3k \frac{(k^r)^2 - (k^s)^2 - (k^t)^2 - \mu^2 - \omega_k^2}{\omega_k} \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right) \\ &\quad + \frac{1}{4} \int d^3k \frac{(k^r)^2 - (k^s)^2 - (k^t)^2 - \mu^2 + \omega_k^2}{\omega_k} \left(a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right) \\ &= \frac{1}{2} \int d^3k \frac{(k^r)^2 - \omega_k^2}{\omega_k} \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2i\omega_k t} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger e^{2i\omega_k t} \right) + \frac{1}{2} \int d^3k \frac{(k^r)^2}{\omega_k} \left(a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \right). \end{aligned} \quad (5.181)$$

This doesn't have the nice cancellation that killed the time dependent terms in the Hamiltonian. Such cancellation also doesn't appear in the off diagonal energy-momentum tensor components, which are

END-REDACTION

Exercise 5.4 A model with $SU(2)_L \times SU(2)_R$ internal global symmetry: chiral symmetry and the Higgs (2018 HW2).

This problem introduces a model to describe the symmetry realization of the nonabelian chiral symmetry in QCD (quantum chromodynamics). The word “chiral” should become clear later in this class, but the “nonabelian” part will be clear below. $SU(2)_L \times SU(2)_R$ is an exact symmetry of QCD in the limit when the “current masses” of the u and d quark, m_u and m_d , are taken to vanish. In the real world, it is an approximate symmetry, in the sense that m_u and m_d are small compared to the intrinsic scale of QCD, given, say, by the proton mass ($m_{u,d} \sim \text{MeV} \ll 1 \text{ GeV}$). This is, thus, an example of an “approximate symmetry”.

Closer to the theory you will study below, the scalar model with $SU(2)_L \times SU(2)_R$ symmetry, is really the same as the Higgs sector in the Standard Model, in the limit when the electromagnetic and weak interactions are turned off. $SU(2)_L \times SU(2)_R$ becomes a symmetry in this limit. It is only an approximate symmetry, as the electromagnetic and weak couplings (which explicitly break it) are dimensionless numbers smaller than unity.

Finally, to end the preaching preamble, the notion of approximate symmetries is not new and you have, for sure, been exposed to its usefulness when studying the hydrogen atom spectrum in quantum mechanics.

- a. The Lagrangian you will study is that of two complex scalar fields, assembled into a column $\Phi = (\phi_1, \phi_2)^T$ (the T is here so I do not have to go through the trouble to write a column instead of a row). It is given by:

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial_\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2. \quad (5.188)$$

Show that eq. (5.188) is invariant under an $SU(2)_L$ global symmetry transformation $\Phi \rightarrow U_L \Phi$, where $U_L^\dagger U_L = 1$ is a 2×2 unitary matrix of unit determinant. In addition, the Lagrangian has a $U(1)$ symmetry, not part of $SU(2)_L$, acting as $\Phi \rightarrow e^{i\alpha} \Phi$. Find the currents and conserved charges under these symmetries.

Hint: recall that an infinitesimal $SU(2)_L$ transformation can be written as $U_L \approx \sigma^0 + i\omega_a \frac{\sigma^a}{2}$, where σ^0 is the unit 2×2 matrix, $\sigma^a, a = 1, 2, 3$ are the Pauli matrices, and ω_a are the three parameters of infinitesimal $SU(2)_L$ transformations.

- b. Show that the charge operators, $\hat{Q}_a^L, a = 1, 2, 3$, conserved due to $SU(2)_L$ invariance, obey the angular momentum algebra, i.e., $[\hat{Q}_1^L, \hat{Q}_2^L] = i\hat{Q}_3^L$ (plus cyclic permutations).

- c. The Lagrangian eq. (5.188) has, however, a larger symmetry than simply the above $SU(2)_L$. To begin seeing this, instead of using $\Phi = (\phi^1, \phi^2)^T$ introduce the real and imaginary parts of $\phi^{1,2}$. Use $\phi^1 = \psi^1 + i\psi^2$, $\phi^2 = \psi^3 + i\psi^4$, and introducing $\Psi = (\psi^1, \psi^2, \psi^3, \psi^4)^T$, show that eq. (5.188) can be written as:

$$\mathcal{L} = a\partial_\mu \Psi^T \partial^\mu \Psi - bm^2 \Psi^T \Psi - c\lambda(\Psi^T \Psi)^2 \quad (5.189)$$

on the way determining the (pure numbers) a, b, c . The Lagrangian eq. (5.189) has, clearly, an $O(4)$ symmetry, i.e., is invariant under $\Psi \rightarrow O\Psi$, where O is a 4×4 orthogonal matrix, $O^T O = 1$. Is there a continuous $U(1)$ allowed in this case?

Comment: I will spare you finding the currents for $SO(4)$ ($SO(4)$ matrices are the restriction of $O(4)$ matrices to the ones with unit determinant). What you will do next, instead, is to use the equivalence of Lie algebras $SO(4) \approx SU(2)_L \times SU(2)_R$, which will come about by another change of variables (see below). Notice also that, as it comes, $SO(4)$ happens to be the Euclidean version of $SO(1, 3)$.

- d. To expose the $SU(2)_L \times SU(2)_R$ symmetry of eq. (5.188), now use the following change of variables. Consider, instead of Φ in eq. (5.188) the 2×2 matrix H made up by components of Φ as follows:

$$H \equiv \frac{1}{\sqrt{2}}(i\sigma^2 \Phi^*, \Phi) = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{bmatrix} \quad (5.190)$$

Show that under $SU(2)$ transformations,

$$\begin{aligned} H &\rightarrow \frac{1}{\sqrt{2}}(i\sigma^2(U_L \Phi)^*, U_L \Phi) \\ &= \frac{1}{\sqrt{2}}(U_L i\sigma^2 \Phi^*, U_L \Phi) \\ &= U_L H. \end{aligned} \quad (5.191)$$

Hint: the tricky part is to show that $i\sigma^2(U_L \Phi)^* = i\sigma^2 U_L^* \Phi^* = U_L i\sigma^2 \Phi^*$. What you need to show, then, is that $\sigma^2 U_L \sigma^2 = U_L^*$ (this fact will be very useful in our future studies of spinors, so make sure you understand it).

- e. Using the change of variables eq. (5.190), show that

$$H^\dagger H = \frac{1}{2} \begin{bmatrix} |\phi_1|^2 + |\phi_2|^2 & 0 \\ 0 & |\phi_1|^2 + |\phi_2|^2 \end{bmatrix}, \quad (5.192)$$

and, hence, that eq. (5.188) can be written as

$$\mathcal{L} = \text{tr}(\partial_\mu H^\dagger \partial^\mu H) - m^2 \text{tr}(H^\dagger H) - \lambda(\text{tr} H^\dagger H)^2 \quad (5.193)$$

where tr denotes the matrix trace. Show that now eq. (5.193) has $SU(2)_L \times SU(2)_R$ symmetry, acting on H as

$$H \rightarrow U_L H U_R^\dagger, \quad (5.194)$$

where the action of U_R^\dagger on the right is pure convention (we could have taken U_R instead). U_L and U_R are two sets of independent $SU(2)$ transformations. The L and R (left and right) names are self-evident in the way eq. (5.194) is written. Show that under $SU(2)_L \times SU(2)_R$

$$\delta H = i\omega_a^L \frac{\sigma^a}{2} H - i\omega_b^R H \frac{\sigma^b}{2}. \quad (5.195)$$

Hint: clearly, the only thing you need to show is $SU(2)_R$ invariance, as $SU(2)_L$ was already shown.

- f. Show that the left and right $SU(2)$ conserved currents can be written as

$$\begin{aligned} j_L^{\mu,a} &= \frac{i}{2} \text{tr} \left(\partial^\mu H^\dagger \sigma^a H - H^\dagger \sigma^a \partial^\mu H \right) \\ j_R^{\mu,b} &= \frac{i}{2} \text{tr} \left(\partial^\mu H \sigma^b H^\dagger - H \sigma^b \partial^\mu H^\dagger \right) \end{aligned} \quad (5.196)$$

and that the corresponding generators $\hat{Q}_a^{L,R}$ obey the commutation relations of two commuting angular momentum algebras.

Hint: notice that both currents are Hermitian and that the left is obtained from the right by interchanging H with H^\dagger .

Answer for Exercise 5.4

PROBLEM SET RELATED MATERIAL REDACTED IN THIS DOCUMENT. PLEASE
FEEL FREE TO EMAIL ME FOR THE FULL VERSION IF YOU AREN'T TAKING PHY2403.

[REDACTED]

END-REDACTION

Exercise 5.5 $SU(2)_L \times SU(2)_R$, realized in the Wigner and Nambu-Goldstone modes. (2018 HW2.III)

Consider now our Lagrangian eq. (5.193) and imagine that $m^2 < 0$, for whatever reason (nobody knows, really), while λ is still positive. This now becomes the Higgs Lagrangian of the Standard Model.

- a. Show that the classical potential in eq. (5.193) now becomes:

$$\begin{aligned} V &= -|m^2| \operatorname{tr} H^\dagger H + \lambda (\operatorname{tr} H^\dagger H)^2 \\ &= \lambda \left(|\phi_1|^2 + |\phi_2|^2 - \frac{|m^2|}{2\lambda} \right)^2 + \text{const.} \end{aligned} \quad (5.197)$$

- b. Clearly, there are extrema of the potential when $|\phi_1|^2 + |\phi_2|^2 = 0$ and when $|\phi_1|^2 + |\phi_2|^2 = \frac{|m^2|}{2\lambda}$. The second one has, clearly, smaller energy density. To quantize the theory, we now have to choose which classical minimum to expand around. Show that, if we expand around $|\phi_1|^2 + |\phi_2|^2 = 0$, we will find that the $\phi_{1,2}$ excitations are tachyons, even classically. This signals an instability, rather than a faster-than-light propagation and shows that we have chosen the wrong value of Φ to build our quantum theory.
- c. Thus, consider the $|\phi_1|^2 + |\phi_2|^2 = \frac{|m^2|}{2\lambda}$ minimum of V . This is really a set of minima. In fact the set parameterized by $|\phi_1|^2 + |\phi_2|^2 = \text{const}$ is also known as a three sphere (S^3 , embedded in a four-dimensional space parameterized by $\psi^{1\dots 4}$ - not the spacetime!). To build the quantum theory, we will choose a point on this three sphere (a.k.a. the “vacuum manifold” - the set of field values that minimize the potential). We will now study the small fluctuations around the chosen point and the spectrum of the theory in this vacuum. There is an infinite number of parameterizations that can be used to do this, but I will suggest one that makes the symmetries the clearest. Thus, use the H -representation and take

$$H(x) = \frac{|m|}{2\sqrt{\lambda}} (1 + h(x)) e^{i\phi^a(x)\sigma^a} \quad (5.198)$$

The logic here is as follows. When $h(x)$ and $\phi^a(x)$ vanish (i.e. there are no excitations), the parameterization eq. (5.198) is equivalent, by eq. (5.192), to taking a specific point on the vacuum manifold, i.e. the one where $\phi_1 = 0$ and $\phi_2 = |m|/\sqrt{2\lambda}$. The fields $h(x)$ and $\phi^a(x)$ parameterize the fluctuations around this ground state (for sure, they can be mapped - the map is nonlinear - to the fluctuations of the fields $\phi_{1,2}$ around the chosen vacuum value for ϕ_2 .⁵ What you will do now is take the form eq. (5.198), plug it into the Lagrangian eq. (5.193) with $m^2 = -|m^2|$, and expand what you find to second order in the fields $h(x)$ and $\phi^a(x)$. Show that the field $h(x)$ has a mass and find an expression

⁵ As in classical mechanics, which variables one uses to describe physics is a matter of choice and convenience. The Euler-Lagrange equations have the property that they are invariant under changes of variables, so long as no singularity occurs in the process. In fact, one of the main motivations of using Lagrangians in classical mechanics is that the change of variables is much easier to do. In other words, it is much easier to first transform the Lagrangian to spherical coordinates and then find the Euler-Lagrange equations than to transform the equations found in Cartesian coordinates to spherical coordinates (in the latter case you need to differentiate twice...). Invariance of physics under nonsingular changes of variables in the Lagrangian is, of course, inherited in field theory.

for it. Show that the fields $\phi^a(x)$ remain massless and that their Lagrangian (not just to quadratic order) only contains derivatives.

The latter point can be seen pretty simply by noting that $H(x)$ from eq. (5.198) can be written as

$$H(x) = \frac{|m|}{2\sqrt{\lambda}} \Omega(x)(1 + h(x)), \quad (5.199)$$

with $\Omega^\dagger \Omega = 1$ and $\det(\Omega(x)) = 1$. In this parameterization $\Omega(x)$ fluctuations correspond to going around the vacuum manifold S^3 , while the $h(x)$ fluctuations are along the “radial” directions away from the minimum. The latter cost energy, hence h is massive (the Higgs field!), while the $\Omega(x)$ only cost energy if the x -dependence is nontrivial. The $\phi^a(x)$ (or $\Omega(x)$) are equivalent parameterizations of the Goldstone fields. What you found here is an example of a general story: if a theory has a continuous symmetry, which is not a symmetry of the ground state, there is a number of massless Goldstone (or Nambu-Goldstone) modes. For internal symmetries like the ones we are considering here, their number is equal to the number of broken generators.

In the Standard Model, $h(x)$ is indeed the Higgs field. The fields $\phi^a(x)$ actually become the longitudinal components of the W and Z -bosons (one usually says that they are “eaten”, a manifestation of the Landau-Anderson-Higgs-Brout-Englert-Guralnik-Hagen... mechanism).

- d. One question that was not discussed and remained a bit obscure is that of the unbroken part of the symmetry. The original Lagrangian has $SU(2)_L \times SU(2)_R$ symmetry. The value of $H(x)$ in the vacuum, denoted by $\langle H \rangle$, is given by eq. (5.198) with $h = \phi^a = 0$ and is $\langle H \rangle \sim$ unit matrix. Show that, while $\langle H \rangle$ is not invariant under $SU(2)_L \times SU(2)_R$ for arbitrary $SU(2)_L$ and $SU(2)_R$ transformations, it is invariant under eq. (5.194) with $U_L = U_R$. Such $SU(2)_L \times SU(2)_R$ transformations with $U_L = U_R$ are called “diagonal” or “vector” $SU(2)_V$ transformations. These remain unbroken in the vacuum. In the electroweak theory, the third component of $SU(2)_V$ is identified with electromagnetic $U(1)$. Show that the current associated with $SU(2)_V$ transformations has the form:

$$j_\mu^{V,a} = \frac{i}{2} \text{tr} \left(\partial_\mu H^\dagger [\sigma^a, H] + \partial_\mu H [\sigma^a, H^\dagger] \right) \quad (5.200)$$

Show also that the other “linear” combination of $SU(2)_L$ and $SU(2)_R$, eq. (5.194) with $U_R = U_L^\dagger$ corresponds to the current (not conserved!) usually called the “axial current”

$$j_\mu^{A,a} = \frac{i}{2} \text{tr} \partial_\mu H^\dagger \{ \sigma^a, H \} - \partial_\mu H \{ \sigma^a, H^\dagger \}, \quad (5.201)$$

where $\{A, B\} = AB + BA$ denotes the anticommutator.

LORENTZ BOOSTS, GENERATORS, LORENTZ INVARIANCE, MICROCAUSALITY.

6.1 LORENTZ TRANSFORM SYMMETRIES.

From last time, recall that an infinitesimal Lorentz transform has the form

$$x^\mu \rightarrow x^\mu + \omega^{\mu\nu} x_\nu, \quad (6.1)$$

where

$$\omega^{\mu\nu} = -\omega^{\nu\mu}. \quad (6.2)$$

We showed last time that ω^{ij} induces a rotation, and will show today that ω^{0i} is a boost.

We introduced a three index current, factoring out explicit dependence on the incremental Lorentz transform tensor $\omega^{\mu\nu}$ as follows

$$J^{\nu\mu\rho} = \frac{1}{2} (x^\rho T^{\nu\mu} - x^\mu T^{\nu\rho}), \quad (6.3)$$

and can easily show that this current has the desired zero four-divergence property

$$\begin{aligned} \partial_\nu J^{\nu\mu\rho} &= \frac{1}{2} ((\partial_\nu x^\rho) T^{\nu\mu} + x^\rho \partial_\nu T^{\nu\mu} - (\partial_\nu x^\mu) T^{\nu\rho} - x^\mu \partial_\nu T^{\nu\rho}) \\ &= \frac{1}{2} (T^{\rho\mu} + -T^{\mu\rho}) \\ &= 0, \end{aligned} \quad (6.4)$$

since the energy-momentum tensor is symmetric.

Defining charge in the usual fashion $Q = \int d^3x j^0$, so we can define a charge for each pair of indexes $\mu\nu$, and in particular

$$\begin{aligned} Q^{0k} &= \int d^3x J^{00k} \\ &= \frac{1}{2} \int d^3x (x^k T^{00} - x^0 T^{0k}) \end{aligned} \quad (6.5)$$

$$\begin{aligned} \dot{Q}^{0k} &= \int d^3x \dot{J}^{00k} \\ &= \frac{1}{2} \int d^3x (x^k \dot{T}^{00} - x^0 \dot{T}^{0k}). \end{aligned} \quad (6.6)$$

However, since $0 = \partial_\mu T^{\mu\nu} = \dot{T}^{0\nu} + \partial_j T^{j\nu}$, or $\dot{T}^{0\nu} = -\partial_j T^{j\nu}$,

$$\begin{aligned}
 \dot{Q}^{0k} &= \frac{1}{2} \int d^3x \left(x^k (-\partial_j T^{j0}) - T^{0k} - x^0 (-\partial_j T^{jk}) \right) \\
 &= \frac{1}{2} \int d^3x \left(\partial_j (-x^k T^{j0}) + (\partial_j x^k) T^{j0} - T^{0k} + x^0 \partial_j T^{jk} \right) \\
 &= \frac{1}{2} \int d^3x \left(\partial_j (-x^k T^{j0}) + \mathcal{T}^{k0} - \mathcal{T}^{0k} + x^0 \partial_j T^{jk} \right) \\
 &= \frac{1}{2} \int d^3x \partial_j \left(-x^k T^{j0} + x^0 T^{jk} \right),
 \end{aligned} \tag{6.7}$$

which leaves just surface terms, so $\dot{Q}^{0k} = 0$.

Quantizing: From our previous identification eq. (5.145), we have

$$T^{\nu\mu} = \partial^\nu \phi \partial^\mu \phi - g^{\nu\mu} \mathcal{L}. \tag{6.8}$$

In particular

$$\begin{aligned}
 T^{00} &= \partial^0 \phi \partial^0 \phi - \frac{1}{2} (\partial_0 \phi \partial^0 \phi + \partial_k \phi \partial^k \phi) \\
 &= \frac{1}{2} \partial^0 \phi \partial^0 \phi - \frac{1}{2} (\nabla \phi)^2,
 \end{aligned} \tag{6.9}$$

and

$$T^{0k} = \partial^0 \phi \partial^k \phi. \tag{6.10}$$

We may quantize these energy momentum tensor components as

$$\begin{aligned}
 \hat{T}^{00} &= \frac{1}{2} \hat{\pi}^2 + \frac{1}{2} (\nabla \hat{\phi})^2 \\
 \hat{T}^{0k} &= \frac{1}{2} \hat{\pi} \partial^k \hat{\phi}.
 \end{aligned} \tag{6.11}$$

We can now start computing the commutators associated with the charge operator. The first of those commutators is

$$[\hat{T}^{00}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = \frac{1}{2} [\hat{\pi}^2(\mathbf{x}), \hat{\phi}(\mathbf{y})], \tag{6.12}$$

which can be evaluated using the field commutator analogue of $[F(p), q] = iF'$ which is

$$[F(\hat{\pi}(\mathbf{x})), \hat{\phi}(\mathbf{y})] = -i \frac{dF}{d\hat{\pi}} \delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{6.13}$$

to give

$$[\hat{T}^{00}(\mathbf{x}), \hat{\phi}(\mathbf{y})] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y})\hat{\pi}(\mathbf{x}). \quad (6.14)$$

The other required commutator is

$$\begin{aligned} [\hat{T}^{0i}(\mathbf{x}), \hat{\phi}(\mathbf{y})] &= [\hat{\pi}(\mathbf{x})\partial^i\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] \\ &= \partial^i\hat{\phi}(\mathbf{x})[\hat{\pi}(\mathbf{x}), \hat{\phi}(\mathbf{y})] \\ &= -i\delta^{(3)}(\mathbf{x} - \mathbf{y})\partial^i\hat{\phi}(\mathbf{x}). \end{aligned} \quad (6.15)$$

The charge commutator with the field can now be computed

$$\begin{aligned} i\epsilon[\hat{Q}^{0k}, \hat{\phi}(\mathbf{y})] &= i\frac{\epsilon}{2} \int d^3x \left(x^k [\hat{T}^{00}, \hat{\phi}(\mathbf{y})] - x^0 [\hat{T}^{0k}, \hat{\phi}(\mathbf{y})] \right) \\ &= \frac{\epsilon}{2} (y^k \hat{\pi}(\mathbf{y}) - y^0 \partial^k \hat{\phi}(\mathbf{y})) \\ &= \frac{\epsilon}{2} (y^k \dot{\hat{\phi}}(\mathbf{y}) - y^0 \partial^k \hat{\phi}(\mathbf{y})), \end{aligned} \quad (6.16)$$

so to first order in ϵ

$$e^{i\epsilon\hat{Q}^{0k}}\hat{\phi}(\mathbf{y})e^{-i\epsilon\hat{Q}^{0k}} = \hat{\phi}(\mathbf{y}) + \frac{\epsilon}{2}y^k\dot{\hat{\phi}}(\mathbf{y}) + \frac{\epsilon}{2}y^0\partial_k\hat{\phi}(\mathbf{y}). \quad (6.17)$$

For example, with $k = 1$

$$\begin{aligned} e^{i\epsilon\hat{Q}^{01}}\hat{\phi}(\mathbf{y})e^{-i\epsilon\hat{Q}^{01}} &= \hat{\phi}(\mathbf{y}) + \frac{\epsilon}{2} \left(y^1 \dot{\hat{\phi}}(\mathbf{y}) + y^0 \frac{\partial \hat{\phi}}{\partial y^1}(\mathbf{y}) \right) \\ &= \hat{\phi}(y^0 + \frac{\epsilon}{2}y^1, y^1 + \frac{\epsilon}{2}y^2, y^3). \end{aligned} \quad (6.18)$$

This is a boost. If we compare explicitly to an infinitesimal Lorentz transformation of the coordinates

$$\begin{aligned} x^0 &\rightarrow x^0 + \omega^{01}x_1 = x^0 - \omega^{01}x^1 \\ x^1 &\rightarrow x^1 + \omega^{10}x_0 = x^1 - \omega^{01}x_0 = x^1 - \omega^{01}x^0, \end{aligned} \quad (6.19)$$

we can make the identification

$$\frac{\epsilon}{2} = -\omega^{01}. \quad (6.20)$$

We now have the explicit form of the generator of a spacetime translation

$$\boxed{\hat{U}(\Lambda) = \exp\left(-i\omega^{0k} \int d^3x (\hat{T}^{00}x^k - \hat{T}^{0k}x^0)\right).} \quad (6.21)$$

An explicit boost along the x-axis has the form

$$\hat{U}(\Lambda)\hat{\phi}(t, \mathbf{x})\hat{U}^\dagger(\Lambda) = \hat{\phi}\left(\frac{t - vx}{\sqrt{1 - v^2}}, \frac{x - vt}{\sqrt{1 - v^2}}, y, z\right), \quad (6.22)$$

and more generally

$$\hat{U}(\Lambda)\hat{\phi}(x)\hat{U}^\dagger(\Lambda) = \hat{\phi}(\Lambda x), \quad (6.23)$$

where x is a four vector, $(\Lambda x)^\mu = \Lambda^\mu_\nu x^\nu$, and $\Lambda^\mu_\nu \approx \delta^\mu_\nu + \omega^\mu_\nu$.

6.2 TRANSFORMATION OF MOMENTUM STATES.

In the momentum space representation

$$\begin{aligned} \hat{\phi}(x) &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{i(\omega_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x})} \hat{a}_{\mathbf{p}} + e^{-i(\omega_{\mathbf{p}} t - \mathbf{p} \cdot \mathbf{x})} \hat{a}_{\mathbf{p}}^\dagger \right) \\ &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{ip^\mu x_\mu} \hat{a}_{\mathbf{p}} + e^{-ip^\mu x_\mu} \hat{a}_{\mathbf{p}}^\dagger \right) \Big|_{p_0 = \omega_{\mathbf{p}}}. \end{aligned} \quad (6.24)$$

$$\begin{aligned} \hat{U}(\Lambda)\hat{\phi}(x)\hat{U}^\dagger(\Lambda) &= \hat{\phi}(\Lambda x) \\ &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{ip^\mu \Lambda^\mu_\nu x^\nu} \hat{a}_{\mathbf{p}} + e^{-ip^\mu \Lambda^\mu_\nu x^\nu} \hat{a}_{\mathbf{p}}^\dagger \right) \Big|_{p_0 = \omega_{\mathbf{p}}}. \end{aligned} \quad (6.25)$$

This can be put into an explicitly Lorentz invariant form

$$\begin{aligned} \hat{\phi}(\Lambda x) &= \int \frac{dp^0 d^3 p}{(2\pi)^3} \delta(p_0^2 - \mathbf{p}^2 - m^2) \Theta(p^0) \sqrt{2\omega_{\mathbf{p}}} e^{ip^\mu \Lambda^\mu_\nu x^\nu} \hat{a}_{\mathbf{p}} + \text{h.c.} \\ &= \int \frac{dp^0 d^3 p}{(2\pi)^3} \left(\frac{\delta(p_0 - \omega_{\mathbf{p}})}{2\omega_{\mathbf{p}}} + \frac{\delta(p_0 + \omega_{\mathbf{p}})}{2\omega_{\mathbf{p}}} \right) \Theta(p^0) \sqrt{2\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}} + \text{h.c.}, \end{aligned} \quad (6.26)$$

which recovers eq. (6.25) by making use of the delta function identity $\delta(f(x)) = \sum_{f(x_*)=0} \frac{\delta(x-x_*)}{f'(x_*)}$, since the $\Theta(p^0)$ kills the second delta function.

We now have a more explicit Lorentz invariant structure

$$\hat{\phi}(\Lambda x) = \int \frac{dp^0 d^3 p}{(2\pi)^3} \delta(p_0^2 - \mathbf{p}^2 - m^2) \Theta(p^0) \sqrt{2\omega_{\mathbf{p}}} e^{ip^\mu \Lambda^\mu_\nu x^\nu} \hat{a}_{\mathbf{p}} + \text{h.c.} \quad (6.27)$$

Recall that a boost moves a spacetime point along a parabola, such as that of fig. 6.1, whereas a rotation moves along a constant “circular” trajectory of a hyper-paraboloid. In general, a

Lorentz transformation may move a spacetime point along any path on a hyper-paraboloid such as the one depicted (in two spatial dimensions) in fig. 6.2. This paraboloid depicts the surfaces of constant energy-momentum $p^0 = \sqrt{\mathbf{p}^2 + m^2}$. Because a Lorentz transformation only shifts points along that energy-momentum surface, but cannot change the sign of the energy coordinate p^0 , this means that $\Theta(p^0)$ is also a Lorentz invariant.

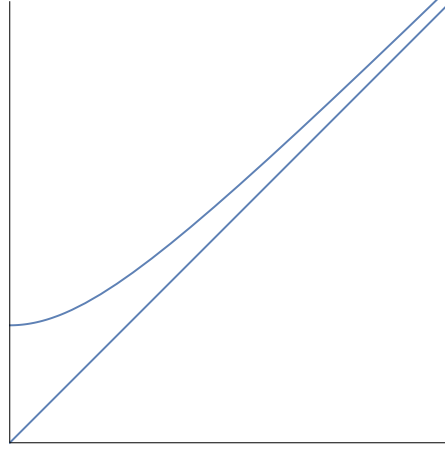


Figure 6.1: One dimensional spacetime surface for constant $(p^0)^2 - \mathbf{p}^2 = m^2$.

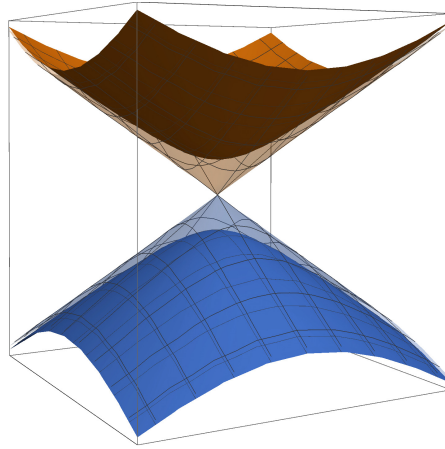


Figure 6.2: Surface of constant squared four-momentum.

Let's change variables

$$p^\lambda = \Lambda^\lambda_\rho p'^\rho, \quad (6.28)$$

so that

$$\begin{aligned}
 p_\mu \Lambda^\mu{}_\nu x^\nu &= \Lambda^\lambda{}_\rho p'^\rho g_{\lambda\nu} \Lambda^\nu{}_\sigma x^\sigma \\
 &= p'^\rho \left(\Lambda^\lambda{}_\rho g_{\lambda\nu} \Lambda^\nu{}_\sigma \right) x^\sigma \\
 &= p'^\rho g_{\rho\sigma} x^\sigma,
 \end{aligned} \tag{6.29}$$

which gives

$$\begin{aligned}
 \hat{\phi}(\Lambda x) &= \int \frac{d p'^0 d^3 p'}{(2\pi)^3} \delta(p'^2_0 - \mathbf{p}'^2 - m^2) \Theta(p'^0) \sqrt{2\omega_{\Lambda \mathbf{p}'}} e^{i p' \cdot x} \hat{a}_{\Lambda \mathbf{p}'} + \text{h.c.} \\
 &= \int \frac{d p^0 d^3 p}{(2\pi)^3} \delta(p^2_0 - \mathbf{p}^2 - m^2) \Theta(p^0) \sqrt{2\omega_{\Lambda \mathbf{p}}} e^{i p \cdot x} \hat{a}_{\Lambda \mathbf{p}} + \text{h.c.}
 \end{aligned} \tag{6.30}$$

Since

$$\hat{\phi}(x) = \int \frac{d p^0 d^3 p}{(2\pi)^3} \delta(p^2_0 - \mathbf{p}^2 - m^2) \Theta(p^0) \sqrt{2\omega_{\mathbf{p}}} e^{i p \cdot x} \hat{a}_{\mathbf{p}} + \text{h.c.} \tag{6.31}$$

we can now conclude that the creation and annihilation operators transform as

$$\boxed{\sqrt{2\omega_{\Lambda \mathbf{p}}} \hat{a}_{\Lambda \mathbf{p}} = \hat{U}(\Lambda) \sqrt{2\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}} \hat{U}^\dagger(\Lambda).} \tag{6.32}$$

If the desired normalization for a momentum state is assumed to be

$$\sqrt{2\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}}^\dagger |0\rangle = |\mathbf{p}\rangle, \tag{6.33}$$

then by noting that $\hat{U}(\Lambda) |0\rangle = |0\rangle$ (i.e. the ground state is Lorentz invariant), we have

$$\begin{aligned}
 \sqrt{2\omega_{\Lambda \mathbf{p}}} \hat{a}_{\Lambda \mathbf{p}}^\dagger |0\rangle &= \hat{U}(\Lambda) \sqrt{2\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}}^\dagger \hat{U}^\dagger(\Lambda) \hat{U}(\Lambda) |0\rangle \\
 &= \hat{U}(\Lambda) \sqrt{2\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}}^\dagger |0\rangle \\
 &= \hat{U}(\Lambda) |\mathbf{p}\rangle.
 \end{aligned} \tag{6.34}$$

The normalization eq. (6.33) means that the Lorentz transformation of a momentum state, takes a particularly simple form

$$\hat{U}(\Lambda) |\mathbf{p}\rangle = |\Lambda \mathbf{p}\rangle. \tag{6.35}$$

In [14], this is argued differently. In particular, it's argued that eq. (6.33) must be the required normalization based on a requirement for Lorentz invariant measure, and then demands $\hat{U}(\Lambda) |\mathbf{p}\rangle = |\Lambda \mathbf{p}\rangle$. After this eq. (6.32) follows as a consequence (albeit, how to conclude that is not spelled out in detail).

6.3 RELATIVISTIC NORMALIZATION.

We will continue looking at the generator of spacetime translation $\hat{U}(\Lambda)$, which has the property

$$\hat{U}(\Lambda) |0\rangle = |0\rangle, \quad (6.36)$$

That is

$$\hat{U}(\Lambda) = \mathbf{1} + \text{operators that annihilate the vacuum state.} \quad (6.37)$$

The action on a field was

$$\hat{U}(\Lambda)\hat{\phi}(x)\hat{U}^\dagger(\Lambda) = \hat{\phi}(\Lambda x), \quad (6.38)$$

and the action on the annihilation operator was

$$\hat{U}(\Lambda) \sqrt{2\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}} \hat{U}^\dagger(\Lambda) = \sqrt{2\omega_{\Lambda\mathbf{p}}} \hat{a}_{\Lambda\mathbf{p}}. \quad (6.39)$$

If $|\mathbf{p}_1\rangle$ is the one particle state with momentum \mathbf{p}_1 , then that momentum state can be generated from the ground state with the following normalized creation operation

$$|\mathbf{p}_1\rangle = \sqrt{2\omega_{\mathbf{p}_1}} \hat{a}_{\mathbf{p}_1}^\dagger |0\rangle. \quad (6.40)$$

We can compute the matrix element between two matrix states using the creation operator representation

$$\begin{aligned} \langle \mathbf{p}_2 | \mathbf{p}_1 \rangle &= \sqrt{2\omega_{\mathbf{p}_1}} \sqrt{2\omega_{\mathbf{p}_2}} \langle 0 | \hat{a}_{\mathbf{p}_2} \hat{a}_{\mathbf{p}_1}^\dagger | 0 \rangle \\ &= \sqrt{2\omega_{\mathbf{p}_1}} \sqrt{2\omega_{\mathbf{p}_2}} \langle 0 | \left(\hat{a}_{\mathbf{p}_1}^\dagger \hat{a}_{\mathbf{p}_2} + i(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \right) \\ &= \sqrt{2\omega_{\mathbf{p}_1}} \sqrt{2\omega_{\mathbf{p}_2}} (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2) \\ &= 2\omega_{\mathbf{p}_1} (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2). \end{aligned} \quad (6.41)$$

6.4 SPACELIKE SURFACES.

If x^μ, p^μ are four vectors, then $p^\mu x_\mu = \text{invariant} = p'^\mu x'_\mu$. The light cone is the surface $p_0^2 = \mathbf{p}^2$, whereas timelike four-momentum form a paraboloid surface $p_0^2 - \mathbf{p}^2 = m^2$ (i.e. $E = \sqrt{m^2 c^4 + \mathbf{p}^2 c^2}$). The surface for constant spacelike points (i.e. all related by a Lorentz transformation) is illustrated in fig. 6.3. A boost moves a point up or down that surface along the energy axis. It is therefore possible to use a sequence of boost and rotation to transform a point $(E, \mathbf{p}) \rightarrow (-E, \mathbf{p}) \rightarrow (-E, -\mathbf{p})$. That is, any spacelike four-vector x may be transformed to $-x$ using a Lorentz transformation.

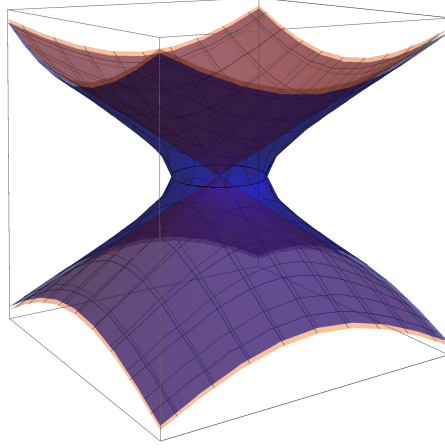


Figure 6.3: Constant spacelike surface.

6.5 CONDITION ON MICROCAUSALITY.

We defined operators $\hat{\phi}(\mathbf{x})$, which was a Hermitian operator for the real scalar field. For the complex scalar field we used $\hat{\phi}(\mathbf{x}) = (\hat{\phi}_1 + \hat{\phi}_2)/\sqrt{2}$, where each of $\hat{\phi}_1, \hat{\phi}_2$ were Hermitian operators. i.e. we can think of these operators as “observables”, that is $\hat{\phi}(\mathbf{x}) = \hat{\phi}^\dagger(\mathbf{x})$.

We now want to show that these operators commute at spacelike separations, and see how this relates to the question of causality. In particular, we want to see that an observation of one operator, will not effect the measurement of the other.

The condition of microcausality is

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0$$

if $x \sim y$, that is $(x - y)^2 < 0$. That is, x, y are spacelike separated.

We wrote

$$\hat{\phi}(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{-ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}} + \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}}^\dagger, \quad (6.42)$$

or $\hat{\phi}(x) = \hat{\phi}_-(x) + \hat{\phi}_+(x)$, where

$$\begin{aligned} \hat{\phi}_-(x) &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{-ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}} \\ \hat{\phi}_+(x) &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}}^\dagger. \end{aligned} \quad (6.43)$$

Compute the commutator

$$\begin{aligned}
 D(x) &= [\hat{\phi}_-(x), \hat{\phi}_+(0)] \\
 &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{-ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \int \frac{d^3 k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} e^{ik \cdot 0} \Big|_{k^0=\omega_{\mathbf{k}}} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{k}}^\dagger] \\
 &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{-ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \int \frac{d^3 k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{k}),
 \end{aligned} \tag{6.44}$$

$$D(x) = \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}}. \tag{6.45}$$

Now about the commutator at two spacetime points

$$\begin{aligned}
 [\hat{\phi}(x), \hat{\phi}(y)] &= [\hat{\phi}_-(x) + \hat{\phi}_+(x), \hat{\phi}_-(y) + \hat{\phi}_+(y)] \\
 &= [\hat{\phi}_-(x), \hat{\phi}_+(y)] + [\hat{\phi}_+(x), \hat{\phi}_-(y)] \\
 &= -D(y - x) + D(x - y).
 \end{aligned} \tag{6.46}$$

Find

$$\begin{aligned}
 [\hat{\phi}(x), \hat{\phi}(y)] &= D(x - y) - D(y - x) \\
 [\hat{\phi}(x), \hat{\phi}(0)] &= D(x) - D(-x).
 \end{aligned} \tag{6.47}$$

Let's look at $D(x)$, eq. (6.45), a bit more closely.

Claim: $D(x)$ is Lorentz invariant (has the same value for all x^μ, x'^μ)

We can see this by writing this out as

$$D(x) = \int \frac{d^3 p}{(2\pi)^3} dp^0 \delta(p_0^2 - \mathbf{p}^2 - m^2) \Theta(p^0) e^{-ip \cdot x}. \tag{6.48}$$

The exponential is Lorentz invariant, and the delta function has been put into a Lorentz invariant form.

Claim 1: $D(x) = D(x')$ where $x^2 = x'^2$.

Claim 2: $x^\mu, -x^\mu$ are related by Lorentz transformations if $x^2 < 0$.

From the figure, we see that $D(x) = D(-x)$ for a spacelike point, which implies that $[\hat{\phi}(x), \hat{\phi}(0)] = 0$ for a spacelike point x .

We've shown this for free fields, but later we will see that this is the case for interacting fields too.

EXTERNAL SOURCES.

7.1 HARMONIC OSCILLATOR.

$$L = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2 - j(t)q. \quad (7.1)$$

The term $j(t)$ shifts the origin in a time dependent fashion (graphical illustration in class wiggling a hockey stick, as a sample of a harmonic oscillator).

$$H = \frac{p^2}{2} + \frac{\omega^2}{2}q^2 + j(t)q \quad (7.2)$$

$$\begin{aligned} i\dot{q}_H(t) &= [q_H, H] = ip_H \\ i\dot{p}_H(t) &= [p_H, H] = -i\omega^2 q_H - ij(t) \end{aligned} \quad (7.3)$$

$$\ddot{q}_H(t) = -\omega^2 q_H(t) - j(t) \quad (7.4)$$

or

$$(\partial_t + \omega^2)q_H(t) = -j(t) \quad (7.5)$$

$$q_H(t) = q_H^0(t) + \int G_R(t-t')j(t')dt'. \quad (7.6)$$

This solves the equation provided $G_R(t-t')$ has the property that

$$(\partial_t + \omega^2)G_R(t-t') = -\delta(t-t'). \quad (7.7)$$

That is

$$(\partial_t + \omega^2)q_H(t) = (\partial_t + \omega^2)q_H^0(t) + (\partial_t + \omega^2) \int G_R(t-t')j(t')dt'. \quad (7.8)$$

This function G_R is called the retarded Green's function. We want to find this function, and as usual, we do this by taking the Fourier transform of eq. (7.7)

$$\begin{aligned} \int dt e^{ipt} (\partial_{tt} + \omega^2) G_R(t - t') &= - \int_{-\infty}^{\infty} dt e^{ipt} \delta(t - t') \\ &= -e^{ipt'}. \end{aligned} \quad (7.9)$$

Let

$$G(t - t') = \int \frac{dp}{2\pi} e^{-ip'(t-t')} \tilde{G}(p'), \quad (7.10)$$

so

$$\begin{aligned} -e^{ipt'} &= \int dt e^{ipt} (\partial_{tt} + \omega^2) \int \frac{dp'}{2\pi} e^{-ip'(t-t')} \tilde{G}(p') \\ &= \int dt e^{ipt} \int \frac{dp'}{2\pi} (-p'^2 + \omega^2) e^{-ip'(t-t')} \tilde{G}(p') \\ &= \int dp' (-p'^2 + \omega^2) e^{ip't'} \delta(p - p') \tilde{G}(p') \\ &= (-p^2 + \omega^2) \tilde{G}(p) e^{ipt'}, \end{aligned} \quad (7.11)$$

so

$$\tilde{G}(p) = \frac{1}{p^2 - \omega^2}. \quad (7.12)$$

Now

$$G(t) = \int \frac{dp}{2\pi} e^{-ipt} \tilde{G}(p). \quad (7.13)$$

Let's write the momentum space Green's function as

$$\tilde{G}(p) = \frac{1}{(p - \omega)(p + \omega)}. \quad (7.14)$$

The solution contained

$$\int G(t - t') j(t') dt'. \quad (7.15)$$

Suppose $j(t) = 0$ for all $t < t_0$. We want the effect of $j(t)$ to be felt in the future, for example, $j(t)$ is an impulse starting at some time. We want $G(t)$ to vanish at negative times.

We want the integral

$$G(t) = \int \frac{dp}{2\pi} e^{-ipt} \frac{1}{(p - \omega)(p + \omega)}, \quad (7.16)$$

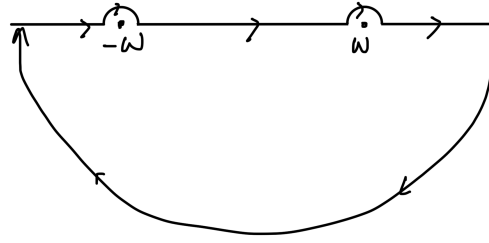


Figure 7.1: Lower plane contour.

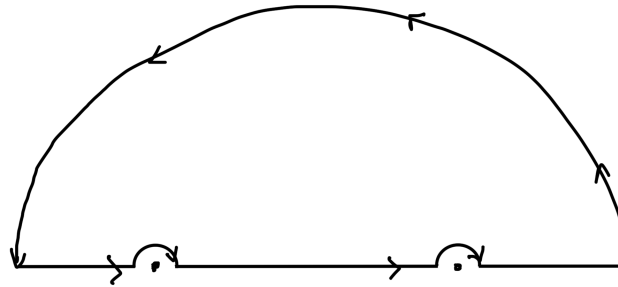


Figure 7.2: Upper plane contour.

to vanish when $t < 0$.

Start with $t > 0$ (that is $t' < t$), so that $e^{-ipt} = e^{-ip|t|}$ which means that we have to integrate over a lower plane contour like fig. 7.1, because the imaginary part of p is negative, but for $t < 0$ (that is $t' > t$), we want an upper plane contour like fig. 7.2.

Question: since we are integrating over the real line, how can we get away with deforming the contour? Answer: it works. If we do this we get a Green's function that makes sense (better answer later?)

We add an infinite circle, so that we can integrate over a closed contour, and pick the contour so that it is zero for $t < 0$ and non-zero (enclosed poles) for $t > 0$.

$$\begin{aligned}
 G_R(t > 0) &= \int_C \frac{dp}{2\pi} e^{-ipt} \frac{1}{(p - \omega)(p + \omega)} \\
 &= \frac{1}{2\pi} (-2\pi i) \left(\frac{e^{-i\omega t}}{2\omega} - \frac{e^{i\omega t}}{2\omega} \right) \\
 &= -\frac{\sin(\omega t)}{\omega}.
 \end{aligned} \tag{7.17}$$

Now we write the Green's function for all time as

$$G_R(t) = -\frac{\sin(\omega t)}{\omega} \Theta(t). \quad (7.18)$$

The question of what contour to pick can now be justified by the result, since this satisfies eq. (7.7). If we wanted a Green's function that selected just future contributions we'd have used a “bumps down” contour. There will be circumstances where we will use some of the other contour possibilities (fig. 7.3). In particular, the bumps up and down contour will be used to derive the “Feynman propagator” that we'll use later.



Figure 7.3: All possible deformations around the poles.

7.2 FIELD THEORY (WHERE WE ARE GOING).

We will consider a massive real scalar field theory with an external source with action

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 + j(x) \phi(x) \right). \quad (7.19)$$

We don't have examples of currents that create scalar fields, but to study such as system, recall that in electromagnetism we added sources to the field by adding a term like

$$\int d^4x A^\mu(x) j_\mu(x), \quad (7.20)$$

to our action.

The equation of motion can be found to be

$$(\partial_\mu \partial^\mu + m^2) \phi(x) = j(x). \quad (7.21)$$

We want to study the Green's function of this Klein-Gordon equation, defined to obey

$$\left(\partial_\mu \partial^\mu + m^2\right)_x G(x-y) = -i\delta^{(4)}(x-y), \quad (7.22)$$

where the $-i$ factor is for convenience. This is analogous to the Green's function that we just studied for the QM harmonic oscillator.

Exercise 7.1 Compute $D(x-y)$ from the commutator.

Generalize the derivation eq. (6.45) by computing the commutator at two different space time points x, y .

Answer for Exercise 7.1

Let

$$\begin{aligned} D(x-y) &= [\hat{\phi}_-(x), \hat{\phi}_+(y)] \\ &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{-ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \int \frac{d^3 k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} e^{ik \cdot y} \Big|_{k^0=\omega_{\mathbf{k}}} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{k}}^\dagger] \\ &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{-ip \cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \int \frac{d^3 k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} e^{ik \cdot y} \Big|_{k^0=\omega_{\mathbf{k}}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{k}) \\ &= \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip \cdot (x-y)} \Big|_{p^0=\omega_{\mathbf{p}}}. \end{aligned} \quad (7.23)$$

Exercise 7.2 Verification of harmonic oscillator Green's function.

Take the derivatives of a convolution of the Green's function eq. (7.18) to show that it satisfies eq. (7.7).

Answer for Exercise 7.2

Let

$$q(t) = \int_{-\infty}^{\infty} G(t-t') j(t') dt' = -\frac{1}{\omega} \int_{-\infty}^{\infty} \sin(\omega(t-t')) \Theta(t-t') j(t') dt'. \quad (7.24)$$

We are free to add any $q_0(t)$ that satisfies the homogeneous wave equation $q_0''(t) + \omega^2 q_0(t) = 0$ to our assumed convolution solution eq. (7.24), but that isn't interesting for this exercise. Since $\Theta(t-t') = 0$ for $t-t' < 0$, or $t' > t$, the convolution can be written as

$$q(t) = -\frac{1}{\omega} \int_{-\infty}^t \sin(\omega(t-t')) j(t') dt', \quad (7.25)$$

which is now in a convenient form to take derivatives. We have contributions from the boundary's time dependence and from the integrand. In particular

$$\frac{d}{dt} \int_{a(t)}^{b(t)} g(x, t) dx = g(b(t))b'(t) - g(a(t))a'(t) + \int_a^b \frac{\partial}{\partial t} g(x, t) dx. \quad (7.26)$$

Assuming that $j(-\infty) = 0$, this gives

$$\begin{aligned} \frac{dq(t)}{dt} &= -\frac{1}{\omega} \sin(\omega(t - t')) j(t') \Big|_{t'=t} - \int_{-\infty}^t \cos(\omega(t - t')) j(t') dt' \\ &= - \int_{-\infty}^t \cos(\omega(t - t')) j(t') dt'. \end{aligned} \quad (7.27)$$

For the second derivative we have

$$\begin{aligned} q''(t) &= -\cos(\omega(t - t')) j(t') \Big|_{t'=t} + \omega \int_{-\infty}^t \sin(\omega(t - t')) j(t') dt' \\ &= -j(t) - \omega^2 \int_{-\infty}^t \frac{-\sin(\omega(t - t'))}{\omega} j(t') dt', \end{aligned} \quad (7.28)$$

or

$$q''(t) = -j(t) - \omega^2 q(t), \quad (7.29)$$

which is our forced Harmonic oscillator equation.

7.3 GREEN'S FUNCTIONS FOR THE FORCED KLEIN-GORDON EQUATION.

The problem we were preparing to do was to study the problem of “particle creation by external classical source”.

We continue with a real scalar field, free, massive, but with an interaction with a source

$$S_{\text{int}} = \int d^4x j(x) \phi(x). \quad (7.30)$$

Modern application: think of ϕ has some SM field and think of j as due to inflaton (i.e. cosmological inflation interaction) oscillation. In the inflationary model, the process of “reheating” creates all the matter in the universe. We won't be talking about inflation, but will be considering a toy model that has some similar characteristics to the inflationary theory.

The equation of motion that we end up with is

$$(\partial_\mu \partial^\mu + m^2) \phi = j, \quad (7.31)$$

and we wish to solve this using Green's function techniques.

Definition 7.1: Klein-Gordon Green's function.

The QFT conventions for the Klein-Gordon Green's function is

$$(\partial_\mu \partial^\mu + m^2) G(x - y) = -i\delta^{(4)}(x - y).$$

As usual, we assume that it is possible to find a solution ϕ by convolution

$$\phi(x) = i \int d^4y G(x - y) j(y). \quad (7.32)$$

Check:

$$\begin{aligned} (\partial_\mu \partial^\mu + m^2) \phi(x) &= i (\partial_\mu \partial^\mu + m^2) \int d^4y G(x - y) j(y) \\ &= i \int d^4y (-i\delta^{(4)}(x - y)) j(y) \\ &= j(x). \end{aligned} \quad (7.33)$$

Also, as usual, we take out our Fourier transforms, the power tool of physics, and determine the structure of the Green's function by inverting the transform equation

$$G(x - y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x - y)} \tilde{G}(p). \quad (7.34)$$

Operating with Klein-Gordon gives

$$(\partial_\mu \partial^\mu + m^2) G(x) = \int \frac{d^4p}{(2\pi)^4} ((-ip_\mu)(-ip^\mu) + m^2) e^{-ip \cdot (x - y)} \tilde{G}(p). \quad (7.35)$$

This must equal

$$-i\delta^{(4)}(x - y) = -i \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x - y)}, \quad (7.36)$$

or

$$(m^2 - p_\mu p^\mu) \tilde{G}(p) = -i. \quad (7.37)$$

The Green's function in the momentum domain is

$$\tilde{G}(p) = \frac{i}{p^2 - m^2}. \quad (7.38)$$

The inverse transform provides the spatial domain representation of the Green's function

$$\begin{aligned}
 G(x) &= \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \frac{i}{(p^0)^2 - \mathbf{p}^2 - m^2} \\
 &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \int \frac{dp_0}{2\pi} e^{-ip_0 x^0} \frac{i}{(p_0 - \omega_{\mathbf{p}})(p_0 + \omega_{\mathbf{p}})}.
 \end{aligned} \tag{7.39}$$

In the p_0 plane, we have two poles at $p_0 = \pm\omega_{\mathbf{p}}$. There are 4 ways to go around the poles, the retarded time deformation that we used to derive the Green's function for the harmonic oscillator, as sketched in fig. 7.4, the advanced time deformation sketched in fig. 7.5, and mixed deformations.

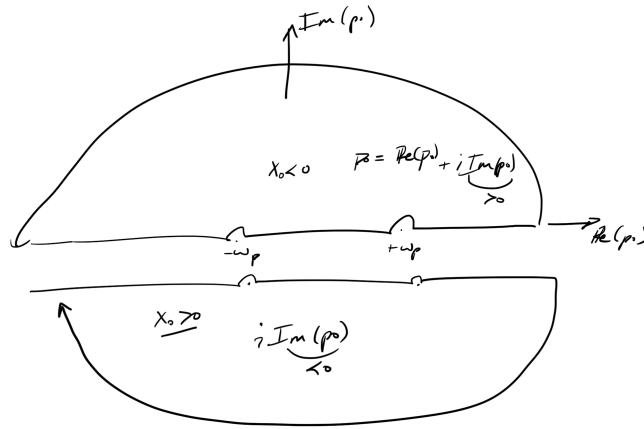


Figure 7.4: Retarded time deformations and contours.

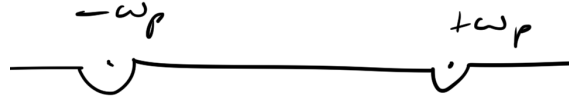


Figure 7.5: Advanced time deformation.

We will evaluate the integral using the “Feynman propagator” contour sketched in fig. 7.6. Why we use the Feynman contour, and not the retarded contour can be justified by how well this works for the perturbation methods that will be developed later.

Consider each contour in turn.

Case I. $x^0 > 0$ For this case, we use the lower half plane contour sketched in fig. 7.7, which vanishes for $\Im(p_0) < 0, x_0 > 0$, where $-i(i\Im(p_0)x_0) < 0$.



Figure 7.6: Feynman propagator deformation path.

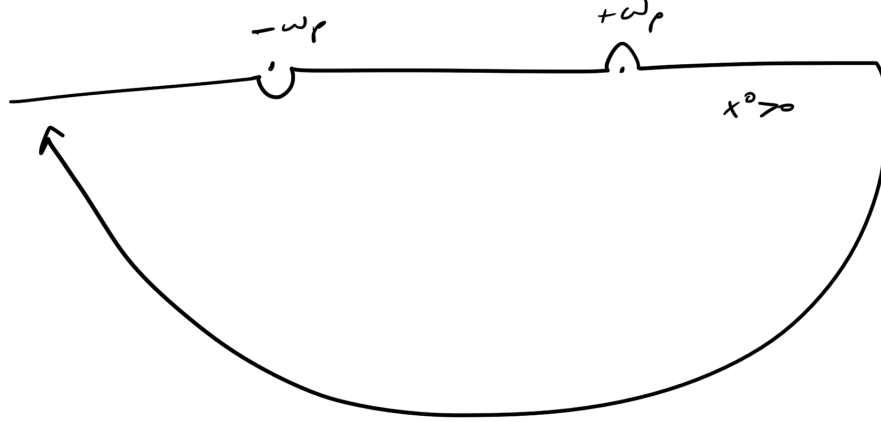


Figure 7.7: Feynman propagator contour for $t > 0$.

Here we pick up just the pole at $p_0 = \omega_{\mathbf{p}}$, and take a negatively oriented path

$$\begin{aligned}
 G_F &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \int \frac{dp_0}{2\pi} e^{-ip_0 x^0} \frac{i}{(p_0 - \omega_{\mathbf{p}})(p_0 + \omega_{\mathbf{p}})} \\
 &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} (-2\pi i) \left(\frac{e^{-ip_0 x^0}}{2\pi} \frac{i}{p_0 + \omega_{\mathbf{p}}} \right) \bigg|_{p_0 = \omega_{\mathbf{p}}} \\
 &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \frac{-2\pi i}{2\pi} \frac{ie^{-ip_0 x^0}}{2\omega_{\mathbf{p}}} \\
 &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \frac{e^{-i\omega_{\mathbf{p}} x^0}}{2\omega_{\mathbf{p}}}.
 \end{aligned} \tag{7.40}$$

Case II. $x^0 < 0$ For $x^0 < 0$ we use an upper half plane contour with the same deformation around the poles. This time

$$\begin{aligned}
 G_F &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \int \frac{dp_0}{2\pi} e^{-ip_0 x^0} \frac{i}{(p_0 - \omega_{\mathbf{p}})(p_0 + \omega_{\mathbf{p}})} \\
 &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} (+2\pi i) \left(\frac{e^{-ip_0 x^0}}{2\pi} \frac{i}{p_0 - \omega_{\mathbf{p}}} \right) \Big|_{p_0 = -\omega_{\mathbf{p}}} \\
 &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{+2\pi i}{2\pi} \frac{ie^{-ip_0 x^0}}{-2\omega_{\mathbf{p}}} \\
 &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{e^{i\omega_{\mathbf{p}} x^0}}{2\omega_{\mathbf{p}}}.
 \end{aligned} \tag{7.41}$$

We've obtained a piecewise representation of the Green's function, where the only difference is the sign of the $i\omega_{\mathbf{p}} x^0$ exponential.

We can combine eq. (7.40) eq. (7.41) by using Θ functions

$$\int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{x}} \left(e^{-i\omega_{\mathbf{p}} x^0} \Theta(x_0) + e^{i\omega_{\mathbf{p}} x^0} \Theta(-x_0) \right). \tag{7.42}$$

The first integral (without the Θ factor) is the Wightman function

$$D(x) = \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip\cdot x} \Big|_{p^0 = \omega_{\mathbf{p}}}. \tag{7.43}$$

For the second integral, we make a change of variables $\mathbf{p} \rightarrow -\mathbf{p}$ leaving

$$\begin{aligned}
 \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{x} + i\omega_{\mathbf{p}} x^0} &\rightarrow \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-i\mathbf{p}\cdot\mathbf{x} + i\omega_{\mathbf{p}} x^0} \\
 &= \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip\cdot x} \\
 &= D(-x),
 \end{aligned} \tag{7.44}$$

so

$$G_F(x) = \Theta(x^0) D(x) + \Theta(-x^0) D(-x). \tag{7.45}$$

7.4 POLE SHIFTING.

Recall that the four dimensional form of the Green's function was

$$D_F = i \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \frac{1}{p^2 - m^2}. \quad (7.46)$$

For the Feynman case, the contour that we were taking around the poles can also be accomplished by shifting the poles strategically, as sketched in fig. 7.8.

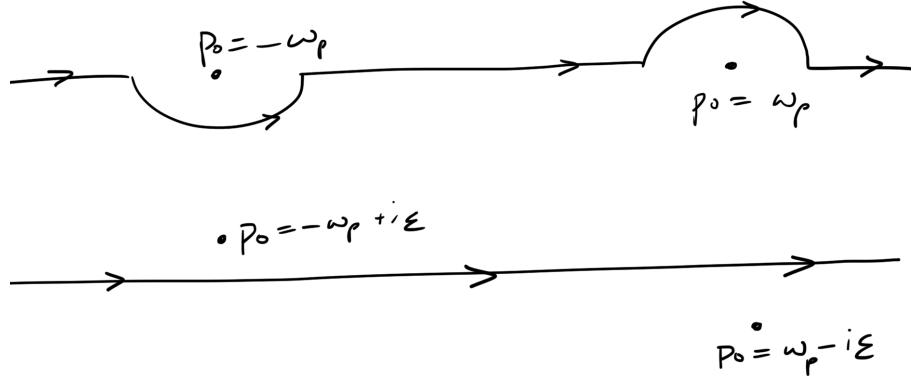


Figure 7.8: Feynman deformation or equivalent shift of the poles.

This shift can be expressed explicit algebraically by introducing an offset

$$D_F = i \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \frac{1}{p^2 - m^2 + i\epsilon}, \quad (7.47)$$

which puts the poles at

$$\begin{aligned} p^0 &= \pm \sqrt{\omega_{\mathbf{p}}^2 - i\epsilon} \\ &= \pm \omega_{\mathbf{p}} \left(1 - \frac{i\epsilon}{\omega_{\mathbf{p}}^2} \right)^{1/2} \\ &= \pm \omega_{\mathbf{p}} \left(1 - \frac{1}{2} \frac{i\epsilon}{\omega_{\mathbf{p}}^2} \right) \\ &= \begin{cases} +\omega_{\mathbf{p}} - \frac{1}{2} i \frac{\epsilon}{\omega_{\mathbf{p}}} \\ -\omega_{\mathbf{p}} + \frac{1}{2} i \frac{\epsilon}{\omega_{\mathbf{p}}} \end{cases} \end{aligned} \quad (7.48)$$

7.5 MATRIX ELEMENT REPRESENTATION OF THE WIGHTMAN FUNCTION.

Recall that the Wightman function eq. (7.43) also had a matrix element representation

$$D(x) = \langle 0 | \phi(x) \phi(0) | 0 \rangle. \quad (7.49)$$

This can be shown by expansion.

$$\langle 0 | \phi(x) \phi(0) | 0 \rangle = \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) \Big|_{p_0=\omega_{\mathbf{p}}} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} (a_{\mathbf{q}}^\dagger + a_{\mathbf{q}}) | 0 \rangle. \quad (7.50)$$

Since $a_{\mathbf{q}} | 0 \rangle = 0 = \langle 0 | a_{\mathbf{p}}^\dagger$, eq. (7.50) reduces to

$$\begin{aligned} \langle 0 | \phi(x) \phi(0) | 0 \rangle &= \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} (a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{-ip \cdot x}) \Big|_{p_0=\omega_{\mathbf{p}}} | 0 \rangle \\ &= \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} ((a_{\mathbf{p}} a_{\mathbf{q}}^\dagger + [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger]) e^{-ip \cdot x}) \Big|_{p_0=\omega_{\mathbf{p}}} | 0 \rangle \\ &= \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} ((2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-ip \cdot x}) \Big|_{p_0=\omega_{\mathbf{p}}} \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-ip \cdot x}}{2\omega_{\mathbf{p}}} \Big|_{p_0=\omega_{\mathbf{p}}}. \end{aligned} \quad (7.51)$$

7.6 RETARDED GREEN'S FUNCTION.

Claim: Retarded Green's function (bumps up contour) can be written

$$D_R(x) = \theta(x_0)(D(x) - D(-x)), \quad (7.52)$$

where $D(x)$ is given by eq. (7.43). Proof: The upper half plane contour ($x_0 < 0$) is zero since it encloses no poles. For the lower half plane contour we have

$$\begin{aligned} D_R(x)|_{x_0 > 0} &= i \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \int \frac{dp_0}{2\pi} e^{-ip_0 x^0} \frac{i}{(p_0 - \omega_{\mathbf{p}})(p_0 + \omega_{\mathbf{p}})} \\ &= i \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \frac{(-2\pi i)}{2\pi} \left(e^{-i\omega_{\mathbf{p}} x^0} \frac{i}{2\omega_{\mathbf{p}}} + e^{i\omega_{\mathbf{p}} x^0} \frac{i}{-2\omega_{\mathbf{p}}} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot \mathbf{x}} \frac{1}{2\omega_{\mathbf{p}}} (e^{-i\omega_{\mathbf{p}} x^0} - e^{i\omega_{\mathbf{p}} x^0}) \\ &= D(x) - D(-x). \end{aligned} \quad (7.53)$$

What does the field look like in terms of the propagator? Assuming that ϕ_0 satisfies the homogeneous equation, we have

$$\begin{aligned}\phi(x) &= \phi_0(x) + i \int d^4y D_R(x-y) j(y) \\ &= \phi_0(x) + i \int d^3y dy_0 \Theta(x_0 - y_0) (D(x-y) - D(y-x)) j(y).\end{aligned}\tag{7.54}$$

Imagine that we have a windowed source function $j(y^0, \mathbf{y})$, as sketched in fig. 7.9.

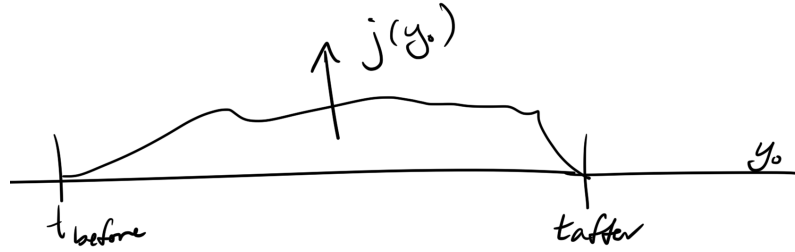


Figure 7.9: Finite window impulse response.

$$\begin{aligned}\phi(x)|_{x^0 > t_{\text{after}}} &= \phi_0(x) \\ &+ i \int d^4y \left(\int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip \cdot (x-y)} j(y) - \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{ip \cdot (x-y)} j(y) \right).\end{aligned}\tag{7.55}$$

Define

$$\tilde{j}(p) = \int d^4y e^{ip \cdot y} j(y),\tag{7.56}$$

which gives

$$\phi(x)|_{x^0 > t_{\text{after}}} = \phi_0(x) + i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left(e^{-ip \cdot x} \tilde{j}(p) - e^{ip \cdot x} \tilde{j}(-p) \right) \Big|_{p_0 = \omega_{\mathbf{p}}}.\tag{7.57}$$

We will interpret this in the next lecture, and start in on Feynman diagrams.

7.7 REVIEW: “PARTICLE CREATION PROBLEM”.

We imagined that we have a windowed source function $j(y^0, \mathbf{y})$, as sketched in fig. 7.9, which is acting as a forcing source for the non-homogeneous Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2) \phi = j.\tag{7.58}$$

Our solution was

$$\phi(x) = \phi(x_0) + i \int d^4 y D_R(x - y) j(y), \quad (7.59)$$

where $\phi(x_0)$ obeys the homogeneous equation, and

$$D_R(x - y) = \Theta(x^0 - y^0) (D(x - y) - D(y - x)), \quad (7.60)$$

and $D(x) = \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip \cdot x} \Big|_{p^0 = \omega_{\mathbf{p}}}$ is the Wightman function.

For $x^0 > t_{\text{after}}$

$$\begin{aligned} \phi(x) &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip \cdot x} a_{\mathbf{p}} + e^{ip \cdot x} a_{\mathbf{p}}^\dagger \right) \Big|_{p^0 = \omega_{\mathbf{p}}} + i \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} \left(e^{-ip \cdot x} \tilde{j}(p) + e^{ip \cdot x} \tilde{j}(p_0, -\mathbf{p}) \right) \Big|_{p^0 = \omega_{\mathbf{p}}} \end{aligned} \quad (7.61)$$

where we have used $\tilde{j}^*(p_0, \mathbf{p}) = \tilde{j}(p_0, -\mathbf{p})$. This gives

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip \cdot x} \left(a_{\mathbf{p}} + i \frac{\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + e^{ip \cdot x} \left(a_{\mathbf{p}}^\dagger - i \frac{\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right) \Big|_{p^0 = \omega_{\mathbf{p}}} \quad (7.62)$$

It was left as an exercise to show that given

$$H = \int d^3 p \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right), \quad (7.63)$$

we obtain

$$H_{\text{after}} = \int d^3 p \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger - i \frac{\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + i \frac{\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right). \quad (7.64)$$

System in ground state

$$\langle 0 | \hat{H}_{\text{before}} | 0 \rangle = \langle E \rangle_{\text{before}} = 0. \quad (7.65)$$

$$\begin{aligned} \langle 0 | \hat{H}_{\text{after}} | 0 \rangle &= \langle E \rangle_{\text{after}} \\ &= \int d^3 p \omega_{\mathbf{p}} \frac{\tilde{j}^*(p) \tilde{j}(p)}{2\omega_{\mathbf{p}}} \\ &= \frac{1}{2} \int d^3 p |\tilde{j}(p)|^2. \end{aligned} \quad (7.66)$$

We can identify

$$N(\mathbf{p}) = \frac{|\tilde{j}(p)|^2}{2\omega_{\mathbf{p}}}, \quad (7.67)$$

as the number density of particles with momentum \mathbf{p} .

7.8 DIGRESSION: COHERENT STATES.

Definition 7.2: Coherent state.

A coherent state is an eigenstate of the destruction operator

$$a|\alpha\rangle = \alpha|\alpha\rangle.$$

For the SHO, if we solve for such a coherent state, we find

$$|\alpha\rangle = \text{constant} \times \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (a^\dagger)^n |0\rangle. \quad (7.68)$$

If we assume the existence of a coherent state

$$a_{\mathbf{p}} \left| \frac{\tilde{j}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}} \right\rangle = \frac{\tilde{j}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}} \left| \frac{\tilde{j}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}} \right\rangle, \quad (7.69)$$

then the expectation value of the number operator with respect to this state is the number density identified in eq. (7.67)

$$\left\langle \frac{\tilde{j}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}} \left| a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right| \frac{\tilde{j}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}} \right\rangle = \frac{|\tilde{j}(\mathbf{p})|^2}{2\omega_{\mathbf{p}}} = N(\mathbf{p}). \quad (7.70)$$

7.9 PROBLEMS.

Exercise 7.3 Spacetime behaviour of various Green's functions. (2018 HW2.I)

Here, you'll study some properties of

$$D(x) \equiv [\hat{\phi}_-(x), \hat{\phi}_+(x)] = \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-i\omega_p t + i\mathbf{p} \cdot \mathbf{x}}. \quad (7.71)$$

a. For $m = 0$ ("photon"), show that:

$$D(x) = -\frac{1}{2\pi^2} \mathcal{P} \frac{1}{t^2 - r^2} - \frac{i}{8\pi} \left(\frac{\delta(t-r)}{r} - \frac{\delta(t+r)}{r} \right), \quad (7.72)$$

where $r = \|\mathbf{x}\|$. Notice that $D(x)$ is singular on the light cone $t = r$. Does it vanish for spacelike separations?

- b. Compute $\langle z'|z\rangle$, and $\langle z|H|z\rangle$.
- c. Show that $|z\rangle$ is an eigenstate of the annihilation operator a and find its eigenvalue. (Don't be disturbed by finding non-orthogonal eigenstates with complex eigenvalues; a is not a Hermitian operator.)
- d. The statement that $|z\rangle$ is an eigenstate of a with well-known eigenvalue is, in the q -representation, a first-order differential equation for $\langle q|z\rangle$, the position-space wave-function of $|z\rangle$. Solve this equation and find and sketch the wave-function. (Don't bother with normalization factors here).
- e. Consider the time evolution of the system (work in the Heisenberg representation). Show that for real z (this just sets the initial conditions) the expectation values of the position and momentum of the coherent state satisfy

$$\langle z|X|z\rangle = \sqrt{\frac{2}{\mu\omega}}z\cos\omega t \quad (7.75)$$

$$\langle z|P|z\rangle = -\sqrt{2\mu\omega}z\sin\omega t \quad (7.76)$$

By contrast, what are the expectation values of X and P for an oscillator in any state of definite excitation number n ? Using a sketch, describe the behavior of the wavepacket as a function of time.

Answer for Exercise 7.4

PROBLEM SET RELATED MATERIAL REDACTED IN THIS DOCUMENT.PLEASE
FEEL FREE TO EMAIL ME FOR THE FULL VERSION IF YOU AREN'T TAKING PHY2403.

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PERTURBATION THEORY.

8.1 FEYNMAN'S GREEN'S FUNCTION.

$$\begin{aligned} D_F(x) &= \Theta(x^0)D(x) + \Theta(-x^0)D(-x) \\ &= \Theta(x^0) \langle 0 | \phi(x)\phi(0) | 0 \rangle + \Theta(x^0) \langle 0 | \phi(-x)\phi(0) | 0 \rangle. \end{aligned} \quad (8.1)$$

Utilizing a translation operation $U(a) = e^{ia_\mu P^\mu}$, where $U(a)\phi(y)U^\dagger(a) = \phi(y+a)$, this second operation can be written as

$$\begin{aligned} \langle 0 | \phi(-x)\phi(0) | 0 \rangle &= \langle 0 | U^\dagger(a)U(a)\phi(-x)U^\dagger(a)U(a)\phi(0)U^\dagger(a)U(a) | 0 \rangle \\ &= \langle 0 | U(a)\phi(-x)U^\dagger(a)U(a)\phi(0)U^\dagger(a) | 0 \rangle \\ &= \langle 0 | \phi(-x+a)\phi(a) | 0 \rangle, \end{aligned} \quad (8.2)$$

In particular, with $a = x$

$$\langle 0 | \phi(-x)\phi(0) | 0 \rangle = \langle 0 | \phi(0)\phi(x) | 0 \rangle, \quad (8.3)$$

so the Feynman's Green function can be written

$$\begin{aligned} D_F(x) &= \Theta(x^0) \langle 0 | \phi(x)\phi(0) | 0 \rangle + \Theta(x^0) \langle 0 | \phi(x)\phi(x) | 0 \rangle \\ &= \langle 0 | \left(\Theta(x^0)\phi(x)\phi(0) + \Theta(-x^0)\phi(0)\phi(x) \right) | 0 \rangle. \end{aligned} \quad (8.4)$$

We define

Definition 8.1: Time ordered product.

The time ordered product of two operators is defined as

$$T(\phi(x)\phi(y)) = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & x^0 < y^0 \end{cases},$$

or

$$T(\phi(x)\phi(y)) = \phi(x)\phi(y)\Theta(x^0 - y^0) + \phi(y)\phi(x)\Theta(y^0 - x^0).$$

Using this helpful construct, the Feynman's Green function can now be written in a very simple fashion

$$D_F(x) = \langle 0 | T(\phi(x)\phi(0)) | 0 \rangle. \quad (8.5)$$

8.2 INTERACTING FIELD THEORY: PERTURBATION THEORY IN QFT.

We perturb the Hamiltonian

$$H = H_0 + H_{\text{int}}, \quad (8.6)$$

where H_0 is the free Hamiltonian and H_{int} is the interaction term (the perturbation).

Example:

$$\begin{aligned} H_0 &= SHO = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \\ H_{\text{int}} &= \lambda q^4. \end{aligned} \quad (8.7)$$

i.e. the anharmonic oscillator.

In QFT

$$\begin{aligned} H_0 &= \int d^3x \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 \right) \\ H_{\text{int}} &= \lambda \int d^3x \phi^4. \end{aligned} \quad (8.8)$$

We will expand the interaction in small λ . Perturbation theory is the expansion in a small dimensionless coupling constant, such as

- λ in $\lambda \phi^4$ theory,
- $\alpha = e^2/4\pi \sim \frac{1}{137}$ in QED, and
- α_s in QCD.

8.3 PERTURBATION THEORY, INTERACTION REPRESENTATION AND DYSON FORMULA.

$$H = H_0 + H_{\text{int}} \quad (8.9)$$

Example interaction

$$H_{\text{int}} = \lambda \int d^3x \phi^4. \quad (8.10)$$

We know all there is to know about H_0 (decoupled SHOs, ...)

$$H_0 |0\rangle = |0\rangle E_{\text{vac}}^0 \quad (8.11)$$

where $E_{\text{vac}}^0 = 0$. Assume

$$(H_0 + H_{\text{int}})|\Omega\rangle = |\Omega\rangle E_{\text{vac}}, \quad (8.12)$$

where the ground state energy of the perturbed system is zero when $\lambda = 0$. That is $E_{\text{vac}}(\lambda = 0) = 0$.

So for

$$\phi(x)|_{x^0 = t_0, \text{some fixed value}} = \int \frac{d^3}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} (e^{-ip \cdot x} a_{\mathbf{p}} + e^{ip \cdot x} a_{\mathbf{p}}^\dagger) \Big|_{p^0 = \omega_{\mathbf{p}}}. \quad (8.13)$$

Let's call $\phi(\mathbf{x}, t_0)$ the free Schrödinger operator, where $\phi(\mathbf{x}, t_0)$ is evaluated at a fixed value of t_0 . At such a point, the Schrödinger and Heisenberg pictures coincide.

$$[\phi(\mathbf{x}, t_0), \pi(\mathbf{y}, t_0)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (8.14)$$

Normally (QM) one defines the Heisenberg operator as

$$O_H = e^{iH(t-t_0)} O_S e^{-iH(t-t_0)}, \quad (8.15)$$

where O_H depends on time, and O_S is defined at a fixed time t_0 , usually 0. From eq. (8.15) we find

$$\frac{dO_H}{dt} = i[H, O_H]. \quad (8.16)$$

The equivalent of eq. (8.15) in QFT is very complicated. We'd like to develop an intermediate picture.

We will define an intermediate picture, called the “interaction representation”, which is equivalent to the Heisenberg picture with respect to H_0 .

Definition 8.2: Interaction picture operator.

$$\phi_I(t, \mathbf{x}) = e^{iH_0(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH_0(t-t_0)}.$$

This is familiar, and is the Heisenberg picture operator that we had in free QFT

$$\phi_I(t, \mathbf{x}) = \int \frac{d^3}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} (e^{-ip \cdot x} a_{\mathbf{p}} + e^{ip \cdot x} a_{\mathbf{p}}^\dagger) \Big|_{p^0 = \omega_{\mathbf{p}}}, \quad (8.17)$$

where $x_0 = t$.

The Heisenberg picture operator is

$$\begin{aligned}\phi_H(t, \mathbf{x}) &= \phi(t, \mathbf{x}) \\ &= e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \left(e^{iH_0(t-t_0)} \phi_S(t_0, \mathbf{x}) e^{-iH_0(t-t_0)} \right) e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi_I(t, \mathbf{x}) e^{-iH_0(t-t_0)} e^{iH(t-t_0)}\end{aligned}\quad (8.18)$$

or

$$\phi_H(t, \mathbf{x}) = U^\dagger(t, t_0) \phi_I(t_0, \mathbf{x}) U(t, t_0), \quad (8.19)$$

where

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}. \quad (8.20)$$

We want to apply perturbation techniques to find $U(t, t_0)$ which is complicated.

$$\begin{aligned}i \frac{\partial}{\partial t} U(t, t_0) &= i e^{iH_0(t-t_0)} i H_0 e^{-iH(t-t_0)} + i e^{iH_0(t-t_0)} e^{-iH(t-t_0)} (-iH) \\ &= e^{iH_0(t-t_0)} (-H_0 + H) e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)} e^{iH_0(t-t_0)} e^{-iH(t-t_0)},\end{aligned}\quad (8.21)$$

so we have

$$\boxed{i \frac{\partial}{\partial t} U(t, t_0) = H_{\text{int}, I}(t) U(t, t_0).} \quad (8.22)$$

For the (Schrödinger) interaction $H_{\text{int}} = \lambda \int d^3 x \phi^4(\mathbf{x}, t_0)$, what we really mean by $H_{\text{int}, I}(t)$ is

$$H_{\text{int}, I}(t) = \lambda \int d^3 x \phi_I^4(\mathbf{x}, t). \quad (8.23)$$

It will be more convenient to remove the explicit λ factor from the interaction Hamiltonian, and write instead

$$H_{\text{int}, I}(t) = \int d^3 x \phi_I^4(\mathbf{x}, t), \quad (8.24)$$

so the equation to solve is

$$i \frac{\partial}{\partial t} U(t, t_0) = \lambda H_{\text{int}, I}(t) U(t, t_0). \quad (8.25)$$

We assume that

$$U(t, t_0) = U_0(t, t_0) + \lambda U_1(t, t_0) + \lambda^2 U_2(t, t_0) + \cdots + \lambda^n U_n(t, t_0). \quad (8.26)$$

Plugging into eq. (8.24) we have

$$\begin{aligned} i\lambda^0 \frac{\partial}{\partial t} U_0(t, t_0) + i\lambda^1 \frac{\partial}{\partial t} U_1(t, t_0) + i\lambda^2 \frac{\partial}{\partial t} U_2(t, t_0) + \cdots + i\lambda^n \frac{\partial}{\partial t} U_n(t, t_0) \\ = \lambda H_{\text{int}, I}(t) \left(1 + \lambda U_1(t, t_0) + \lambda^2 U_2(t, t_0) + \cdots + \lambda^n U_n(t, t_0) \right), \end{aligned} \quad (8.27)$$

so equating equal powers of λ on each side gives a recurrence relation for each $U_k, k > 0$

$$\frac{\partial}{\partial t} U_k(t, t_0) = -iH_{\text{int}, I}(t)U_{k-1}(t, t_0). \quad (8.28)$$

Let's consider each power in turn.

$O(\lambda^0)$: Solving eq. (8.22) to $O(\lambda^0)$ gives

$$i \frac{\partial}{\partial t} U_0(t, t_0) = 0, \quad (8.29)$$

or

$$U(t, t_0) = 1 + O(\lambda). \quad (8.30)$$

$O(\lambda^1)$:

$$\frac{\partial U_1(t, t_0)}{\partial t} = -iH_{\text{int}, I}(t), \quad (8.31)$$

which has solution

$$U_1(t, t_0) = -i \int_{t_0}^t H_{\text{int}, I}(t') dt'. \quad (8.32)$$

$O(\lambda^2)$:

$$\begin{aligned} \frac{\partial U_2(t, t_0)}{\partial t} &= -iH_{\text{int}, I}(t)U_1(t, t_0) \\ &= (-i)^2 H_{\text{int}, I}(t) \int_{t_0}^t H_{\text{int}, I}(t') dt', \end{aligned} \quad (8.33)$$

which has solution

$$\begin{aligned} U_2(t, t_0) &= (-i)^2 \int_{t_0}^t H_{\text{int}, I}(t'') dt'' \int_{t_0}^{t''} H_{\text{int}, I}(t') dt' \\ &= (-i)^2 \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' H_{\text{int}, I}(t'') H_{\text{int}, I}(t'). \end{aligned} \quad (8.34)$$

$O(\lambda^3)$:

$$\frac{\partial U_3(t, t_0)}{\partial t} = -iH_{\text{int},I}(t)U_2(t, t_0), \quad (8.35)$$

so

$$\begin{aligned} U_3(t, t_0) &= -i \int_{t_0}^t dt''' H_{\text{int},I}(t''') U_2(t''', t_0) \\ &= (-i)^3 \int_{t_0}^t dt''' H_{\text{int},I}(t''') \int_{t_0}^{t'''} dt'' \int_{t_0}^{t''} dt' H_{\text{int},I}(t'') H_{\text{int},I}(t') \\ &= (-i)^3 \int_{t_0}^t dt''' \int_{t_0}^{t'''} dt'' \int_{t_0}^{t''} dt' H_{\text{int},I}(t''') H_{\text{int},I}(t'') H_{\text{int},I}(t'). \end{aligned} \quad (8.36)$$

Simplifying the integration region. For the two fold integral, the integration range is the upper triangular region sketched in fig. 8.1.

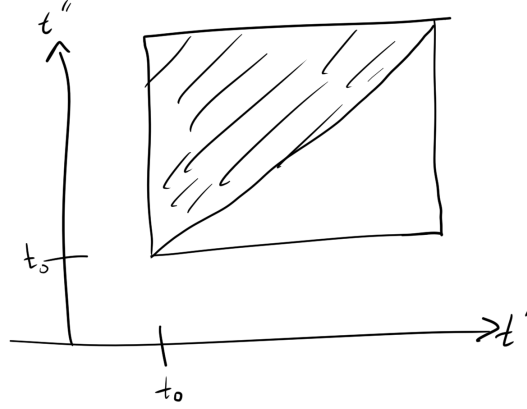


Figure 8.1: Upper triangular integration region.

Claim: We can integrate over the entire square, and divide by two, provided we keep the time ordering

$$U_2(t, t_0) = \frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' T(H_{\text{int},I}(t'') H_{\text{int},I}(t')) \quad (8.37)$$

Demonstration:

$$\begin{aligned} &\frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' T(H_I(t'') H_I(t')) \\ &= \frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' \Theta(t'' - t') H_I(t'') H_I(t') + \frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' \Theta(t' - t'') H_I(t') H_I(t''), \end{aligned} \quad (8.38)$$

but the $\Theta(t'' - t')$ function is non-zero only for $t'' - t' > 0$, or $t' < t''$, and the $\Theta(t' - t'')$ function is non-zero only for $t' - t'' > 0$, or $t'' < t'$, so we can adjust the integration ranges for

$$\begin{aligned}
 & \frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' T(H_I(t'')H_I(t')) \\
 &= \frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' H_I(t'')H_I(t') + \frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t''}^t dt' H_I(t')H_I(t'') \\
 &= \frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' H_I(t'')H_I(t') + \frac{(-i)^2}{2} \int_{t_0}^t dt'' \int_{t''}^t dt' H_I(t'')H_I(t') \\
 &= U_2(t, t_0),
 \end{aligned} \tag{8.39}$$

where we swapped integration variables in second integral. We can clearly do the same thing for the higher order repeated integrals, but instead of a $1/2 = 1/2!$ adjustment for the number of orderings, we will require a $1/n!$ adjustment for an n -fold integral.

Summary:

$$\begin{aligned}
 U_0 &= 1 \\
 U_1 &= -i \int_{t_0}^t dt_1 H_I(t_1) \\
 U_2 &= \frac{(-i)^2}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 T(H_I(t_1)H_I(t_2)) \\
 U_3 &= \frac{(-i)^3}{3!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 T(H_I(t_1)H_I(t_2)H_I(t_3)) \\
 U_n &= \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_n T(H_I(t_1)H_I(t_2) \cdots H_I(t_n))
 \end{aligned} \tag{8.40}$$

Summing we find

$$\begin{aligned}
 U(t, t_0) &= T \exp \left(-i \int_{t_0}^t dt_1 H_I(t_1) \right) \\
 &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \cdots dt_n T(H_I(t_1) \cdots H_I(t_n)).
 \end{aligned} \tag{8.41}$$

This is called Dyson's formula.

8.4 NEXT TIME.

Our goal is to compute: $\langle \Omega | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle$.

8.5 REVIEW.

Given a field $\phi(t_0, \mathbf{x})$, satisfying the commutation relations

$$[\pi(t_0, \mathbf{x}), \phi(t_0, \mathbf{y})] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad (8.42)$$

we introduced an interaction picture field given by

$$\phi_I(t, x) = e^{iH_0(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH_0(t-t_0)}, \quad (8.43)$$

related to the Heisenberg picture representation by

$$\begin{aligned} \phi_H(t, x) &= e^{iH(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH(t-t_0)} \\ &= U^\dagger(t, t_0) \phi_I(t, \mathbf{x}) U(t, t_0), \end{aligned} \quad (8.44)$$

where $U(t, t_0)$ is the time evolution operator.

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \quad (8.45)$$

We argued that

$$i \frac{\partial}{\partial t} U(t, t_0) = H_{\text{I, int}}(t) U(t, t_0), \quad (8.46)$$

and found the “glorious expression”

$$\begin{aligned} U(t, t_0) &= T \exp \left(-i \int_{t_0}^t H_{\text{I, int}}(t') dt' \right) \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 dt_2 \cdots dt_n T (H_{\text{I, int}}(t_1) H_{\text{I, int}}(t_2) \cdots H_{\text{I, int}}(t_n)). \end{aligned} \quad (8.47)$$

However, what we are really after is

$$\langle \Omega | T(\phi(x_1) \cdots \phi(x_n)) | \Omega \rangle. \quad (8.48)$$

Such a product has many labels and names, and we’ll describe it as “vacuum expectation values of time-ordered products of arbitrary #s of local Heisenberg operators”.

8.6 PERTURBATION.

Following §4.2, [14].

$$\begin{aligned} H &= \text{exact Hamiltonian} = H_0 + H_{\text{int}} \\ H_0 &= \text{free Hamiltonian.} \end{aligned} \tag{8.49}$$

We know all about H_0 and assume that it has a lowest (ground state) $|0\rangle$, the “vacuum” state of H_0 .

H has eigenstates, in particular H is assumed to have a unique ground state $|\Omega\rangle$ satisfying

$$H|\Omega\rangle = |\Omega\rangle E_0, \tag{8.50}$$

and has states $|n\rangle$, representing excited (non-vacuum states with energies $> E_0$). These states are assumed to be a complete basis

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_n |n\rangle\langle n| + \int dn |n\rangle\langle n|. \tag{8.51}$$

The latter terms may be written with a superimposed sum-integral notation as

$$\sum_n + \int dn = \oint_n, \tag{8.52}$$

so the identity operator takes the more compact form

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \oint_n |n\rangle\langle n|. \tag{8.53}$$

For some time T we have

$$e^{-iHT} |0\rangle = e^{-iHT} \left(|\Omega\rangle\langle\Omega|0\rangle + \oint_n |n\rangle\langle n|0\rangle \right). \tag{8.54}$$

We now wish to argue that the \oint_n term can be ignored.

Argument 1: This is something of a fast one, but one can consider a formal transformation $T \rightarrow T(1 - i\epsilon)$, where $\epsilon \rightarrow 0^+$, and consider very large T . This gives

$$\begin{aligned}
 \lim_{T \rightarrow \infty, \epsilon \rightarrow 0^+} e^{-iHT(1-i\epsilon)} |0\rangle &= \lim_{T \rightarrow \infty, \epsilon \rightarrow 0^+} e^{-iHT(1-i\epsilon)} \left(|\Omega\rangle \langle \Omega|0\rangle + \sum_n |n\rangle \langle n|0\rangle \right) \\
 &= \lim_{T \rightarrow \infty, \epsilon \rightarrow 0^+} e^{-iE_0T - E_0\epsilon T} |\Omega\rangle \langle \Omega|0\rangle + \sum_n e^{-iE_nT - \epsilon E_nT} |n\rangle \langle n|0\rangle \\
 &= \lim_{T \rightarrow \infty, \epsilon \rightarrow 0^+} e^{-iE_0T - E_0\epsilon T} \left(|\Omega\rangle \langle \Omega|0\rangle + \sum_n e^{-i(E_n - E_0)T - \epsilon T(E_n - E_0)} |n\rangle \langle n|0\rangle \right).
 \end{aligned} \tag{8.55}$$

The limits are evaluated by first taking T to infinity, then only after that take $\epsilon \rightarrow 0^+$. Doing this, the sum is dominated by the ground state contribution, since each excited state also has a $e^{-\epsilon T(E_n - E_0)}$ suppression factor (in addition to the leading suppression factor).

Argument 2: With the hand waving required for the argument above, it's worth pointing other (less formal) ways to arrive at the same result. We can write

$$\sum_n |n\rangle \langle n| \rightarrow \sum_k \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}, k\rangle \langle \mathbf{p}, k|, \tag{8.56}$$

where k is some unknown quantity that we are summing over. If we have

$$H |\mathbf{p}, k\rangle = E_{\mathbf{p},k} |\mathbf{p}, k\rangle, \tag{8.57}$$

then

$$e^{-iHT} \sum_n |n\rangle \langle n| = \sum_k \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}, k\rangle \langle \mathbf{p}, k| e^{-iE_{\mathbf{p},k}T}. \tag{8.58}$$

If we take matrix elements

$$\begin{aligned}
 \langle A| e^{-iHT} \sum_n |n\rangle \langle n| |B\rangle &= \sum_k \int \frac{d^3p}{(2\pi)^3} \langle A|\mathbf{p}, k\rangle e^{-iE_{\mathbf{p},k}T} \langle \mathbf{p}, k|B\rangle \\
 &= \sum_k \int \frac{d^3p}{(2\pi)^3} e^{-iE_{\mathbf{p},k}T} f(\mathbf{p}).
 \end{aligned} \tag{8.59}$$

If we assume that $f(\mathbf{p})$ is a well behaved smooth function, we have “infinite” frequency oscillation within the envelope provided by the amplitude of that function, as depicted in fig. 8.2. The Riemann-Lebesgue lemma [18] describes such integrals, the result of which is that such an integral goes to zero. This is a different sort of hand waving argument, but either way, we can argue that only the ground state contributes to the sum eq. (8.54) above.

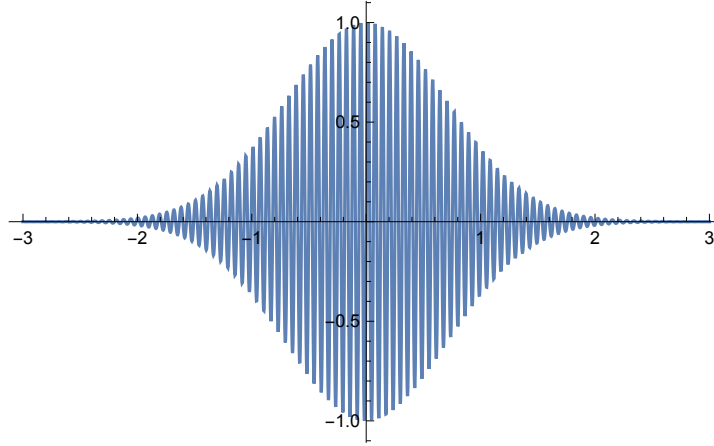


Figure 8.2: High frequency oscillations within envelope of well behaved function.

Ground state of the perturbed Hamiltonian. With the excited states ignored, we are left with

$$e^{-iHT} |0\rangle = e^{-iE_0 T} |\Omega\rangle \langle \Omega|0\rangle, \quad (8.60)$$

in the $T \rightarrow \infty(1 - i\epsilon)$ limit. We can now write the ground state as

$$\begin{aligned} |\Omega\rangle &= \frac{e^{iE_0 T - iHT} |0\rangle}{\langle \Omega|0\rangle} \Big|_{T \rightarrow \infty(1 - i\epsilon)} \\ &= \frac{e^{-iHT} |0\rangle}{e^{-iE_0 T} \langle \Omega|0\rangle} \Big|_{T \rightarrow \infty(1 - i\epsilon)}. \end{aligned} \quad (8.61)$$

Shifting the very large $T \rightarrow T + t_0$ shouldn't change things, so

$$|\Omega\rangle = \frac{e^{-iH(T+t_0)} |0\rangle}{e^{-iE_0(T+t_0)} \langle \Omega|0\rangle} \Big|_{T \rightarrow \infty(1 - i\epsilon)}. \quad (8.62)$$

A bit of manipulation shows that the operator in the numerator has the structure of a time evolution operator.

Claim: (DIY): Equation (8.45), eq. (8.47) may be generalized to

$$U(t, t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)} = T \exp \left(-i \int_{t'}^t H_{\text{I, int}}(t'') dt'' \right). \quad (8.63)$$

Observe that we recover eq. (8.47) when $t' = t_0$. Using eq. (8.63) we find

$$\begin{aligned} U(t_0, -T) |0\rangle &= e^{iH_0(t_0-t_0)} e^{-iH(t_0+T)} e^{-iH_0(-T-t_0)} |0\rangle \\ &= e^{-iH(t_0+T)} e^{-iH_0(-T-t_0)} |0\rangle \\ &= e^{-iH(t_0+T)} |0\rangle, \end{aligned} \quad (8.64)$$

where we use the fact that $e^{iH_0\tau} |0\rangle = (1 + iH_0\tau + \cdots) |0\rangle = 1 |0\rangle$, since $H_0 |0\rangle = 0$.

We are left with

$$|\Omega\rangle = \frac{U(t_0, -T) |0\rangle}{e^{-iE_0(t_0 - (-T))} \langle\Omega|0\rangle}. \quad (8.65)$$

We are close to where we want to be. Wednesday we finish off, and then start scattering and Feynman diagrams.

8.7 REVIEW.

We developed the interaction picture representation, which is really the Heisenberg picture with respect to H_0 .

Recall that we found

$$U(t, t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}, \quad (8.66)$$

with solution

$$U(t, t') = T \exp \left(-i \int_{t'}^t H_{\text{I, int}}(t'') dt'' \right), \quad (8.67)$$

$$\begin{aligned} U(t, t')^\dagger &= T \exp \left(i \int_{t'}^t H_{\text{I, int}}(t'') dt'' \right) \\ &= T \exp \left(-i \int_t^{t'} H_{\text{I, int}}(t'') dt'' \right) \\ &= U(t', t), \end{aligned} \quad (8.68)$$

and can use this to calculate the time evolution of a field $\phi(\mathbf{x}, t) = U^\dagger(t, t_0) \phi_I(\mathbf{x}, t) U(t, t_0)$ and found the ground state ket for H was

$$|\Omega\rangle = \frac{U(t_0, -T) |0\rangle}{e^{-iE_0(T-t_0)} \langle\Omega|0\rangle} \Big|_{T \rightarrow \infty(1-i\epsilon)}. \quad (8.69)$$

Question: What's the point of this, since it is self referential?

Answer: We will see, and also see that it goes away. Alternatively, you can write it as

$$|\Omega\rangle \langle\Omega|0\rangle = \frac{U(t_0, -T) |0\rangle}{e^{-iE_0(T-t_0)}} \Big|_{T \rightarrow \infty(1-i\epsilon)}.$$

We can also show that

$$\langle \Omega | = \frac{\langle 0 | U(T, t_0)}{e^{-iE_0(T-t_0)} \langle 0 | \Omega \rangle} \Big|_{T \rightarrow \infty(1-i\epsilon)}. \quad (8.70)$$

Our goal is still to calculate

$$\langle \Omega | T(\phi(x)\phi(y)) | \Omega \rangle. \quad (8.71)$$

Claim: the “LSZ” theorem (a neat way of writing this) relates this to S matrix elements.

Assuming $x^0 > y^0$

$$\langle \Omega | \phi(x)\phi(y) | \Omega \rangle = \frac{\langle 0 | U(T, t_0) U^\dagger(x^0, t^0) \phi_I(x) U(x^0, t^0) U^\dagger(y^0, t^0) \phi_I(y) U(y^0, t^0) U(t_0, -T) | 0 \rangle}{e^{-i2E_0T} |\langle 0 | \Omega \rangle|^2}. \quad (8.72)$$

Normalize $\langle \Omega | \Omega \rangle = 1$, gives

$$\begin{aligned} 1 &= \frac{\langle 0 | U(T, t_0) U(t_0, -T) | 0 \rangle}{e^{-i2E_0T} |\langle 0 | \Omega \rangle|^2} \\ &= \frac{\langle 0 | U(T, -T) | 0 \rangle}{e^{-i2E_0T} |\langle 0 | \Omega \rangle|^2}, \end{aligned} \quad (8.73)$$

so that

$$\langle \Omega | \phi(x)\phi(y) | \Omega \rangle = \frac{\langle 0 | U(T, t_0) U^\dagger(x^0, t^0) \phi_I(x) U(x^0, t^0) U^\dagger(y^0, t^0) \phi_I(y) U(y^0, t^0) U(t_0, -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle}. \quad (8.74)$$

For $t_1 > t_2 > t_3$

$$\begin{aligned} U(t_1, t_2) U(t_2, t_3) &= T e^{-i \int_{t_2}^{t_1} H_I} T e^{-i \int_{t_3}^{t_2} H_I} \\ &= T \left(e^{-i \int_{t_2}^{t_1} H_I} e^{-i \int_{t_3}^{t_2} H_I} \right) \\ &= T(e^{-i \int_{t_3}^{t_1} H_I}), \end{aligned} \quad (8.75)$$

with an end result of

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3). \quad (8.76)$$

(DIY: work through the details – this is a problem in [14])

This gives

$$\langle \Omega | \phi(x)\phi(y) | \Omega \rangle = \frac{\langle 0 | U(T, x^0) \phi_I(x) U(x^0, y^0) \phi_I(y) U(y^0, -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle}. \quad (8.77)$$

If $y^0 > x^0$ we have the same result, but the y ’s will come first.

Claim:

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \frac{\langle 0 | T \left(\phi_I(x) \phi_I(y) e^{-i \int_{-T}^T H_{\text{I, int}}(t') dt'} \right) | 0 \rangle}{\langle 0 | T(e^{-i \int_{-T}^T H_{\text{I, int}}(t') dt'}) | 0 \rangle}. \quad (8.78)$$

More generally

$$\langle \Omega | \phi_I(x_1) \cdots \phi_I(x_n) | \Omega \rangle = \frac{\langle 0 | T \left(\phi_I(x_1) \cdots \phi_I(x_n) e^{-i \int_{-T}^T H_{\text{I, int}}(t') dt'} \right) | 0 \rangle}{\langle 0 | T(e^{-i \int_{-T}^T H_{\text{I, int}}(t') dt'}) | 0 \rangle}. \quad (8.79)$$

This is the holy grail of perturbation theory.

In QFT II you will see this written in a path integral representation

$$\langle \Omega | \phi_I(x_1) \cdots \phi_I(x_n) | \Omega \rangle = \frac{\int [\mathcal{D}\phi] \phi(x_1) \phi(x_2) \cdots \phi(x_n) e^{-iS[\phi]}}{\int [\mathcal{D}\phi] e^{-iS[\phi]}}. \quad (8.80)$$

8.8 UNPACKING IT.

$$\begin{aligned} \int_{-T}^T H_{\text{I, int}}(t) dt &= \int_{-T}^T \int d^3 \mathbf{x} \frac{\lambda}{4} (\phi_I(\mathbf{x}, t))^4 \\ &= \int d^4 x \frac{\lambda}{4} (\phi_I)^4, \end{aligned} \quad (8.81)$$

so we have

$$\frac{\langle 0 | T \left(\phi_I(x_1) \cdots \phi_I(x_n) e^{-i \frac{\lambda}{4} \int d^4 x \phi_I^4(x)} \right) | 0 \rangle}{\langle 0 | T e^{-i \frac{\lambda}{4} \int d^4 x \phi_I^4(x)} | 0 \rangle}. \quad (8.82)$$

The numerator expands as

$$\begin{aligned} &\langle 0 | T (\phi_I(x_1) \cdots \phi_I(x_n)) | 0 \rangle - i \frac{\lambda}{4} \int d^4 x \langle 0 | T (\phi_I(x_1) \cdots \phi_I(x_n) \phi_I^4(x)) \\ &+ \frac{1}{2} \left(-i \frac{\lambda}{4} \right)^2 \int d^4 x d^4 y \langle 0 | T (\phi_I(x_1) \cdots \phi_I(x_n) \phi_I^4(x) \phi_I^4(y)) | 0 \rangle + \cdots \end{aligned} \quad (8.83)$$

so we see that the problem ends up being the calculation of time ordered products.

8.9 CALCULATING PERTURBATION.

Let's simplify notation, dropping interaction picture suffixes, writing $\phi(x_i) = \phi_i$.

Let's calculate $\langle 0 | T(\phi_1 \cdots \phi_n) | 0 \rangle$. For $n = 2$ we have

$$\begin{aligned} \langle 0 | T(\phi_1 \cdots \phi_n) | 0 \rangle &= D_F(x_1 - x_2) \\ &\equiv D_F(1 - 2). \end{aligned} \quad (8.84)$$

8.10 WICK CONTRACTIONS.

Here's a double dose of short hand, first an abbreviation for the Feynman propagator

$$D_F(1 - 2) \equiv D_F(x_1, x_2), \quad (8.85)$$

and second

$$\overline{\phi_i \phi_j} = D_F(i - j), \quad (8.86)$$

which is called a contraction.

Contractions allow time ordered products to be written in a compact form

Theorem 8.1: Wick's theorem (stub).

The rough idea (from the example below) is that the time ordering of the fields has all the combinations of the pairwise contractions and normal ordered fields.

See exercise 8.2 (Hw4) for full details.

Illustrating by example for the time ordering of $n = 4$ fields, we have

$$\begin{aligned} T(\phi_1 \phi_2 \phi_3 \phi_4) &= :\phi_1 \phi_2 \phi_3 \phi_4: + \overline{\phi_1 \phi_2} : \phi_3 \phi_4: + \overline{\phi_1 \phi_3} : \phi_2 \phi_4: + \overline{\phi_1 \phi_4} : \phi_2 \phi_3: + \overline{\phi_2 \phi_3} : \phi_1 \phi_4: \\ &\quad + \overline{\phi_2 \phi_4} : \phi_1 \phi_3: + \overline{\phi_3 \phi_4} : \phi_1 \phi_2: + \overline{\phi_1 \phi_2} \overline{\phi_3 \phi_4} + \overline{\phi_1 \phi_3} \overline{\phi_2 \phi_4} + \overline{\phi_1 \phi_4} \overline{\phi_2 \phi_3}. \end{aligned} \quad (8.87)$$

Theorem 8.2: Corollary: Vacuum expectation of Wick's theorem expansion

For n even

$$\langle 0 | T(\phi_1 \phi_2 \cdots \phi_n) | 0 \rangle = \overline{\phi_1 \phi_2} \overline{\phi_3 \phi_4} \overline{\phi_5 \phi_6} \cdots \overline{\phi_{n-1} \phi_n} + \text{all other terms.}$$

For n odd, this vanishes.

8.11 SIMPLEST FEYNMAN DIAGRAMS.

For $n = 4$ we have

$$\langle 0 | T(\phi_1 \phi_2 \phi_3 \phi_4) | 0 \rangle = \overbrace{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_3 \phi_2 \phi_4} + \overbrace{\phi_1 \phi_4 \phi_2 \phi_3}, \quad (8.88)$$

the set of Wick contractions can be written pictorially fig. 8.3, and are called Feynman diagrams

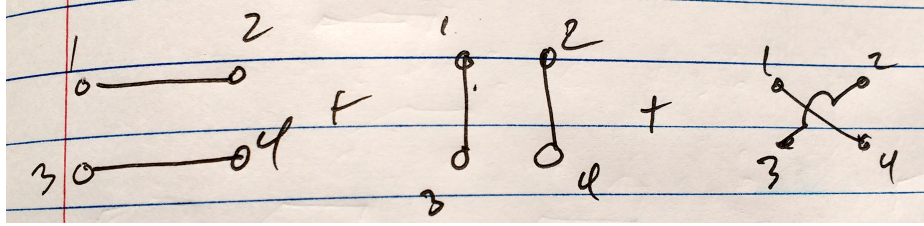


Figure 8.3: Simplest Feynman diagrams.

These are the very simplest Feynman diagrams.

8.12 ϕ^4 INTERACTION.

Introducing another shorthand, we will use an expectation like notation to designate the matrix element for the vacuum state

$$\langle \text{blah} \rangle = \langle 0 | \text{blah} | 0 \rangle. \quad (8.89)$$

For the ϕ^4 theory, this allows us to write the numerator of the perturbed ground state interaction as

$$\begin{aligned} \langle \Omega | \phi(x) \phi(y) | \Omega \rangle &\sim \langle 0 | T \left(\phi_I(x) \phi_I(y) e^{-i \int_{-T}^T H_{\text{int}}(t') dt'} \right) | 0 \rangle \\ &= \left\langle \phi_I(x) \phi_I(y) e^{-i \int d^4 z \phi^4(z)} \right\rangle. \end{aligned} \quad (8.90)$$

To first order, this is

$$\langle T \phi_x \phi_y \rangle - i \frac{\lambda}{4} \int d^4 z \langle T \phi_x \phi_y \phi_z \phi_z \phi_z \phi_z \rangle, \quad (8.91)$$

The first bracket has the pictorial representation sketched in fig. 8.4. whereas the second has the diagrams sketched in fig. 8.5.



Figure 8.4: First integral diagram.

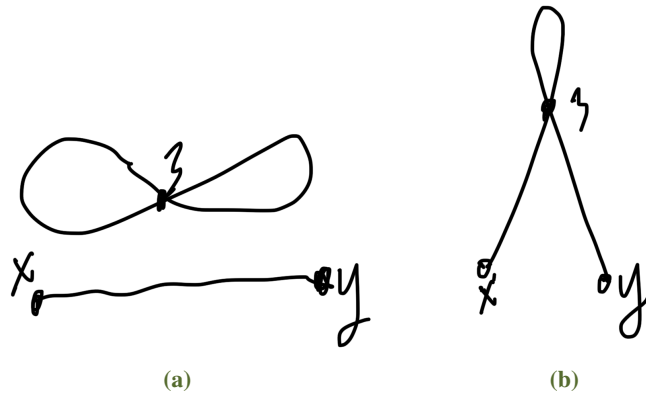


Figure 8.5: Second integral diagrams.

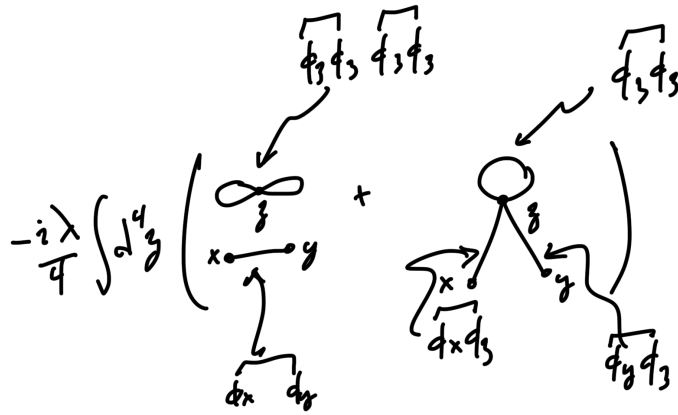


Figure 8.6: Integrals as diagrams.

We can depict the entire second integral in diagrams as sketched in fig. 8.6.

Solving for the perturbed ground state can now be thought of as reduced to drawing pictures. Each line from $x \rightarrow x'$ represents a propagator $D_F(x - x')$, and each vertex $-i\lambda \int d^4z \times$ symmetry coefficients.¹

We may also translate back from the diagrams to an algebraic representation. For the first order ϕ^4 interaction, that is

$$\langle T\phi_x\phi_y \rangle - \frac{i\lambda}{4} \int d^4z D_F(x-y)D_F^2(z-z) + D_F(x-z)D_F(y-z). \quad (8.92)$$

Other diagrams can be similarly translated. For example fig. 8.7. represents



Figure 8.7: Figure eight diagram.

$$\int d^4z D_F^2(z-z) = V_3 T \left(\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \right)^2. \quad (8.93)$$

Clearly, additional interpretation will be required, since this diverges. The resolution of this unfortunately has to be deferred to QFT II, where renormalization is covered.

8.13 TREE LEVEL DIAGRAMS.

We would like to only discuss tree level diagrams, which exclude diagrams like fig. 8.8².

For the bracket³

$$\left\langle \int d^4z \phi_1 \phi_2 \phi_3 \phi_4 \phi_z \phi_z \phi_z \phi_z \right\rangle \quad (8.94)$$

we draw diagrams like those of fig. 8.9, the first of which is a tree level diagram.

- 1 Symmetry coefficients weren't discussed until the next lecture. This means making combinatorial arguments to count the number of equivalent diagrams.
- 2 I think this is what is referred to as connected, amputated graphs in the next lecture. Such diagrams are the ones of interest for scattering and decay problems.
- 3 I'd written: $\langle \int \phi_1 \phi_2 \phi_3 \phi_4 \lambda \int d^4z \phi_z \phi_z \phi_z \phi_z \rangle$. Is this two fold integral what was intended, or my correction in eq. (8.94)?

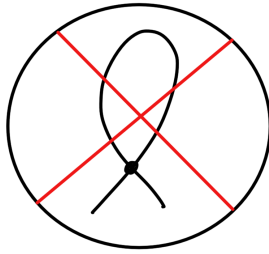


Figure 8.8: Not a tree level diagram.

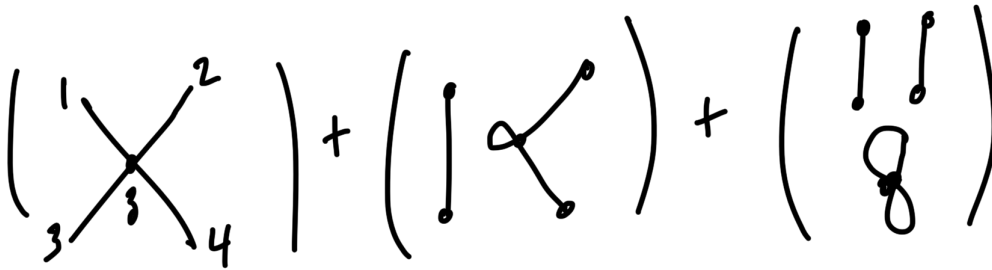


Figure 8.9: First order interaction diagrams.

8.14 PROBLEMS.

Exercise 8.1 Hamiltonian with forcing term.

Prove eq. (7.64).

Answer for Exercise 8.1

In class we derived the field for the non-homogeneous Klein-Gordon equation

$$\begin{aligned}
 \phi(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip \cdot x} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + e^{ip \cdot x} \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right) \Big|_{p^0=\omega_{\mathbf{p}}} \\
 &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{-i\omega_{\mathbf{p}}t + i\mathbf{p} \cdot \mathbf{x}} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + e^{i\omega_{\mathbf{p}}t - i\mathbf{p} \cdot \mathbf{x}} \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right).
 \end{aligned} \tag{8.95}$$

This means that we have

$$\begin{aligned}\pi = \dot{\phi} &= \int \frac{d^3 p}{(2\pi)^3} \frac{i\omega_{\mathbf{p}}}{\sqrt{2\omega_{\mathbf{p}}}} \left(-e^{-i\omega_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + e^{i\omega_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right) \\ (\nabla\phi)_k &= \int \frac{d^3 p}{(2\pi)^3} \frac{ip_k}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{-i\omega_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) - e^{i\omega_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right),\end{aligned}\tag{8.96}$$

and could plug these into the Hamiltonian

$$H = \int d^3 p \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla\phi)^2 + \frac{m^2}{2} \phi^2 \right),\tag{8.97}$$

to find H in terms of \tilde{j} and $a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger$. The result was mentioned in class, and it was left as an exercise to verify.

There's an easy way and a dumb way to do this exercise. I did it the dumb way, and then after suffering through two long pages, where the equations were so long that I had to write on the paper sideways, I realized the way I should have done it.

The easy way is to observe that we've already done exactly this for the case $\tilde{j} = 0$, which had the answer

$$H = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger).\tag{8.98}$$

To handle this more general case, all we have to do is apply a transformation

$$a_{\mathbf{p}} \rightarrow a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}},\tag{8.99}$$

to eq. (8.98), which gives

$$\begin{aligned}H &= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(\left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right)^\dagger \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right)^\dagger \right) \\ &= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(\left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right).\end{aligned}\tag{8.100}$$

Like the $\tilde{j} = 0$ case, we can use normal ordering. This is easily seen by direct expansion:

$$\begin{aligned} \left(a_{\mathbf{p}}^{\dagger} - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) &= a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} - \frac{i\tilde{j}^*(p)a_{\mathbf{p}}}{\sqrt{2\omega_{\mathbf{p}}}} + \frac{a_{\mathbf{p}}^{\dagger}i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} + \frac{|\tilde{j}|^2}{2\omega_{\mathbf{p}}} \\ \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}}^{\dagger} - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) &= a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \frac{i\tilde{j}^*(p)a_{\mathbf{p}}^{\dagger}}{\sqrt{2\omega_{\mathbf{p}}}} - \frac{a_{\mathbf{p}}i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} + \frac{|\tilde{j}|^2}{2\omega_{\mathbf{p}}}. \end{aligned} \quad (8.101)$$

Because \tilde{j} is just a complex valued function, it commutes with $a_{\mathbf{p}}, a_{\mathbf{p}}^{\dagger}$, and these are equal up to the normal ordering, allowing us to write

$$: H := \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^{\dagger} - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right), \quad (8.102)$$

which is the result mentioned in class (albeit without the explicit normal ordering syntax.)

Exercise 8.2 The Wick theorem(s) (2018 HW4.I)

The mother of all Wick theorem(s): Let A_1, A_2, \dots and B denote a set of either creation or annihilation operators. In other words, $A_i = a_{k_i}$ or $a_{k_i}^{\dagger}$ (as well as B ; B is just like one of the A 's, but we'll use the letter B to denote an operator which is singled out, as it is needed in the proof). Next, define a contraction $A_i A_k$ as follows:

$$O_1 \overline{A_i O_2 A_k} = O_1 O_2 \overline{A_i A_k}, \quad (8.103)$$

where O_1, O_2 are arbitrary strings of operators. The above equation signifies the fact that the "contraction" is a c -number, i.e. commutes with all operators. It is defined as follows:

$$\overline{A_i A_j} = \begin{cases} 0, & \text{if } A_i = a_{k_i}, A_j = a_{k_j} \text{ or } A_i = a_{k_i}^{\dagger}, A_j = a_{k_j}^{\dagger} \\ 0, & \text{if } A_i = a_{k_i}^{\dagger} \text{ and } A_j = a_{k_j} \\ (2\pi)^3 \delta^{(3)}(k_i - k_j), & \text{if } A_i = a_{k_i} \text{ and } A_j = a_{k_j}^{\dagger} \end{cases} \quad (8.104)$$

Put in words, the contraction vanishes if both A 's are creation (or both are annihilation operators), as indicated in the first line in eq. (8.104). The contraction is also zero if the operator to the right is an annihilation one, as per the second line in eq. (8.104). Finally, the contraction is equal to the commutator of a_{k_i} with $a_{k_j}^{\dagger}$ in the case when the creation operator is to the left of the annihilation operator.

Finally, we use $: A \dots B :$ to denote the expression where all annihilation operators appear to the right of all creation operators, i.e. the usual normal ordered expression. Then, Wick's theorem—as used in many body physics—is formulated as follows:

$$\begin{aligned} A_1 \dots A_n &= :A_1 \dots A_n: \\ &+ \overbrace{:A_1 A_2 A_3 \dots A_n:} + \dots + \overbrace{:A_1 \dots A_{n-1} A_n:} + \overbrace{:A_1 \dots A_n:} \\ &+ \overbrace{:A_1 A_2 A_3 A_4 \dots A_n:} + \dots \end{aligned} \quad (8.105)$$

The first line contains the normal-ordered product of all operators without contractions, the second line—all possible terms with one contraction (not involving only A_1 of course, but all single-contraction terms, which would be painful to indicate), the third line has all possible two-contraction terms, etc.

Now, you will prove eq. (8.105) in steps.

a. Prove the following Lemma:

$$: A_1 A_2 \dots A_n : B = : A_1 A_2 \dots A_n B : + \sum_{1 \leq k \leq n} : A_1 \dots \overbrace{A_k \dots A_n} B : \quad (8.106)$$

Argue that if B is an annihilation operator, the Lemma is trivial. Thus, consider B to be a creation operator. Notice that if any of the $A_{1,\dots,n}$ are creation operators, they can be taken to the left of the normal products in eq. (8.106) (because all their contractions with B are zero). Thus, if the eq. (8.106) is proven for arbitrary n for the case when all A_i 's are annihilation operators, the general case is obtained by multiplying on the left with the desired number of creation operators. Thus, it suffices to prove the Lemma for the case when all A_i 's are annihilation operators. Also notice Thus, after proving the Lemma for $n = 1$, use induction to show that it holds for any n . Assuming it holds for some number n , go to the case $n + 1$ by multiplying eq. (8.106) by some annihilation operator A_0 on the left and show that the Lemma holds for $n + 1$ operators.

By the chain of logic described above, you have proven eq. (8.106).

Notice also that the lemma eq. (8.106) holds also if the product

$$: A_1 A_2 \dots A_n :$$

is replaced by

$$: \overbrace{A_1 A_2 \dots A_p} \dots A_n :, \quad (8.107)$$

i.e. with the product of operators with an arbitrary number of contractions (one, as written above), with a trivial modification of the last term (since, obviously, you can not contract B with contractions).

- b. Now prove the actual Wick theorem eq. (8.105). Assuming that it holds for $n = 2$. Imagine that eq. (8.105) holds for n operators and prove that it holds for $n + 1$, using eq. (8.106).
- c. **An intermediate step:** Let now A_i and B be operators expressed as some linear combinations of creation and annihilation operators. In particular the subscripts i may now indicate spatial dependence, rather than momentum eigenvalues. Now, define the contraction as follows:

$$\overline{A_i A_j} = \langle 0 | A_i A_j | 0 \rangle, \quad (8.108)$$

where $|0\rangle$ is the Fock vacuum. Notice that eq. (8.108) is equivalent to eq. (8.104) when A_i 's are either creation or annihilation operators. Argue that eq. (8.105) holds verbatim.

- d. **The time-ordered Wick theorem:** Use the above Wick theorem to prove the time-ordered version. Notice that, despite appearances, there is not much left to do. Now, we have space-time rather than momentum space arguments and the theorem is now formulated as follows:

$$\begin{aligned} T(A_1 \dots A_n) = & A_1 \dots A_n : \\ & + : \overline{A_1 A_2} A_3 \dots A_n : + \dots + : \overline{A_1 \dots A_{n-1}} A_n : + : \overline{A_1 \dots A_n} : \\ & + : \overline{A_1 A_2} \overline{A_3 A_4} \dots A_n : + \dots, \end{aligned} \quad (8.109)$$

with the difference that A_i are fields (we are considering real scalar fields), $1 \dots n$ denote space-time points, and the contraction is now the Feynman propagator, e.g. $D_F(x_1 - x_2)$, etc.

Notice that to prove (8.109) one can consider a particular time ordering. Then the T product becomes the normal product of operators (as they are assumed ordered). The space-time dependence can be taken out by Fourier transform which multiplies every term. Every operator is a sum of creation and annihilation operators. Their commutators are exactly the ones giving rise to the contraction in eq. (8.104), on one hand, and to the function $D(x_i - x_j)$ after Fourier transform, on the other (recall that this function appears in the Feynman propagator). Convince yourselves, using eq. (8.108), that this proves the theorem.

- e. For extra bonus, generalize all theorems above to anti commuting fields.

Answer for Exercise 8.2

PROBLEM SET RELATED MATERIAL REDACTED IN THIS DOCUMENT. PLEASE
FEEL FREE TO EMAIL ME FOR THE FULL VERSION IF YOU AREN'T TAKING PHY2403.

END-REDACTION

Exercise 8.3 $U(T, t_0)U(t_0, -T)$

Show that

$$U(T, t_0)U(t_0, -T) = U(T, -T).$$

Answer for Exercise 8.3

We can see that from

$$\begin{aligned} U(T, t_0) &= e^{iH_0(T-t_0)} e^{-iH(T-t_0)} e^{-iH_0(t_0-T)} \\ U(t_0, -T) &= e^{iH_0(t_0-T)} e^{-iH(t_0-T)} e^{-iH_0(-T-t_0)}, \end{aligned} \quad (8.110)$$

so

$$\begin{aligned} U(T, t_0)U(t_0, -T) &= e^{iH_0(T-t_0)} e^{-iH(T-t_0)} e^{-iH(t_0-T)} e^{-iH_0(-T-t_0)} \\ &= e^{iH_0(T-t_0)} e^{-iH2T} e^{-iH_0(-T-t_0)}, \end{aligned} \quad (8.111)$$

whereas

$$\begin{aligned} U(T, -T) &= e^{iH_0(T-t_0)} e^{-iH(T-T)} e^{-iH_0(-T-t_0)} \\ &= e^{iH_0(T-t_0)} e^{-iH2T} e^{-iH_0(-T-t_0)}. \end{aligned} \quad (8.112)$$

Exercise 8.4 Pondering the ground state bra formula.

Prove eq. (8.70). What is wrong with conjugating eq. (8.69) to find

$$\langle \Omega | = \frac{\langle 0 | U(-T, t_0)}{e^{+iE_0(T-t_0)} \langle 0 | \Omega \rangle} \Big|_{T \rightarrow \infty(1-i\epsilon)}.$$

Answer for Exercise 8.4

While there is nothing wrong with stating

$$\left(\frac{U(t_0, -T) | 0 \rangle}{e^{-iE_0(T-t_0)} \langle \Omega | 0 \rangle} \right)^\dagger = \frac{\langle 0 | U(-T, t_0)}{e^{+iE_0(T-t_0)} \langle 0 | \Omega \rangle}, \quad (8.113)$$

the limit point $\infty(1 - i\epsilon)$ also needs to be changed with this conjugation. So eq. (8.113) is correct, but it is only part of the story, and should really be stated as

$$\langle \Omega | = \frac{\langle 0 | U(-T, t_0)}{e^{+iE_0(T-t_0)} \langle 0 | \Omega \rangle} \Big|_{T \rightarrow \infty(1+i\epsilon)}. \quad (8.114)$$

This is awkward because now our expressions for $\langle \Omega |$ and $|\Omega\rangle$ approach T from different directions, and we want to evaluate both with a single limiting argument.

To resolve this, we really have to start back with the identity expansion we used in lecture 14, and write

$$\begin{aligned} \langle 0 | e^{-iHT} &= \left(\langle 0 | \Omega \rangle \langle \Omega | + \sum_n \langle 0 | n \rangle \langle n | \right) e^{-iHT} \\ &= \langle 0 | \Omega \rangle \langle \Omega | e^{-iE_0 T} + \sum_n \langle 0 | n \rangle \langle n | e^{-iE_n T}. \end{aligned} \quad (8.115)$$

We argued (as does the text) that approaching to as $T(1 - i\epsilon)$ kills off the energetic states since

$$\langle n | e^{-iE_n T} \rightarrow \langle n | e^{-iE_n T} e^{-E_n T \epsilon} \quad (8.116)$$

and the exponential damping factor is smaller for each $E_n > E_0$, so it can be neglected in the large T limit, leaving

$$\langle 0 | e^{-iHT} = \lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0 | \Omega \rangle \langle \Omega |. \quad (8.117)$$

As we did for $|\Omega\rangle$ we can shift the large time T by a small constant (this time $-t_0$ instead of t_0), to give

$$\begin{aligned} \langle \Omega | &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | e^{-iHT}}{\langle 0 | \Omega \rangle e^{-iE_0 T}} \\ &\approx \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | e^{-iH(T-t_0)}}{\langle 0 | \Omega \rangle e^{-iE_0(T-t_0)}} \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | e^{iH_0(T-t_0)} e^{-iH(T-t_0)}}{\langle 0 | \Omega \rangle e^{-iE_0(T-t_0)}} \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | U(T, t_0)}{\langle 0 | \Omega \rangle e^{-iE_0(T-t_0)}}, \end{aligned} \quad (8.118)$$

where the projective property $\langle 0 | e^{iH_0 \alpha} = \langle 0 |$ has been used to insert a no-op (i.e. $\langle 0 | H_0 = 0$). This recovers the result stated in class (also: [14] eq. (4.29).)

Exercise 8.5 **Interaction energy between static external charges (2018 HW3.D)**

- a. Calculate the vacuum expectation value of the time ordered exponential

$$\langle 0 | T e^{i \int d^4x g j(x) \phi(x)} | 0 \rangle \quad (8.119)$$

for the case of a massive free real scalar field. Here, g is a coupling constant, which we shall call the “Yukawa coupling”. Show, e.g. using Wick’s theorem, that the answer is

$$e^{-\frac{g^2}{2} \int d^4x d^4y j(x) D_F(x-y) j(y)}, \quad (8.120)$$

which is really the exponential of the second order term and D_F is the Feynman propagator.

- b. Consider the case where $j(t, \mathbf{x}) = \theta(T - t)\theta(T + t) (\delta^{(3)}(\mathbf{x}) - \delta^{(3)}(\mathbf{x} - \mathbf{R}))$. This source term represents two external opposite “charges”⁴ a distance $R = |\mathbf{R}|$ apart, created at $t = -T$ and existing for time $2T$. Show that, in the limit $T \gg R \gg 1/m$, eq. (8.120) is proportional to:

$$e^{-i2TV(R)}, \quad (8.121)$$

where $V(R)$ is the Yukawa potential.

Hint: Recall that $\lim_{T \rightarrow \infty} \int_{-T}^T dx e^{ipx} = 2\pi\delta(p)$ as well as the usual relation $(2\pi\delta(p))^2 = 2\pi\delta(p)2T$.

The result (8.121) means that “two static sources of scalar field a distance R apart interact via the Yukawa potential.” This is because (8.121) is the evolution operator (it is $\sim e^{-iHt}$, for $t = 2T$) of the field theory in the presence of the static external sources (or, more appropriately, (8.121) is the contribution to the evolution operator that has to do with the interaction between the sources induced by the field). Thus, it is natural to call the quantity multiplying $-i2T$ and depending on R , the interaction potential $V(R)$ between the sources.

Do opposite-sign “charges” attract or repel? How about same-sign?

Notice that when the “charges” are also considered as part of a QFT and, therefore, $j(x)$ in (8.119) is replaced by an appropriate QFT expression, one finds more interesting results. Namely, the Yukawa interaction between two fermions is always attractive—whether it is between two particles, two anti-particles, or between a particle and an anti-particle. The way to establish this, as well as an alternative derivation of the expression for $V(R)$ you found in (8.121), is to start from the scattering of (anti)fermions via scalar exchange and then take the nonrelativistic limit. A comparison with quantum-mechanical Born scattering yields then an expression for $V(R)$.

This result quoted above is of great interest in nuclear physics, where single-pion exchange operates via $V(R)$, and turns out to be attractive between nucleons and between nucleons and anti-nucleons.

⁴ In other words, classical particles linearly coupled to ϕ (if ϕ was the electrostatic potential A^0 , this would really be the electromagnetic charge.) For a discussion of whether an interaction like you will study can arise from a realistic QFT, see comment in 2. below.

- d. Find the probability that the source creates one particle of momentum \mathbf{k} . First, compute this to order j and then to all orders, using the trick above to sum the series.
- e. Show that the probability of producing n particles is $P(n) = \frac{1}{n!} \lambda^n e^{-\lambda}$, the Poisson distribution.
- f. Show that $\sum_{n=0}^{\infty} P(n) = 1$ and that $\langle N \rangle = \sum_{n=0}^{\infty} n P(n) = \lambda$, where λ is given in (8.124). Notice that the expression for the mean particle number $\langle N \rangle$ created exactly reproduces (when dimensionally reduced to $d = 1$) the one from quantum mechanics given in class. Finally, compute the mean square fluctuation $\langle (N - \langle N \rangle)^2 \rangle$.

Answer for Exercise 8.6

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[REDACTED]

END-REDACTION

Exercise 8.7 Where is the particle? (2018 HW3.IV)

In class, we did mention that, by analogy with non relativistic quantum mechanics, the state $\hat{\phi}(\mathbf{x}, t = 0)|0\rangle$ allows us to say something along the lines that “the operator $\hat{\phi}(\mathbf{x})_+$ creates a particle at \mathbf{x} ”. These words are based on noticing that in QM, we have

$$|\mathbf{x}\rangle \sim \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle,$$

where $|\mathbf{x}\rangle$ is an eigenstate of the position operator with eigenvalue \mathbf{x} and $|\mathbf{p}\rangle$ is, likewise, an eigenstate of momentum. On the other hand, in free massive scalar theory, the state $\hat{\phi}(\mathbf{x}, t = 0)|0\rangle$ can be similarly expressed as

$$\hat{\phi}(\mathbf{x}, t = 0)|0\rangle = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{p}\cdot\mathbf{x}} \hat{a}_{\mathbf{p}}^\dagger |0\rangle = \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle,$$

where $|\mathbf{p}\rangle$ is the relativistically normalized momentum eigenstate. Comparing the above two equations, reading from left to right, we are compelled to utter the words quoted in the beginning.

Accepting this interpretation literally, we are next faced with explaining the following. Consider the state $|\mathbf{0}, 0\rangle = \hat{\phi}(\mathbf{0}, t=0)|0\rangle$, interpreted (as per the above discussion) as a particle created at $\mathbf{x} = 0$ at $t = 0$. Similarly, the state

$$|\mathbf{y}, t\rangle = \hat{\phi}(\mathbf{y}, t)|0\rangle$$

is that of a particle at \mathbf{y} at t . Notice that these are free fields so their time evolution is trivial. Then, by the usual Born rule of quantum mechanics (which we accept in QFT), the inner product

$$\langle \mathbf{y}, t | \mathbf{0}, 0 \rangle$$

would be “the amplitude that the particle created at $\mathbf{0}$ at $t = 0$ is found at \mathbf{y} at t ”. Notice that this is exactly the kind of answer that the quantum-mechanical propagator, often denoted precisely by $\langle \mathbf{y}, t | \mathbf{0}, 0 \rangle$, gives. A problem with this arises when one realizes that

$$\langle \mathbf{y}, t | \mathbf{0}, 0 \rangle = \langle 0 | \hat{\phi}(\mathbf{y}, t) \hat{\phi}(\mathbf{0}, 0) | 0 \rangle = D(\mathbf{y}, t) \neq 0 \text{ for } (\mathbf{y}, t) \sim (\mathbf{0}, 0) .$$

In other words, this amplitude is nonzero for spacelike separations (as you explicitly showed in Homework 2, Problem 1, Part 2). The point of the simple exercise below is to argue that the above interpretation of this amplitude should be taken with a grain of salt, i.e. not too literally, as far as relativity is concerned, of course.

The question we will ask is: to what extent is this particle at $\mathbf{x} = 0$ localized? In quantum mechanics, we answer this question by pointing out that for an eigenstate of \hat{x} , whose wave function is $\delta(x - x')$, the probability to find the particle anywhere but at $x = x'$ is zero. Trying to pursue this in QFT, a conundrum that arises is that we do not have wave functions for particles. Recall that we have wave functionals, which determine the probability that *the field* has this or that value. The coordinate, on the other hand, is an argument, not an operator (hence “observable”) in the theory—just like time in QM, which is also not an operator; after all we said “QM=QFT in $d = 1$ ”. The best we can do is to consider the state $|\mathbf{y}, 0\rangle$ and ask where its properties identifiable in QFT—energy or momentum—are localized.

Thus, consider the expectation value of $\hat{T}_{00}(\mathbf{x}, t)$ (assumed normal-ordered) in this state:

$$\rho(\mathbf{y}, \mathbf{x}, t) \equiv \langle \mathbf{y}, 0 | T_{00}(\mathbf{x}, t) | \mathbf{y}, 0 \rangle .$$

From the Born rule, the natural interpretation of the above quantity is the value of the energy density of the state $|\mathbf{y}, 0\rangle$ observed at (\mathbf{x}, t) —spacelike or not w.r.t. $(\mathbf{y}, 0)$.

- Show, using the translation operator, that $\rho(\mathbf{y}, \mathbf{x}, t) = \rho(0, \mathbf{x} - \mathbf{y}, t) \equiv \tilde{\rho}(\mathbf{x} - \mathbf{y}, t)$, where the last equality defines the new energy density $\tilde{\rho}(\mathbf{x}, t)$.
- Using Wick’s theorem—really, a baby-version thereof—express $\tilde{\rho}(\mathbf{x}, t)$ in terms of $D(\mathbf{x}, t)$ and its derivatives.

- c. Using the knowledge acquired from Homework 2, study how well is the particle's energy localized, already at $t = 0$.

Are you surprised by the result? Are you comforted?

We didn't have time, apart from Problem 4 of Homework 2, to dwell much on the nonrelativistic limit. This limit can be achieved by forgetting the antiparticles and then defining non-relativistic fields. This is very well described in either Tong's or Luke's notes. For those of you studying cold atoms, it is definitely a must-read!

My final comment is that the most concise formulation of causality that goes beyond simply stating that the commutators vanish for spacelike separations is the one first due to Stueckelberg (1940's) and then finessed by Bogoljubov (1950's).

They consider the expectation value of an operator $\hat{O}(x)$ in a state prepared by the action of an operator $U[g]|0\rangle$. $U[g]$ is an evolution operator (see below) which is a functional of some classical fields $g(y)$ used to prepare the state of the field (e.g. external e.m. fields used to focus, accelerate, etc., the particles; $g(y)$ could also be used to turn on and off the interactions in different space time regions). Thus the object of study is:

$$\langle \hat{O}(x) \rangle = \langle 0 | U^\dagger[g] \hat{O}(x) U[g] | 0 \rangle .$$

The causality condition, then, is that

$$\frac{\delta \langle \hat{O}(x) \rangle}{\delta g[y]} = 0 \text{ for } x \sim y .$$

Now, recalling the form of the evolution operator, $U[g] = T e^{i \int dt d^3x L_I(t, \mathbf{x}, g(\mathbf{x}, t))}$, and the Baker-Campbell-Hausdorff formula, it should be clear how the vanishing of the commutators outside the light cone becomes relevant for the above condition to hold. For Bogoljubov, the vanishing commutators are a *consequence* of the causality condition given in terms of variational derivatives, as expressed above; he derives the S -matrix expansion from that requirement along with a few others (locality and Lorentz invariance, basically).

The reason to include this comment was to close the loop on something that I mentioned in class, now that we've seen what $U[g]$ may look like.

Answer for Exercise 8.7

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SCATTERING AND DECAY.

9.1 ADDITIONAL RESOURCES.

The video [10] does an excellent job explaining these concepts, covering the same material, but doing so in a very structured fashion. He also nicely highlights which parts we are basically taking on faith in order to gain some calculation experience.

9.2 DEFINITIONS AND MOTIVATION.

In QM we did lots of scattering problems as sketched in fig. 9.1, and were able to compute the reflected and transmitted wave functions and quantities such as the reflection and transmission coefficients

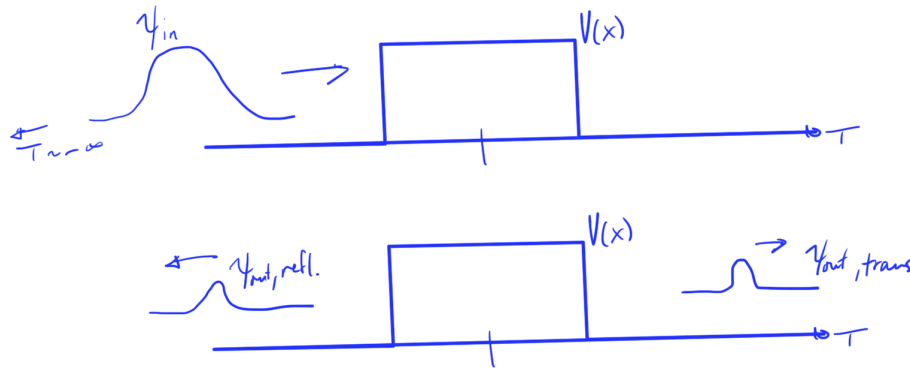


Figure 9.1: Reflection and transmission of wave packets.

$$R = \frac{|\Psi_{\text{ref}}|^2}{|\Psi_{\text{in}}|^2}$$

$$T = \frac{|\Psi_{\text{trans}}|^2}{|\Psi_{\text{in}}|^2}.$$
(9.1)

We'd like to consider scattering in some region of space with a non-zero potential, such as the scattering of a plane wave with known electron flux rate as sketched in fig. 9.2. We can imagine

that we have a detector capable of measuring the number of electrons with momentum \mathbf{p}_{out} per unit time.

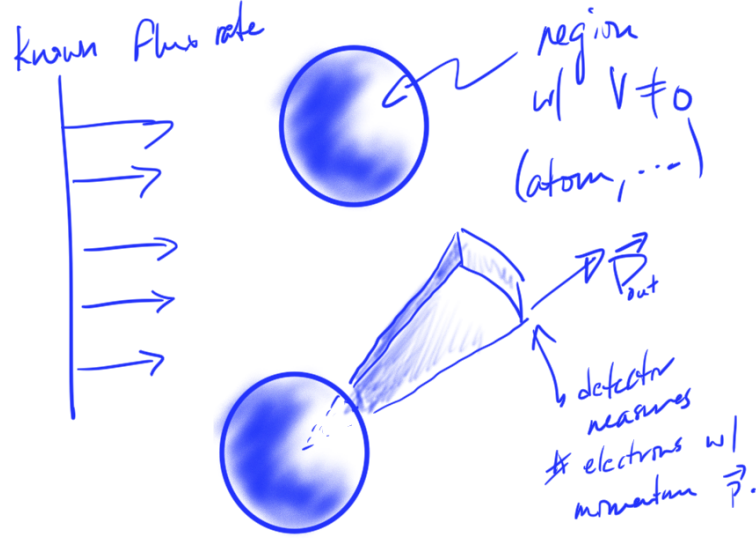


Figure 9.2: Plane wave scattering off a potential.

Definition 9.1: Total cross section (X-section).

$$\sigma_{\text{total}} = \frac{\text{number of scattering events with } \mathbf{p}_{\text{out}} \neq \mathbf{k}_{\text{in}} \text{ per unit time}}{\text{Flux of incoming particles}},$$

where the flux is the number of particles crossing a unit area in unit time.

Units of the x-section are (with $\hbar = c = 1$)

$$[\sigma] = \text{area} = \frac{1}{M^2}. \quad (9.2)$$

The concept of scattering cross section may not be new, as it can even be encountered in classical mechanics. One such scenario is sketched in fig. 9.3 where the cross section is just the area

$$\sigma = \pi R^2. \quad (9.3)$$

Other classical fields where cross section is encountered includes antenna theory (radar scattering profiles, ...).

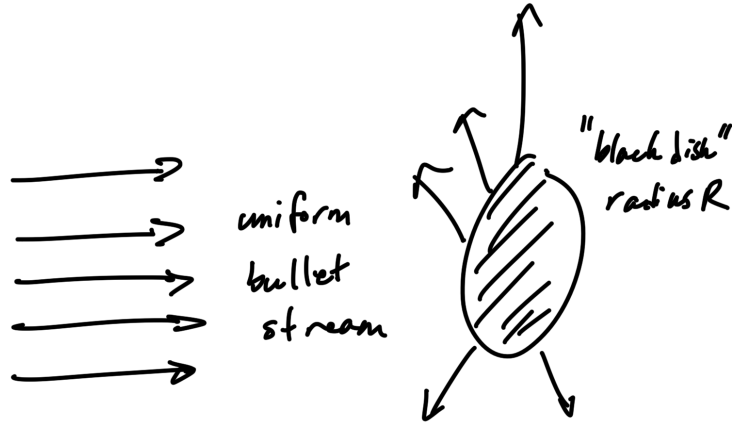


Figure 9.3: Classical scattering.

Definition 9.2: Differential cross section.

$$\frac{d^3\sigma}{dp_x dp_y dp_z} = \frac{\text{number of scattering events with } \mathbf{p}_{\text{out}} \text{ between } (\mathbf{p}, \mathbf{p} + \Delta\mathbf{p})}{\text{flux}}.$$

In QFT we typically study $2 \rightarrow n$ inelastic scattering. Most commonly the nature of the final state particles are different from the nature of the incoming state.

For example, we can collide an electron and anti-electron, and can get muon and anti-muon particles as sketched in fig. 9.4, or pions as sketched in fig. 9.5, or even both as sketched in fig. 9.6.

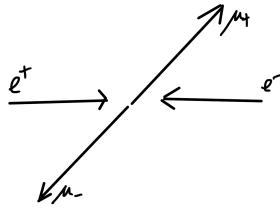


Figure 9.4: Muon pair production.

In the $\lambda\phi^4$ theory we can have scattering events such as $2 \rightarrow 2$ and $2 \rightarrow 2n$ production as sketched in fig. 9.7.

How to calculate in QFT. Initial state of 2 particles A, B with initial state

$$|\mathbf{k}_A, \mathbf{k}_B\rangle_{\text{in}, T \rightarrow -\infty} \tag{9.4}$$

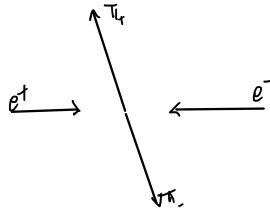


Figure 9.5: Pion pair production.

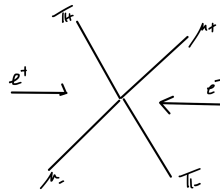


Figure 9.6: Muon and pion pair production.

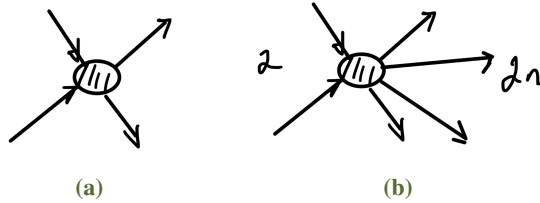


Figure 9.7: lambda fourth scattering events.

and final n-particle state

$$|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle_{\text{out}, T \rightarrow +\infty} \quad (9.5)$$

The QM transition amplitude from the initial to the final state is

$${}_{\text{out}}\langle \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n | \mathbf{k}_A, \mathbf{k}_B \rangle_{\text{in}} = \langle \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n | e^{-2iHT} | \mathbf{k}_A, \mathbf{k}_B \rangle. \quad (9.6)$$

This is the amplitude for $AB \rightarrow 1 \dots n$. Ultimately, we want the scattering x-section.

We will also be interested in decay rates, as there are unstable particles in QFT that can decay. This doesn't happen in $\lambda\phi^4$ theory. In a theory with 2 scalar fields Φ, φ with $m_\Phi > 2m_\varphi$. A possible interaction for such a theory is

$$H_{\text{int}} = \mu\Phi\varphi^2, \quad (9.7)$$

which would permit $\Phi \rightarrow \varphi\varphi$ decays. HW4 has a coupling like $(h/V)\partial_\mu\phi^a\partial^\mu\phi^a$ for which a $h \rightarrow \phi^a\phi^a$ decay is possible.

Definition 9.3: Decay rate.

The decay rate is defined as

$$\Gamma = \frac{\text{Number of decays } \Phi \rightarrow \varphi\varphi \text{ in unit time}}{\text{Number of } \Phi \text{ particles present}}$$

What is the amplitude for such a decay transition?

$$\langle \mathbf{k}_\phi |_{\text{in}, T \rightarrow -\infty} \rightarrow \langle \mathbf{k}_1, \mathbf{k}_2 |_{\text{out}, T \rightarrow +\infty}. \quad (9.8)$$

The amplitude for $\mathbf{k}_\phi \rightarrow \mathbf{k}_1, \mathbf{k}_2$.

$$\langle \mathbf{k}_1, \mathbf{k}_2 | e^{-i2HT} | \mathbf{k}_\phi \rangle = {}_{\text{out}}\langle \mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_\phi \rangle \quad (9.9)$$

mysterious seeming statement something like : “The decays are essentially due to interactions with vacuum fluctuations.”

9.3 CALCULATING INTERACTIONS.

We write

$$\begin{aligned} {}_{\text{out}}\langle \mathbf{p}_1, \dots, \mathbf{p}_n | \mathbf{k}_A, \mathbf{k}_B \rangle_{\text{in}} &= \lim_{T \rightarrow \infty} \langle \mathbf{p}_1, \dots, \mathbf{p}_n | e^{-i2HT} | \mathbf{k}_A, \mathbf{k}_B \rangle \\ &= \langle \mathbf{p}_1, \dots, \mathbf{p}_n | \hat{S} | \mathbf{k}_A, \mathbf{k}_B \rangle \\ &= \langle \mathbf{p}_1, \dots, \mathbf{p}_n | \mathbf{1} + i\hat{T} | \mathbf{k}_A, \mathbf{k}_B \rangle, \end{aligned} \quad (9.10)$$

where \hat{S} is called the S-matrix or scattering matrix, which is decomposed into a unit portion **1** which is a convenient way to exclude events with no scattering. **1** contributes for $n = 2$ only, but is an n scattering amplitude. We are really interested in the $i\hat{T}$ portion of this amplitude

$$\langle \mathbf{p}_1, \dots, \mathbf{p}_n | i\hat{T} | \mathbf{k}_A, \mathbf{k}_B \rangle = (2\pi)^4 \delta^{(4)}(\mathbf{k}_A + \mathbf{k}_B - \sum_{i=1}^n \mathbf{p}_i) \times iM(\mathbf{k}_A + \mathbf{k}_B \rightarrow \mathbf{p}_1 \dots \mathbf{p}_n). \quad (9.11)$$

This amounts to a definition of M . Recall that we found

$$\begin{aligned} U(T, -T) &= T \left(e^{-i \int_{-T}^T H_I(t') dt'} \right) \\ &= e^{iH_0(T-t_0)} e^{-i2HT} e^{-iH_0(-T-t_0)}. \end{aligned} \quad (9.12)$$

We want to replace the e^{-i2HT} in the matrix element above with U .

In perturbation theory, we assume (conjecture) that

$$\begin{aligned} |\mathbf{k}_A, \mathbf{k}_B\rangle &\sim |\mathbf{k}_A, \mathbf{k}_B\rangle_0 \\ &\sim \text{const } a_{\mathbf{k}_A}^\dagger a_{\mathbf{k}_B}^\dagger |0\rangle \end{aligned} \quad (9.13)$$

Because we'll be squaring the amplitudes, we can assume that the $e^{iH_0(T-t_0)}$ will result in just phase factors that won't survive, so in eq. (9.10) we can insert U

$$\text{out} \langle \mathbf{p}_1, \dots, \mathbf{p}_n | \mathbf{k}_A, \mathbf{k}_B \rangle_{\text{in}} = \lim_{T \rightarrow \infty} \langle \mathbf{p}_1, \dots, \mathbf{p}_n | U(T, -T) | \mathbf{k}_A, \mathbf{k}_B \rangle \quad (9.14)$$

$$\langle \mathbf{p}_1, \dots, \mathbf{p}_n | i\hat{T} | \mathbf{k}_A, \mathbf{k}_B \rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} {}_0 \langle \mathbf{p}_1, \dots, \mathbf{p}_n | T(e^{-i \int_{-T}^T H_I(t') dt'}) | \mathbf{k}_A, \mathbf{k}_B \rangle_0 \quad (9.15)$$

We will see that evaluating this beastie amounts to summing all the connected and amputated diagrams, a subset of all the possible graphs. Why this is true is not covered in this course (i.e. see QFT II and/or §7.2 [14]). Using the ϕ^4 th theory, [9] provides a very nice example that shows how the first order disconnected diagrams happen to cancel out perfectly (even though they both represent infinities!).

What is “connected and amputated”? Explaining by example. $n = 2, \lambda\phi^4/4!$.

$$\langle 0 | a_{\mathbf{p}_1} a_{\mathbf{p}_2} \left(\lambda - \frac{i\lambda}{4!} \int d^4x \phi_I^4(x) + \frac{1}{2} \left(\frac{i\lambda}{4!} \right)^2 \int d^4x d^4y \phi_I^4(x) \phi_I^4(y) + \dots \right) a_{\mathbf{k}_A}^\dagger a_{\mathbf{k}_B}^\dagger | 0 \rangle \quad (9.16)$$

Here time ordering operations are implied, but not written explicitly. Also, the “amputated” indicates that we are going to be dropping the **1** portion of the exponential expansion (as we've also dropped that in eq. (9.15)). We will also be using a relativistic normalization so that the $a_{\mathbf{k}_A}^\dagger a_{\mathbf{k}_B}^\dagger$ terms include $\sqrt{2\omega_{\mathbf{k}_A} 2\omega_{\mathbf{k}_B}}$ contributions and the $a_{\mathbf{p}_1} a_{\mathbf{p}_2}$ include $\sqrt{2\omega_{\mathbf{p}_1} 2\omega_{\mathbf{p}_2}}$ contributions.

$$\overline{T\phi_I(x_1)\phi_I(x_2)} = D_F(x_1 - x_2) \quad (9.17)$$

When we look at

$$\begin{aligned} \overline{\phi_I(x_1)a_{\mathbf{k}}^\dagger} \sqrt{2\omega_{\mathbf{k}}} &= \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot x}}{\sqrt{2\omega_{\mathbf{p}}}} \overline{a_{\mathbf{p}}a_{\mathbf{k}}^\dagger} \sqrt{2\omega_{\mathbf{k}}} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip \cdot x}}{\sqrt{2\omega_{\mathbf{p}}}} \delta^{(3)}(\mathbf{p} - \mathbf{k}) \sqrt{2\omega_{\mathbf{k}}} \\ &= e^{-ik \cdot x}. \end{aligned} \quad (9.18)$$

Similarly

$$\begin{aligned} \overline{a_{\mathbf{p}}\phi_I(x_1)} \sqrt{2\omega_{\mathbf{p}}} &= \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot x}}{\sqrt{2\omega_{\mathbf{k}}}} \overline{a_{\mathbf{p}}a_{\mathbf{k}}^\dagger} \sqrt{2\omega_{\mathbf{k}}} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot x}}{\sqrt{2\omega_{\mathbf{k}}}} \delta^{(3)}(\mathbf{p} - \mathbf{k}) \sqrt{2\omega_{\mathbf{k}}} \\ &= e^{+ip \cdot x}. \end{aligned} \quad (9.19)$$

Summarizing

$$\begin{aligned} \overline{\phi_I(x)a_{\mathbf{p}}^\dagger} &= e^{-ip \cdot x} \\ \overline{a_{\mathbf{p}}\phi_I(x)} &= e^{ip \cdot x}. \end{aligned}$$

(9.20)

9.4 EXAMPLE DIAGRAMS.

We want to examine the relevant diagrams corresponding to a transition amplitudes for the ϕ^4 theory. Contractions such as

$$\langle \overline{a_{\mathbf{p}_1}a_{\mathbf{p}_2}} \overline{a_{\mathbf{k}_A}^\dagger a_{\mathbf{k}_B}^\dagger} \rangle_0. \quad (9.21)$$

result in diagrams that are not connected as sketched in fig. 9.8.

There are no other possibilities for the first order (and these ones are not interesting). For the second order transition amplitudes we want to sum of all the contractions for the expectation

$$\langle a_{\mathbf{p}_1}a_{\mathbf{p}_2}\phi_I^4(x)a_{\mathbf{k}_A}^\dagger a_{\mathbf{k}_B}^\dagger \rangle = -i\frac{\lambda}{4!} \sum \text{all contractions}. \quad (9.22)$$

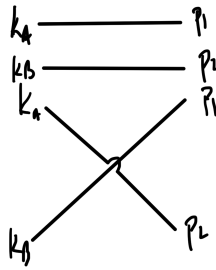


Figure 9.8: Not connected diagrams.

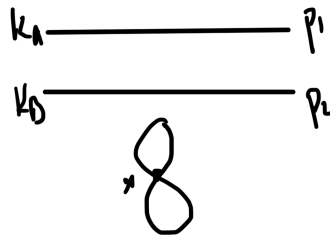


Figure 9.9: Not connected second order interactions, including vacuum fluctuations.

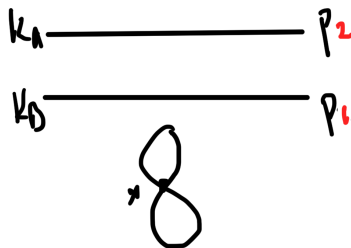


Figure 9.10: Another second order diagram.

Our diagrams include fig. 9.9, which are not connected. The figure eight is a vacuum fluctuation that represents virtual processes. Another diagram is fig. 9.10, also not connected.

We want diagrams that we will describe as “connected and amputated”. We are clearly discarding non-connected diagrams like those above, but will need to demonstrate what is meant by amputated, and will continue to consider examples to make that clear.

Here’s another diagram fig. 9.11 that is also not connected. From the diagrams we can con-

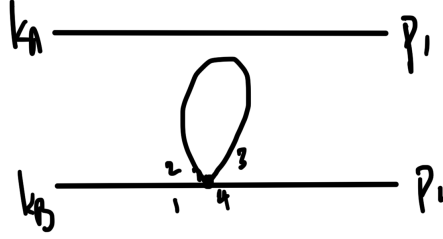


Figure 9.11: Another not-connected diagram.

struct the functionals that they represent. The single line in this one is a $\delta^{(3)}(\mathbf{p}_1 - \mathbf{k}_A)$ whereas the balloon with two strings is

$$\int d^4x e^{-ik_B \cdot x} D_F(x - x) e^{ip_2 \cdot x}. \quad (9.23)$$

There are similar not-connected variations of the possible diagrams that we will also discard. The connected diagrams all come from contractions such as

$$\langle 0 | a_{\mathbf{p}_1} a_{\mathbf{p}_2} \phi_I^4(x) a_{\mathbf{k}_A}^\dagger a_{\mathbf{k}_B}^\dagger | 0 \rangle \quad (9.24)$$

The diagram for this interaction now has a vertex representing the contractions with $\phi_I^4(x)$ with four edges from that vertex as sketched in fig. 9.12. The algebraic expression for this diagram is

$$4! \left(\frac{-i\lambda}{4!} \right) \int d^4x e^{-i(k_A + k_B) \cdot x} e^{i(p_1 + p_2) \cdot x} = -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_A - k_B). \quad (9.25)$$

Such a diagram has the general form

$$(2\pi)^4 \delta^{(4)} \left(\sum \text{in} - \sum \text{final} \right) \times iM(A, B \rightarrow 1, 2), \quad (9.26)$$

so

$$M(A, B \rightarrow 1, 2) = -\lambda. \quad (9.27)$$

Here the “symmetry factor” $4!$ was added in to count all possible ways of constructing such a diagram.

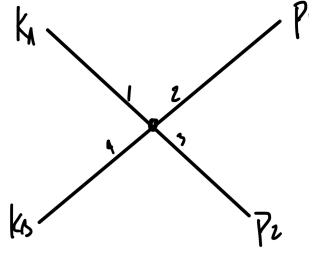


Figure 9.12: Not non-connected diagram.

Next order How about an amplitude like

$$\langle 0 | a_{\mathbf{p}_1} a_{\mathbf{p}_2} \frac{1}{2} \left(\frac{-i\lambda}{4!} \right)^2 \int d^4x \int d^4y \phi_I^4(x) \phi_I^4(y) a_{\mathbf{k}_A}^\dagger a_{\mathbf{k}_B}^\dagger | 0 \rangle \quad (9.28)$$

Disconnected diagrams include fig. 9.13. However, we have connected diagrams like fig. 9.14.

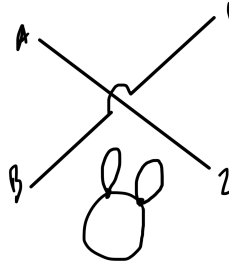


Figure 9.13: Disconnected third order interaction.

The loop in this diagram represents an interaction with “vacuum fluctuation”. Such an interaction is not relevant to scattering, and we may consider just the portion of the diagram that leaves off this vacuum fluctuation. This is what is meant by amputated. Amputated diagrams do not include such factors. Other example interactions that may also be amputated include fig. 9.15.

At the next order we can have fun interactions like that of fig. 9.16, which is not amputatable (it connects branches), and must be considered.

At the λ^2 order, the relevant diagrams are sketched in fig. 9.17. At this order $\phi^4(x)$, $\phi^4(y)$ each contribute a vertex with 4 edges.

Definition 9.4: Amputated

Omit anything that only effects input or output lines.

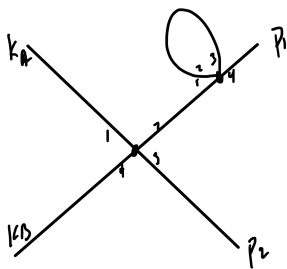


Figure 9.14: Connected diagram.

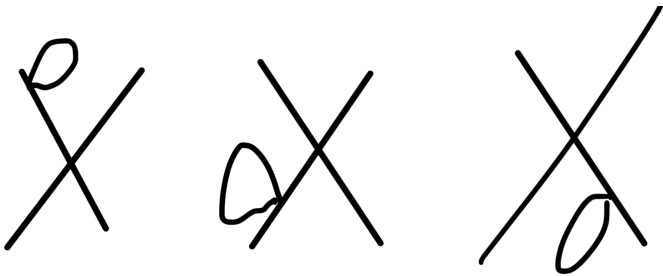


Figure 9.15: Other amputatable diagrams.

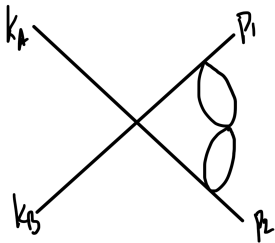


Figure 9.16: Fun interaction.

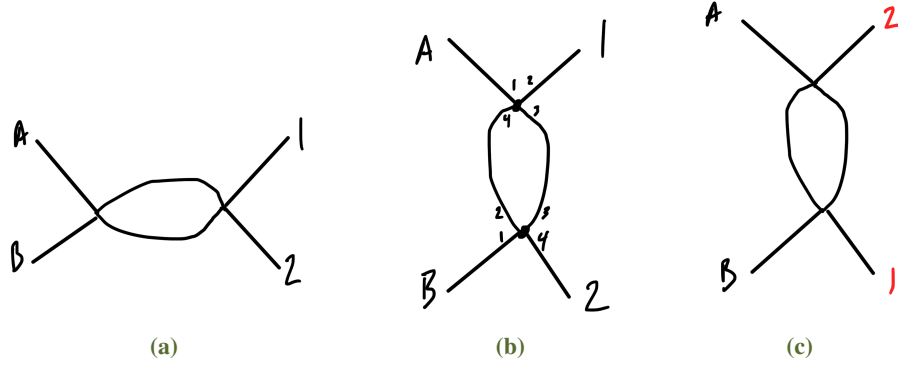


Figure 9.17: Second order connected amputated diagrams.

9.5 THE RECIPE.

The general transition amplitude for a $2 \rightarrow n$ event has the form

$$\langle p_1 \cdots p_n | k_A k_B \rangle = (2\pi)^4 \delta^{(4)} \left(\sum k_{\text{in}} - \sum p_{\text{out}} \right) iM(A, B \rightarrow 1, \cdots n). \quad (9.29)$$

Our recipe is

1. $iM = \sum$ of all connected amputated diagrams, lines and vertices.
2. to every internal line (not connected to input or final particle)
3. associate a propagator

$$\frac{i}{p^2 - m^2 - i\epsilon}, \quad (9.30)$$

where p is the 4-momentum of the line. External lines are $\equiv 1$.

4. Impose non-conservation with every vertex.
5. integrate $\int d^4 p / (2\pi)^4$ over all momenta not fixed.
6. symmetry factors
7. vertex: $(-i\lambda)$.

9.6 BACK TO OUR SCALAR THEORY.

Applying these rules to the diagram fig. 9.18, we get

$$-i\lambda = iM, \quad (9.31)$$

or

$$M = -\lambda. \quad (9.32)$$

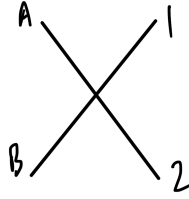


Figure 9.18: First order interaction.

For the second order diagrams The first diagram gives

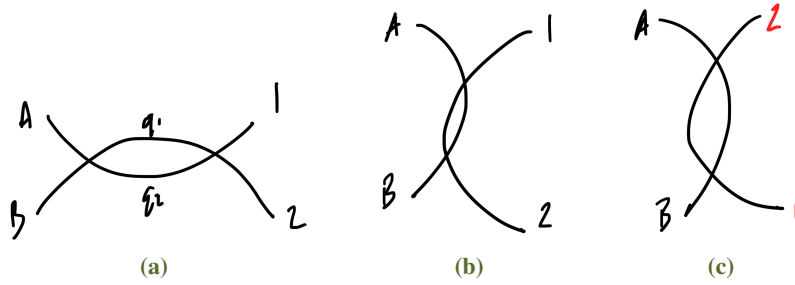


Figure 9.19: Second order diagrams.

$$(-i\lambda)^2 \frac{i}{q_1^2 - m^2 - i\epsilon} \frac{i}{q_2^2 - m^2 - i\epsilon}, \quad (9.33)$$

where $q_1 + q_2 = k_A + k_B$, so we can let $q_2 = k_A + k_B - q_1$, which gives

$$\int \frac{d^4 q_1}{(2\pi)^4} (-i\lambda)^2 \frac{i}{q_1^2 - m^2 - i\epsilon} \frac{i}{(k_A + k_B - q_1)^2 - m^2 - i\epsilon} \quad (9.34)$$

Calculating the symmetry coefficients is a counting game, illustrated roughly in fig. 9.20, where the $1/2$ factor was eliminated by the two choices, and the rest by factorial counting (4 ways to pick first, leaving 3 ways for the next choice, two for the next, until the last.) In the end we have a symmetry factor of $(4 \times 3) \times 2 \times (4 \times 3)$.

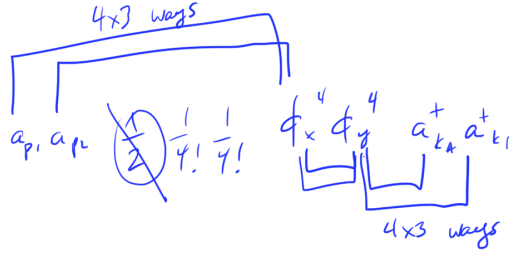


Figure 9.20: Symmetry coefficient counting.

9.7 REVIEW: S-MATRIX.

We defined an S -matrix

$$\langle f | S | i \rangle = S_{fi} = (2\pi)^4 \delta^{(4)} \left(\sum \left(p_i - \sum_{p_f} \right) \right) iM_{fi}, \quad (9.35)$$

where

$$iM_{fi} = \sum \text{all connected amputated Feynman diagrams.} \quad (9.36)$$

The matrix element $\langle f | S | i \rangle$ is the amplitude of the transition from the initial to the final state. In general this can get very complicated, as the number of terms grows factorially with the order.

We also talked about decays.

9.8 SCATTERING IN A SCALAR THEORY.

Suppose that we have a scalar theory with a light field Φ, M and a heavy field ϕ, m , where $m > 2M$. Perhaps we have an interaction with a z^2 symmetry so that the interaction potential is quadratic in Φ

$$V_{\text{int}} = \mu \phi \Phi \Phi. \quad (9.37)$$

We may have $\Phi\Phi \rightarrow \Phi\Phi$ scattering.

We will denote diagrams using a double line for ϕ and a single line for Φ , as sketched in fig. 9.21.

There are three possible diagrams:

The first we will call the s-channel, which has amplitude

$$\begin{aligned} A(\text{s-channel}) &\sim \frac{i}{p^2 - m^2 + i\epsilon} \\ &= \frac{i}{s - m^2 + i\epsilon}, \end{aligned} \quad (9.38)$$

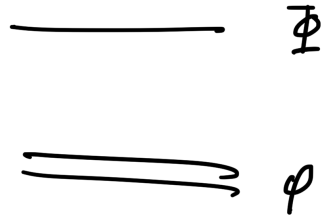


Figure 9.21: Particle line convention.

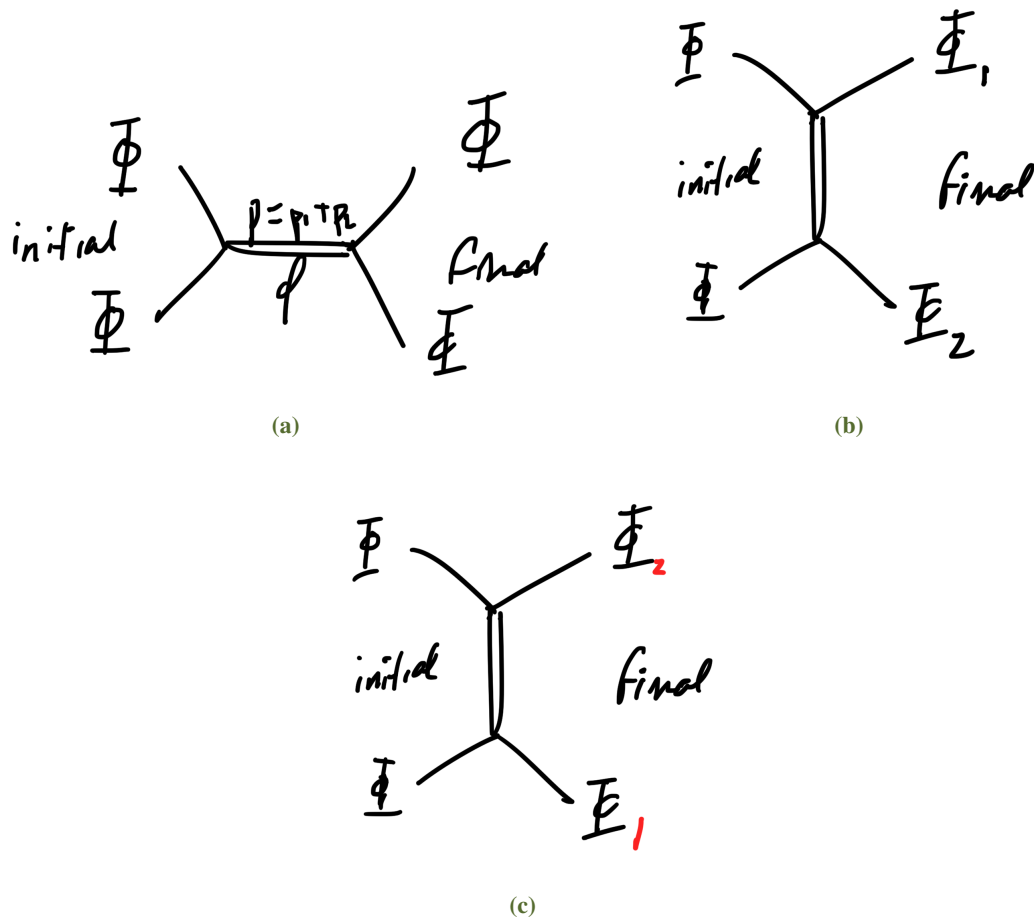


Figure 9.22: Possible diagrams.

where we designate the total squared four-momentum as

$$(p_1 + p_2)^2 = s. \quad (9.39)$$

In the centre of mass frame

$$\mathbf{p}_1 = -\mathbf{p}_2, \quad (9.40)$$

so

$$s = (p_1^0 + p_2^0)^2 = E_{\text{cm}}^2. \quad (9.41)$$

To the next order we have a diagram like fig. 9.23. and can have additional virtual particles

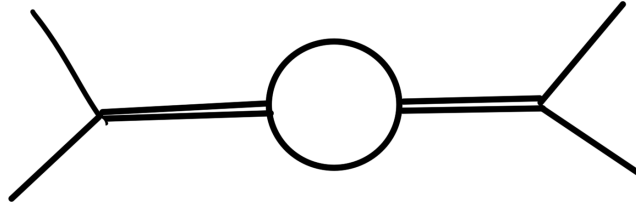


Figure 9.23: Higher order.

created, with diagrams like fig. 9.24.

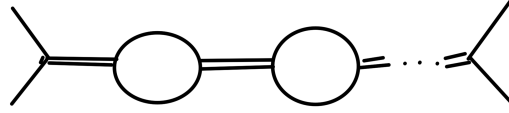


Figure 9.24: More virtual particles.

We will see (QFT II) that this leads to an addition imaginary $i\Gamma$ term in the propagator

$$\frac{i}{s - m^2 + i\epsilon} \rightarrow \frac{i}{s - m^2 - im\Gamma + i\epsilon}. \quad (9.42)$$

If we choose to zoom into the such a figure, as sketched in fig. 9.25, we find that it contains the interaction of interest for our diagram, so we can (looking forward to currently unknown material) know that our diagram also has such an imaginary $i\Gamma$ term in its propagator.

Assuming such a term, the squared amplitude becomes

$$\begin{aligned} \sigma|_{s \text{ near } m^2} &\sim |A_s|^2 \\ &\sim \frac{1}{(s - m^2)^2 + m^2\Gamma^2} \end{aligned} \quad (9.43)$$

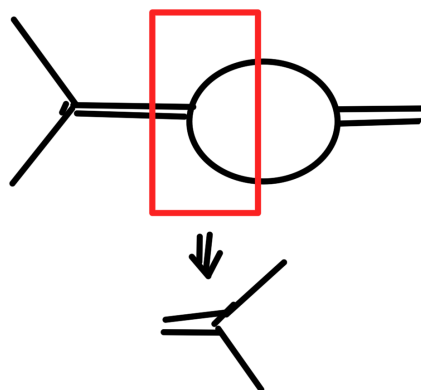


Figure 9.25: Zooming into the diagram for a higher order virtual particle creation event.

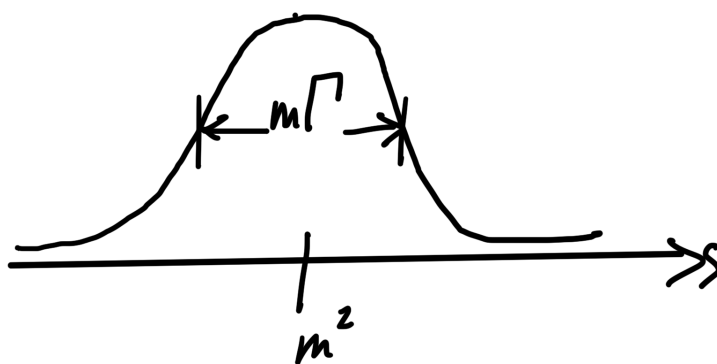


Figure 9.26: Resonance.

This is called a resonance, and is sketched in fig. 9.26.
Where are the poles of the modified propagator?

$$\frac{i}{s - m^2 - im\Gamma + i\epsilon} = \frac{i}{p_0^2 - \mathbf{p}^2 - m^2 - im\Gamma + i\epsilon} \quad (9.44)$$

The pole is found, neglecting $i\epsilon$, is found at

$$\begin{aligned} p_0 &= \sqrt{\omega_{\mathbf{p}}^2 + im\Gamma} \\ &= \omega_{\mathbf{p}} \sqrt{1 + \frac{im\Gamma}{\omega_{\mathbf{p}}^2}} \\ &\approx \omega_{\mathbf{p}} + \frac{im\Gamma}{2\omega_{\mathbf{p}}} \end{aligned} \quad (9.45)$$

9.9 DECAY RATES.

We have an initial state

$$|i\rangle = |k\rangle, \quad (9.46)$$

and final state

$$|f\rangle = |p_1^f, p_2^f \cdots p_n^f\rangle. \quad (9.47)$$

We defined decay rate as the ratio of the number of initial particles to the number of final particles.

The probability is proportional to

$$\begin{aligned} \rho &\sim |\langle f | S | i \rangle|^2 \\ &= (2\pi)^4 \delta^{(4)}(p_{\text{in}} - \sum p_f) (2\pi)^4 \delta^{(4)}(p_{\text{in}} - \sum p_f) \times |M_{fi}|^2, \end{aligned} \quad (9.48)$$

where the proportionality is because we will have to divide by all the norms of the final states¹.

Saying that $\delta(x)f(x) = \delta(x)f(0)$ we can set the argument of one of the delta functions to zero, which gives us a vacuum volume element factor

$$(2\pi)^4 \delta^{(4)}(p_{\text{in}} - \sum p_f) = (2\pi)^4 \delta^{(4)}(0) = V_3 T, \quad (9.49)$$

so

$$\frac{\text{probability for } i \rightarrow f}{\text{unit time}} \sim (2\pi)^4 \delta^{(4)}(p_{\text{in}} - \sum p_f) V_3 \times |M_{fi}|^2, \quad (9.50)$$

¹ Required for the probability to be no greater than one.

For the norms, we use the relativistic normalization

$$\langle k|p\rangle = (2\pi)^3 2\omega_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{k}) \quad (9.51)$$

and our volume element interpretation of $\delta^{(3)}(0)^2$, which is

$$\langle p|p\rangle = 2\omega_{\mathbf{p}} \int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} \Big|_{\mathbf{x}=0} = 2\omega_{\mathbf{p}} V_3. \quad (9.52)$$

We now have the full expression for the probability per unit time

$$\frac{\text{probability for } i \rightarrow f}{\text{unit time}} = \frac{(2\pi)^4 \delta^{(4)}(p_{\text{in}} - \sum p_f) |M_{fi}|^2 V_3}{2\omega_{\mathbf{k}} V_3 2\omega_{\mathbf{p}_1} \cdots 2\omega_{\mathbf{p}_n} V_3^n} \quad (9.53)$$

In terms of number of states in a small momentum space volume. If we multiply the number of final states with $p_i^f \in (p_i^f, p_i^f + dp_i^f)$ for a particle in a box

$$p_x = \frac{2\pi n_x}{L} \quad (9.54)$$

$$\Delta p_x = \frac{2\pi}{L} \Delta n_x \quad (9.55)$$

$$\Delta n_x = \frac{L}{2\pi} \Delta p_x \quad (9.56)$$

and

$$\Delta n_x \Delta n_y \Delta n_z = \frac{V_3}{(2\pi)^3} \Delta p_x \Delta p_y \Delta p_z \quad (9.57)$$

$$\begin{aligned} \Gamma &= \frac{\text{number of events } i \rightarrow f}{\text{unit time}} \\ &= \prod_f \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}^f}} \frac{(2\pi)^4 \delta^{(4)}(k - \sum_f p^f) |M_{fi}|^2}{2\omega_{\mathbf{k}}} \end{aligned} \quad (9.58)$$

Note that everything here is Lorentz invariant except for the denominator of the second term ($2\omega_{\mathbf{k}}$). This is a well known result (the decay rate changes in different frames).

2 Originally seen in eq. (4.76).

9.10 CROSS SECTION.

For $2 \rightarrow$ many transitions

$$\begin{aligned} & \frac{\text{probability } i \rightarrow f}{\text{unit time}} \times \left(\text{number of final states with } p_f \in (p_f, p_f + dp_f) \right) \\ &= \frac{(2\pi)^4 \delta^{(4)}(\sum p_i - \sum_f p_f) |M_{fi}|^2 V_3}{2\omega_{\mathbf{k}_1} V_3 2\omega_{\mathbf{k}_2} V_3} \prod_f \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}_f}} \end{aligned} \quad (9.59)$$

We need to divide by the flux to obtain the cross section.

In the CM frame, as sketched in fig. 9.27, the current is

$$\mathbf{j} = n\mathbf{v}_1 - n\mathbf{v}_2, \quad (9.60)$$

so if the density is

$$n = \frac{1}{V_3}, \quad (9.61)$$

(one particle in V_3), then

$$\mathbf{j} = \frac{\mathbf{v}_1 - \mathbf{v}_2}{V_3}. \quad (9.62)$$



Figure 9.27: Centre of mass frame.

$$\sigma = \frac{(2\pi)^4 \delta^{(4)}(\sum p_i - \sum_f p_f) |M_{fi}|^2}{2\omega_{\mathbf{k}_1} 2\omega_{\mathbf{k}_2} \|\mathbf{v}_1 - \mathbf{v}_2\|} \prod_f \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}_f}} \quad (9.63)$$

This is where [14] stops.

There is, however, a nice Lorentz invariant generalization

$$j = \frac{1}{V_3 \omega_{k_A} \omega_{k_B}} \sqrt{(k_A k_B)^2 - m_A^2 m_B^2} \quad (9.64)$$

(Claim: DIY)

$$\begin{aligned}
j|_{CM} &= \frac{1}{V_3} \left(\frac{\|\mathbf{k}_A\|}{\omega_{k_A}} + \frac{\|\mathbf{k}_B\|}{\omega_{k_B}} \right) \\
&= \frac{1}{V_3} (\|\mathbf{v}_A\| + \|\mathbf{v}_B\|) \\
&= \frac{1}{V_3} \|\mathbf{v}_1 - \mathbf{v}_2\|
\end{aligned} \tag{9.65}$$

$$\sigma = \frac{(2\pi)^4 \delta^{(4)}(\sum p_i - \sum_f p^f) |M_{fi}|^2}{4 \sqrt{(k_A k_B)^2 - m_A^2 m_B^2}} \prod_f \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}^f}}. \tag{9.66}$$

9.11 MORE ON CROSS SECTION.

$$d\sigma(AB \rightarrow 1 \cdots n) = \frac{1}{4 \sqrt{(k_A k_B)^2 - m_A^2 m_B^2}} |M_{fi}|^2 \times (2\pi)^4 \delta^{(4)}(\sum p_i - \sum_f p^f) \prod_f \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}^f}}. \tag{9.67}$$

For two particles, the particle data book this factor has the identification

$$d(LIPS)_2 = (2\pi)^4 \delta^{(4)}(\sum p_i - \sum_f p^f) \prod_f \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}^f}}, \tag{9.68}$$

$$\sigma = \frac{d(LIPS)_2 |M_{fi}|^2}{4 \sqrt{(k_A k_B)^2 - m_A^2 m_B^2}}. \tag{9.69}$$

Digression

Digression: One to three muon decay I can't recall what prompted this, but an example of muon (a negative charged "heavy electron") decay was given, sketched in fig. 9.28. In such decay lepton number and charge both have to be conserved. This is why such a decay must have a mu-neutrino (ν_μ), as well as an antineutrino ($\bar{\nu}_e$) along with the electron (e^-).

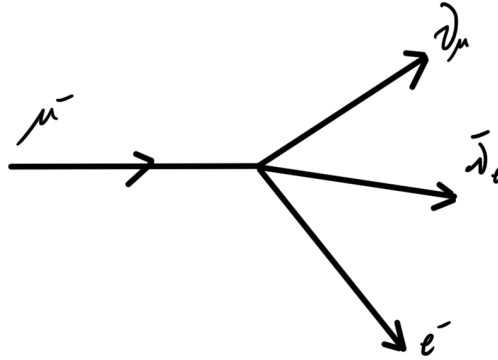


Figure 9.28: Muon decay: One to three scattering.

Example: $\phi\phi \rightarrow \phi\phi$: Let's calculate the flux factor for the $2 \rightarrow 2$ scattering sketched in fig. 9.29 from the CM frame. Our four-momenta are

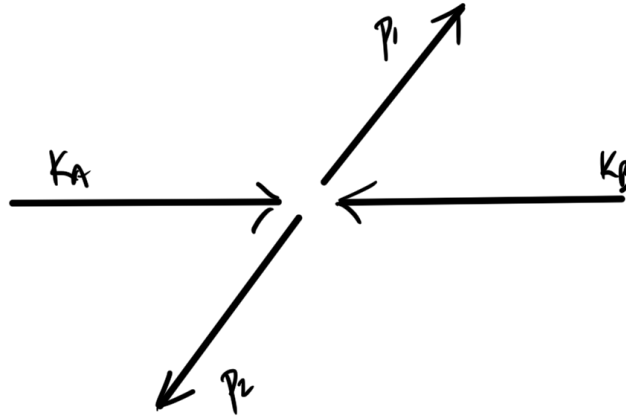


Figure 9.29: Two to two scattering in CM frame.

$$k_A = \left(\frac{\sqrt{s}}{2}, -\mathbf{k} \right) \quad (9.70)$$

$$k_B = \left(\frac{\sqrt{s}}{2}, +\mathbf{k} \right),$$

where

$$\sqrt{s} = E_A + E_B, \quad (9.71)$$

and s is known as a Mandelstam variable (see for example: [17]). The particles are each on shell, so

$$\begin{aligned} k_A^2 &= \frac{s}{4} - \mathbf{k}^2 = m^2 \\ k_B^2 &= \frac{s}{4} - \mathbf{k}^2 = m^2. \end{aligned} \quad (9.72)$$

The flux factor is

$$\begin{aligned} 4 \sqrt{(k_A k_B)^2 - m_A^2 m_B^2} &= 4 \sqrt{\left(\frac{s}{4} + \mathbf{k}^2\right)^2 - (m^2)^2} \\ &= 4 \sqrt{\left(\frac{s}{4} + \mathbf{k}^2 - m^2\right)\left(\frac{s}{4} + \mathbf{k}^2 + m^2\right)}. \end{aligned} \quad (9.73)$$

Using eq. (9.72), gives

$$4 \sqrt{(k_A k_B)^2 - m_A^2 m_B^2} = 4 \sqrt{(2\mathbf{k}^2) \frac{s}{2}} = 4 \|\mathbf{k}\| \sqrt{s}. \quad (9.74)$$

9.12 $d(LIPS)_2$.

In the CM frame the delta function simplifies and we have

$$\begin{aligned} d(LIPS)_2 &= \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2\omega_1 2\omega_2} 2\pi (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2) \delta(2\omega_1 - \sqrt{s}) \\ &= \frac{d^3 p_1}{(2\pi)^3} \frac{1}{4\omega_1^2} 2\pi \delta(2\omega_1 - \sqrt{s}) \\ &= \frac{d^3 p_1}{(2\pi)^3} \frac{1}{4\omega_1^2} 2\pi \delta(2\sqrt{\mathbf{p}_1^2 + m^2} - \sqrt{s}) \end{aligned} \quad (9.75)$$

$$p_1 = \left(\frac{\sqrt{s}}{2}, \mathbf{p}_1 \right) \quad (9.76)$$

The square of this four-momentum is

$$p_1^2 = m^2 = \frac{s}{4} - \mathbf{p}_1^2, \quad (9.77)$$

so

$$\mathbf{p}_1^2 = \frac{s}{4} - m^2 \quad (9.78)$$

Using the delta function identity

$$\delta(f(x)) = \frac{\delta(x^*)}{f'(x^*)} \Big|_{f(x^*)=0}, \quad (9.79)$$

and letting $d^3 p_1 = d\Omega p^2 dp$, $p = \|\mathbf{p}_1\|$, and $f(p) = 2\sqrt{p^2 + m^2} = \sqrt{s}$ we have a zero at

$$x^* = p - \sqrt{s/4 - m^2}, \quad (9.80)$$

$$p^2 = \frac{s}{4} - m^2. \quad (9.81)$$

$$\begin{aligned} f'(x^*) &= \frac{d}{dp} 2\sqrt{p^2 + m^2} \\ &= 2 \frac{1}{2} 2 \frac{p}{\omega_1} \\ &= \frac{2p}{\omega_1}. \end{aligned} \quad (9.82)$$

$$\begin{aligned} d(LIPS)_2 &= \frac{d^3 p_1}{(2\pi)^3} \frac{1}{4\omega_1^2} 2\pi \delta(p - \sqrt{s/4 - m^2}) \\ &= \frac{d^2 \Omega p^2 dp}{(2\pi)^2 4\omega_1^2} \frac{\delta(p - \sqrt{s/4 - m^2})}{2p/\omega_1} \\ &= \frac{d^2 \Omega p dp}{(2\pi)^2 8\omega_1} \delta(p - \sqrt{s/4 - m^2}), \end{aligned} \quad (9.83)$$

but $\int dx x \delta(x - x^*) = x^*$, so

$$\begin{aligned} d(LIPS)_2 &= \frac{d^2 \Omega p}{(2\pi)^2 8\omega_1} \\ &= \frac{d^2 \Omega p}{16\pi^2 \sqrt{s}}, \end{aligned} \quad (9.84)$$

since $\omega_1 = \sqrt{s}/2$.

Plugging eq. (9.74) and eq. (9.84) into eq. (9.69) we have

$$\begin{aligned} \frac{d\sigma}{d^2 \Omega} &= \frac{d(LIPS)_2}{d^2 \Omega} |M_{fi}|^2 \frac{1}{4 \sqrt{(k_A k_B)^2 - m_A^2 m_B^2}} \\ &= \frac{p}{16\pi^2 \sqrt{s}} |-i\lambda|^2 \frac{1}{4 \|\mathbf{k}\| \sqrt{s}} \\ &= \frac{\lambda^2}{64\pi^2 s}. \end{aligned} \quad (9.85)$$

Since

$$\int d^2\Omega = 4\pi, \quad (9.86)$$

the total cross section is

$$\begin{aligned} \sigma_{\text{total}} &= \frac{\lambda^2}{s\pi^2} \frac{4\pi}{264/16} \\ &= \frac{\lambda^2}{32s}. \end{aligned} \quad (9.87)$$

There was a counting adjustment made here that I didn't catch.

9.13 PROBLEMS.

Exercise 9.1 The " $h \rightarrow WW, ZZ$ " Higgs-decay width. (2018 HW4.II)

From the $SU(2)_L \times SU(2)_R$ model of Homework 2—really, the Higgs Lagrangian of the Standard Model, find the coupling of the h -particle (the Higgs boson) to the ϕ^a particles (these are now Goldstone bosons, in the electroweak theory, they become the longitudinal components of the W and Z particles). Canonically normalizing h and ϕ^a , this coupling has the form

$$\mathcal{L}_{\text{int}} = \text{const. } h \partial_\mu \phi^a \partial^\mu \phi^a. \quad (9.88)$$

- Determine the value of *const.* for canonically normalized h and ϕ^a .
- Use this coupling to compute the width $\Gamma(h \rightarrow \phi^3 \phi^3)$ of the Higgs particle to decay to two longitudinal (say) Z -bosons (hence the index 3).
- Plug in some numbers. Use the fact that the vacuum expectation value $|m|/\sqrt{\lambda} = 246$ GeV and the fact that $m_h = 125$ GeV to get a number for the lifetime. Compare to the total width of the Higgs from <http://pdg.lbl.gov/2012/reviews/rpp2012-rev-higgs-boson.pdf>, see figure 5 there, as well to the partial width to WW given in Figure 4 there.
- At the same time, determine the values of $|m|$ and λ separately. Is $\lambda \ll 1$ (i.e. perturbative)?

Notice that this calculation would have been physically relevant had the Higgs been heavy, $m_h \gg m_W \sim 100$ GeV. This is because the $h \rightarrow WW$ decay then is dominated (in this limit) by the decay into the longitudinal component, which is really the Goldstone boson field ϕ^a (in this limit, the result is independent of the gauge couplings $g_{1,2}$ of the Standard Model). Nonetheless, having some real numbers in this class is good.

PROBLEM SET RELATED MATERIAL REDACTED IN THIS DOCUMENT.PLEASE
FEEL FREE TO EMAIL ME FOR THE FULL VERSION IF YOU AREN'T TAKING PHY2403.

END-REDACTION

This problem has:

- Now, to the concrete stuff:

- For energies of the ϕ^a quanta greater than the mass of the W and Z bosons (roughly 100 GeV), this scattering amplitude via h -exchange can be shown [you got to believe me here] to be the same as the scattering of *longitudinal* W, Z -bosons.

$$M(\phi^1\phi^1 \rightarrow \phi^3\phi^3)|_{h\text{-exchange}} = const. \frac{s^2}{v^2(s-m_h^2)}, \quad (9.89)$$

Exercise 9.3 “Radiation” by accelerated source and “IR catastrophe” (2018 HW3.III)

This is a baby problem having to do with radiation of scalar particles. (As we will not have too much time to study the radiation of electromagnetic fields this term, it is a good opportunity.) Consider the coupling of a classical particle to a scalar field (remember Homework 1, Problem 1, where a similar coupling to the electromagnetic field was considered):

$$S_{int} = e \int_{\text{worldline}} ds \phi(x(s)), \quad (9.91)$$

where $x(s)$ is the worldline of the particle and e is its scalar charge (what is its dimension?). The coupling (9.91) corresponds to a “current” $j(x)$ coupling to ϕ as in Problem II. above:

$$S_{int} = e \int_{\text{worldline}} ds \phi(x(s)) = \int d^4x j(x) \phi(x), \text{ where } j(x) = e \int_{\text{worldline}} ds \delta^{(4)}(x - x(s)), \quad (9.92)$$

is the current.

- a. Consider a particle of mass M , whose worldline is given by:

$$x^\mu(s) = \frac{p_i^\mu}{M} s, \text{ for } s < 0 \text{ and } x^\mu(s) = \frac{p_f^\mu}{M} s, \text{ for } s > 0, \quad (9.93)$$

where p_i^μ and p_f^μ are the initial and final four-momenta of the particle (both obeying $p^\mu p_\mu = M^2$, with $p^0 > 0$, of course). The physical meaning of this trajectory is that the particle undergoes acceleration at $x^0 = 0$, suddenly changing its four-momentum from p_i to p_f . Show that the Fourier transform of the current, as defined in (8.124) above, is given by:

$$\tilde{j}(p) = \frac{ieM}{p \cdot p_f} - \frac{ieM}{p \cdot p_i} \quad (9.94)$$

To make the TA’s life (and yours) easier, in getting (9.94), consider without loss of generality, trajectories with $p_i = (M, 0, 0, 0)$ and $p_f = (\sqrt{M^2 + q^2}, q, 0, 0)$.⁴

- b. Now study the expression for the average number of particles produced, λ , or $\langle N \rangle$, of eq. (8.124), as well as the average energy $\langle E \rangle$, which you can easily come up with, from (8.124). From now on, consider the case where the mass of the produced particles (ϕ -quanta) is zero. This has two advantages: simplifications in the various formulae as well as giving us the feeling that we are actually looking at something close to radiation of photons.

Show that the integrals over the momenta of the emitted “photons” in $\langle N \rangle$ and $\langle E \rangle$ diverge at large p .

⁴ Recall the “half-delta function” integrals from Homework 2, Problem 1 and ignore the $i\epsilon$ factors which should be present in the denominators in (9.94) as they will not be important for what follows.

FERMIONS, AND SPINORS.

10.1 FERMIONS: \mathbb{R}^3 ROTATIONS.

Given a real vector \mathbf{x} and the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (10.1)$$

We may form a Pauli matrix representation of a vector

$$\boldsymbol{\sigma} \cdot \mathbf{x} = \begin{bmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{bmatrix}, \quad (10.2)$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$. This matrix, like the Pauli matrices, is a 2×2 Hermitian traceless matrix. We find that the determinant is

$$\begin{aligned} \det(\boldsymbol{\sigma} \cdot \mathbf{x}) &= -(x^3)^2 - (x^1)^2 - (x^2)^2 \\ &= -\mathbf{x}^2. \end{aligned} \quad (10.3)$$

We may form

$$U(\boldsymbol{\sigma} \cdot \mathbf{x})U^\dagger, \quad (10.4)$$

where U is a unitary 2×2 unit determinant matrix, satisfying

$$\begin{aligned} U^\dagger U &= 1 \\ \det U &= 1. \end{aligned} \quad (10.5)$$

Further

$$\begin{aligned} \det(U\boldsymbol{\sigma} \cdot \mathbf{x})U^\dagger &= \det U \det(\boldsymbol{\sigma} \cdot \mathbf{x}) \det U^\dagger \\ &= \det(\boldsymbol{\sigma} \cdot \mathbf{x}). \end{aligned} \quad (10.6)$$

Moral: $U(\boldsymbol{\sigma} \cdot \mathbf{x})U^\dagger = \boldsymbol{\sigma} \cdot \mathbf{x}'$, where \mathbf{x}' has the same length of \mathbf{x} .

We may use this to represent an arbitrary rotation

$$U(\boldsymbol{\sigma} \cdot \mathbf{x})U^\dagger = R^i_j x^j \sigma^i \quad (10.7)$$

We say that $U \in SU(2)$ and $R \in SU(3)$, and $SU(2)$ is called the “universal cover of $SO(3)$ ”.

Pauli figured out that, in non-relativistic QM, that this type of transformation also applies to (spin) wave functions (spinors)

$$\Psi(\mathbf{x}) \rightarrow \Psi'(\mathbf{x}') = U\Psi(\mathbf{x}) \quad (10.8)$$

where

$$\mathbf{x} \rightarrow \mathbf{x}' = R\mathbf{x}, \quad (10.9)$$

and $R^T R = 1$. Here Ψ is a two element vector

$$\Psi(\mathbf{x}) = \begin{bmatrix} \Psi_{\uparrow}(\mathbf{x}) \\ \Psi_{\downarrow}(\mathbf{x}) \end{bmatrix}, \quad (10.10)$$

so the transformation should be thought of as a matrix operation

$$\begin{bmatrix} \Psi_{\uparrow}(\mathbf{x}) \\ \Psi_{\downarrow}(\mathbf{x}) \end{bmatrix} \rightarrow \begin{bmatrix} \Psi'_{\uparrow}(\mathbf{x}') \\ \Psi'_{\downarrow}(\mathbf{x}') \end{bmatrix} = U \begin{bmatrix} \Psi_{\uparrow}(\mathbf{x}) \\ \Psi_{\downarrow}(\mathbf{x}) \end{bmatrix}. \quad (10.11)$$

Having seen such representations and their $SU(2)$ transformations in NRQM, we want to know what the relativistic generalization is.

10.2 LORENTZ GROUP.

Let

$$\begin{aligned} (x^0, \mathbf{x}) &= x^0 \sigma^0 - \boldsymbol{\sigma} \cdot \mathbf{x} \\ &= \begin{bmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{bmatrix}. \end{aligned} \quad (10.12)$$

This has determinant

$$\begin{aligned} \det(x^0, \mathbf{x}) &= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \\ &= x^\mu x_\mu. \end{aligned} \quad (10.13)$$

We therefore identify (x^0, \mathbf{x}) as a four vector

$$\begin{aligned} (x^0, \mathbf{x}) &= x_\mu \sigma^\mu \\ &= x^\mu \sigma^\nu g_{\mu\nu}. \end{aligned} \quad (10.14)$$

We say that $SL(2, \mathbb{C})$ is a double cover of $SO(1, 3)$.

Note that the matrix U can be built explicitly. For example, it may be built up using Euler angles as sketched in fig. 10.1. or algebraically

$$U = e^{i\psi\sigma_3/2} e^{i\theta\sigma_1/2} e^{i\phi\sigma_3/2}. \quad (10.15)$$

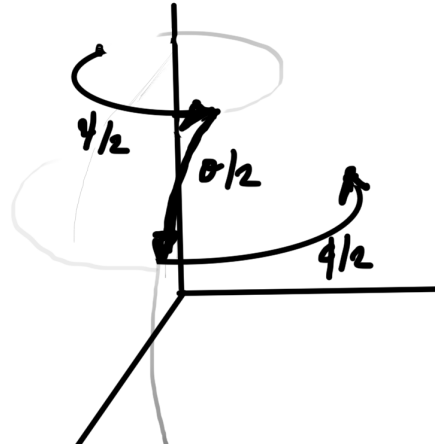


Figure 10.1: Euler angle rotations.

10.3 WEYL SPINORS.

We will see that there is generalization of Pauli spinors, called Weyl spinors, but we will have to introduce 4 component objects.

We'd like to argue that there is a correspondence (also $2 \rightarrow 1$) between $SL(2, \mathbb{C}) \rightarrow SO(1, 3)$. Here:

- S : special
- L : linear
- 2 : 2×2
- \mathbb{C} : complex.

and we say that $M \in SL(2, \mathbb{C})$ if $\det M = 1$, where M is a complex 2×2 , but not necessarily unitary. The $SU(2)$ group is a subset of $SL(2, \mathbb{C})$. In this representation $SU(2)$ matrices are $SL(2, \mathbb{C})$ matrices, but not necessarily the opposite.

We introduce a special notation for the identity matrix

$$\sigma^0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (10.16)$$

and can now form four vectors in a matrix representation

$$\begin{aligned}
 x \cdot \sigma &\equiv x^\mu \sigma_\mu \\
 &= x^0 \sigma^0 - \boldsymbol{\sigma} \cdot \mathbf{x} \\
 &= \begin{bmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{bmatrix}.
 \end{aligned} \tag{10.17}$$

Such 2×2 matrices are Hermitian. Notice that the space of 2×2 Hermitian matrices is 4 dimensional.

We found that

$$\det(x^\mu \sigma_\mu) = (x^0)^2 - \mathbf{x}^2. \tag{10.18}$$

The transformation

$$x^\mu \sigma_\mu \rightarrow M (x^\mu \sigma_\mu) M^\dagger, \tag{10.19}$$

maps 2×2 Hermitian matrices to 2×2 Hermitian matrices using a unit determinant transformation M . Note that M is not unitary, as it is an arbitrary (Hermitian) matrix. In particular $MM^\dagger \neq 1$! Also note that the determinant of the transformed object is

$$\det(M (x^\mu \sigma_\mu) M^\dagger) = 1 \times \det(x^\mu \sigma_\mu) \times 1, \tag{10.20}$$

since $\det M = 1$, so that we see that the Lorentz invariant length is preserved by such a transformation. This can be expressed as

$$x \cdot \sigma \rightarrow M x \cdot \sigma M^\dagger = x' \cdot \sigma, \tag{10.21}$$

where $(x')^2 = x^2$.

Motivated by this $SL(2, \mathbb{C}) \rightarrow SO(1, 3)$ correspondence, postulate that we study two component objects

$$U(x) = \begin{bmatrix} U_1(x) \\ U_2(x) \end{bmatrix}, \tag{10.22}$$

where $x = (x^0, x^1, x^2, x^3)$ is a four-vector, and assume that such objects transform as follows in $SO(1, 3)$

$$\begin{aligned}
 U(x) &\rightarrow U'(x') = M^\dagger U(x) \\
 x^\mu &\rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu,
 \end{aligned} \tag{10.23}$$

where M^\dagger is the one giving rise to Λ . To understand what is meant by “giving rise to”, consider

$$\begin{aligned}
 M x^\mu \sigma_\mu M^\dagger &= x'^\nu \sigma_\nu \\
 &= \sigma_\nu \Lambda^\nu{}_\mu x^\mu,
 \end{aligned} \tag{10.24}$$

and this holds for all x^μ , we must have

$$M\sigma_\mu M^\dagger = \sigma_\nu \Lambda^\nu_\mu. \quad (10.25)$$

Theorem 10.1: Transformation of $U^\dagger(x)\sigma_\mu U(x)$

$U^\dagger(x)\sigma_\mu U(x)$ transforms as a four vector.

Proof:

$$\begin{aligned} U^\dagger(x)\sigma_\mu U(x) &\rightarrow U'^\dagger(x')\sigma_\mu U'(x') \\ &= (M^\dagger U(x))^\dagger \sigma_\mu M^\dagger U(x) \\ &= U^\dagger(x) (M\sigma_\mu M^\dagger) U(x) \\ &= U^\dagger(x) \sigma_\nu U(x) \Lambda^\nu_\mu \end{aligned} \quad (10.26)$$

so we find that $U^\dagger(x)\sigma_\mu U(x)$ transforms as a four vector as claimed.

Theorem 10.2: Transformation of partials.

The four-gradient coordinates transform as a four vector

$$(\partial_\mu)' = (\Lambda^{-1})^\sigma_\mu \partial_\sigma.$$

Proof¹: Inverting the transformation relation

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \quad (10.27)$$

gives

$$x^\sigma = (\Lambda^{-1})^\sigma_\mu \Lambda^\mu_\nu x^\nu = (\Lambda^{-1})^\sigma_\mu x'^\mu, \quad (10.28)$$

so

$$\begin{aligned} \partial_\mu &\rightarrow (\partial_\mu)' \\ &= \frac{\partial}{\partial x'^\mu} \\ &= \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial}{\partial x^\sigma} \\ &= (\Lambda^{-1})^\sigma_\mu \frac{\partial}{\partial x^\sigma} \\ &= (\Lambda^{-1})^\sigma_\mu \partial_\sigma. \end{aligned} \quad (10.29)$$

¹ In class we proved this by considering the transformation properties of a direction derivative $dx^\mu \cdot \partial_\mu$, but that isn't the method that seems most intuitive to me.

Theorem 10.3: Transformation of $U^\dagger \sigma^\mu \partial_\mu U$

$U^\dagger \sigma^\mu \partial_\mu U$ transforms as a four vector.

Proof:

$$\begin{aligned}
 U^\dagger(x) \sigma^\mu \frac{\partial}{\partial x^\mu} U(x) &\rightarrow U'^\dagger(x') \sigma^\mu \frac{\partial}{\partial x'^\mu} U'(x') \\
 &= \Lambda^\mu{}_\nu U^\dagger(x) \sigma^\nu \frac{\partial}{\partial x'^\mu} (\Lambda^{-1})^{\mu'}{}_\mu U(x) \\
 &= U^\dagger(x) \sigma^\nu \frac{\partial}{\partial x'^\mu} \delta^{\mu'}{}_\nu U(x) \\
 &= U^\dagger(x) \sigma^\nu \frac{\partial}{\partial x^\nu} U(x)
 \end{aligned} \tag{10.30}$$

We can now define

Definition 10.1: Weyl action (name?)

We may construct the following Lorentz invariant action

$$S_{\text{Weyl}} = \int d^4x i U^\dagger(x) \sigma^\mu \partial_\mu U(x),$$

where $U(x)$ is a Weyl spinor.

The i factor here is so that the action is real. This can be seen by noting that $(i\sigma^\mu)^\dagger = -i\sigma^\mu$ and integrating the Hermitian conjugate by parts

$$(i\sigma^0)^\dagger = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}^\dagger = -i\sigma^0 \tag{10.31a}$$

$$(i\sigma^1)^\dagger = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}^\dagger = -i\sigma^1 \tag{10.31b}$$

$$(i\sigma^2)^\dagger = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^\dagger = -i\sigma^2 \tag{10.31c}$$

$$(i\sigma^3)^\dagger = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^\dagger = -i\sigma^3 \quad (10.31d)$$

$$\begin{aligned} S_{\text{Weyl}}^\dagger &= \int d^4x \partial_\mu U^\dagger(x) (i\sigma^\mu)^\dagger U(x) \\ &= - \int d^4x \partial_\mu U^\dagger(x) i\sigma^\mu U(x) \\ &= - \int d^4x \partial_\mu (U^\dagger(x) i\sigma^\mu U(x)) + \int d^4x U^\dagger(x) i\sigma^\mu \partial_\mu U(x) \\ &= \int d^4x U^\dagger(x) i\sigma^\mu \partial_\mu U(x) \\ &= S_{\text{Weyl}}, \end{aligned} \quad (10.32)$$

where it was assumed that any boundary terms vanish.

Theorem 10.4: Weyl equation.

Variation of the action definition 10.1 gives rise to the equations of motion

$$\sigma^\mu \frac{\partial}{\partial x^\mu} U = 0$$

which is called the Weyl equation.

Proof:

$$\begin{aligned} \delta S &= i \int d^4x (\delta U^\dagger \sigma^\mu \partial_\mu U + U^\dagger \sigma^\mu \partial_\mu \delta U) \\ &= i \int d^4x (\delta U^\dagger \sigma^\mu \partial_\mu U + \partial_\mu (U^\dagger \sigma^\mu \delta U) - (\partial_\mu U^\dagger) \sigma^\mu \delta U) \\ &= i \int d^4x (\delta U^\dagger (\sigma^\mu \partial_\mu U) - ((\partial_\mu U^\dagger) \sigma^\mu) \delta U) \\ &= \int d^4x (\delta U^\dagger (i\sigma^\mu \partial_\mu U) + (\delta U^\dagger (i\sigma^\mu \partial_\mu U))^\dagger). \end{aligned} \quad (10.33)$$

Requiring this to vanish for all variations δU^\dagger proves the result.

Written out explicitly in matrix form, the Weyl equation is

$$\begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0, \quad (10.34)$$

or

$$(\partial_0 + \partial_3)U_1 + (\partial_1 - i\partial_2)U_2 = 0 \quad (10.35a)$$

$$(\partial_1 + i\partial_2)U_1 + (\partial_0 - \partial_3)U_2 = 0. \quad (10.35b)$$

Theorem 10.5: Weyl equation relation to the massless Klein-Gordon equation.

The Weyl equation is equivalent to a set of massless Klein-Gordon equations.

$$\partial_\mu \partial^\mu U_k = 0,$$

for $k = 1, 2$.

Proof:

Multiplying eq. (10.35a) by $\partial_1 + i\partial_2$ gives

$$\begin{aligned} (\partial_1 + i\partial_2) \left((\partial_0 + \partial_3)U_1 + (\partial_1 - i\partial_2)U_2 \right) &= (\partial_0 + \partial_3)(\partial_1 + i\partial_2)U_1 + (\partial_1 + i\partial_2)(\partial_1 - i\partial_2)U_2 \\ &= -(\partial_0 + \partial_3)(\partial_0 - \partial_3)U_2 + (\partial_1 + i\partial_2)(\partial_1 - i\partial_2)U_2 \\ &= (-\partial_{00} + \partial_{33} + \partial_{11} + \partial_{22})U_2 \\ &= (-\partial_0^2 - \partial_3^2 - \partial_1^2 - \partial_2^2)U_2 \\ &= -\partial_\mu \partial^\mu U_2. \end{aligned} \quad (10.36)$$

Similarly, multiplying eq. (10.35b) by $\partial_1 - i\partial_2$ we find

$$\begin{aligned} 0 &= (\partial_1 - i\partial_2) \left((\partial_1 + i\partial_2)U_1 + (\partial_0 - \partial_3)U_2 \right) \\ &= (\partial_{11} + \partial_{22})U_1 + (\partial_0 - \partial_3) \left(\underbrace{(\partial_1 - i\partial_2)U_2}_{-(\partial_0 + \partial_3)U_1} \right) \\ &= (\partial_{11} + \partial_{22} - \partial_{00} + \partial_{33})U_1 \\ &= -\partial_\mu \partial^\mu U_1. \end{aligned} \quad (10.37)$$

Because S_{Weyl} results in a massless Klein-Gordon equation, this is no good for electrons, and we have to look for a different action.

Claim: $U^T \sigma_2 U$ is the only bilinear Lorentz invariant that we can add to the action.

An action like:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} m U^T \sigma_2 U + \frac{1}{2} m^* U^\dagger \sigma_2 (U^\dagger)^T, \quad (10.38)$$

may exist in nature (we don't know), and are called Majorana neutrino masses. The problem with such a Lagrangian density is that it breaks $U(1)$ symmetry. In particular $U \rightarrow e^{i\alpha} U$ symmetry of the kinetic term. This means that the particle associated with such a Lagrangian cannot be charged.

Recall that we introduced electromagnetic potentials into NRQM with

$$i \hbar \frac{\partial}{\partial t} \Psi = \frac{1}{2m} (\nabla - e\mathbf{A})^2 \Psi \quad (10.39)$$

which is a gauge transformation. We'd like to have this capability.

What we can do instead and maintain $U(1)$ symmetries, is to introduce two U 's, like

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} m U_1^T \sigma_2 U_2 + \frac{1}{2} m^* U_2^\dagger \sigma_2 (U_1^\dagger)^T \quad (10.40)$$

What we are really doing is assembling a four component spinor out of the two U 's.

10.4 LORENTZ SYMMETRY.

We want to examine the Lorentz invariance of $U^T \sigma_2 U$, but need an intermediate result first.

Lemma 10.1: Transpose of Pauli vector representation

For any $\mathbf{x} \in \mathbb{R}^3$

$$(\boldsymbol{\sigma} \cdot \mathbf{x})^T = -\sigma^2 (\boldsymbol{\sigma} \cdot \mathbf{x}) \sigma^2,$$

or more compactly

$$\boldsymbol{\sigma}^T = -\sigma^2 \boldsymbol{\sigma} \sigma^2.$$

Geometrically, this transposition operation reflects \mathbf{x} about the y-axis.

Proving lemma 10.1 is well suited to software [diracWeylMatrixRepresentationAndIdentities.nb](#), but can also be done algebraically with ease. First note that

$$\begin{aligned} \sigma_1^T &= \sigma_1 \\ \sigma_2^T &= -\sigma_2 \\ \sigma_3^T &= \sigma_3 \end{aligned} \quad (10.41)$$

which means that

$$\begin{aligned}
 (\boldsymbol{\sigma} \cdot \mathbf{x})^T &= \sigma^1 x^1 - \sigma^2 x^2 + \sigma^3 x^3 \\
 &= \sigma^2 \sigma^2 (\sigma^1 x^1 - \sigma^2 x^2 + \sigma^3 x^3) \\
 &= \sigma^2 (-\sigma^1 x^1 - \sigma^2 x^2 - \sigma^3 x^3) \sigma^2 \\
 &= -\sigma^2 (\boldsymbol{\sigma} \cdot \mathbf{x})^T \sigma^2.
 \end{aligned} \tag{10.42}$$

Now we are ready to proceed.

Theorem 10.6: $U^T \sigma_2 U$ invariance

$U^T \sigma_2 U$ is Lorentz invariant.

Proof:

$$\begin{aligned}
 U^T \sigma_2 U &\rightarrow U'^T \sigma_2 U' \\
 &= U^T M^{\dagger T} \sigma_2 M^{\dagger} U,
 \end{aligned} \tag{10.43}$$

where $U' = M^{\dagger} U$ and $U'^T = U^T M^{\dagger T}$.

Note that if we can show that $M^{\dagger T} \sigma_2 M^{\dagger} = \sigma_2$, then we are done.

It is simple to show that any

$$U = e^{i\boldsymbol{\sigma} \cdot \mathbf{a}}, \tag{10.44}$$

for $\mathbf{a} \in \mathbb{R}^3$, has eigenvalues $\pm i \|\mathbf{a}\|$. The determinant of such a matrix is

$$\det U = \begin{vmatrix} e^{i\|\mathbf{a}\|} & 0 \\ 0 & e^{-i\|\mathbf{a}\|} \end{vmatrix} = 1, \tag{10.45}$$

so we see that such a matrix has the $U^{\dagger} U = 1$ and $\det U = 1$ properties that we desire for elements of $SU(2)$ ². We haven't shown that all matrices $U \in SU(2)$ can be written in this form, but let's assume that's the case.

² In class the suitability of $e^{i\boldsymbol{\sigma} \cdot \mathbf{a}}$ as an element of $SU(2)$ was demonstrated with an argument that diagonalizable matrices satisfy $\det e^A = e^{\text{tr} A}$

Claim: Generalizing from the exponential form of $SU(2)$ elements seen above, we assume that any $SL(2, \mathbb{C})$ matrix M can be written as

$$M^\dagger = e^{i\sigma \cdot (\mathbf{a} + i\mathbf{b})}, \quad (10.46)$$

for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

The transpose of an exponential of a sigma matrix goes like

$$\begin{aligned} (e^{\sigma \cdot \mathbf{u}})^T &= \sum_{k=0}^{\infty} \frac{1}{k!} ((\sigma \cdot \mathbf{u})^k)^T \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (-\sigma_2 (\sigma \cdot \mathbf{u}) \sigma_2)^k \\ &= \sigma_2 \left(\sum_{k=0}^{\infty} \frac{1}{k!} (-\sigma \cdot \mathbf{u})^k \right) \sigma_2 \\ &= \sigma_2 e^{-\sigma \cdot \mathbf{u}} \sigma_2, \end{aligned} \quad (10.47)$$

so

$$\begin{aligned} M^{\dagger T} \sigma_2 M^\dagger &= \left(e^{i\sigma \cdot (\mathbf{a} + i\mathbf{b})} \right)^T \sigma_2 e^{i\sigma \cdot (\mathbf{a} + i\mathbf{b})} \\ &= \left(\sigma_2 e^{-i\sigma \cdot (\mathbf{a} + i\mathbf{b})} \sigma_2 \right) \sigma_2 e^{i\sigma \cdot (\mathbf{a} + i\mathbf{b})} \\ &= \sigma_2, \end{aligned} \quad (10.48)$$

which is the result required to finish the proof of theorem 10.6³.

10.5 DIRAC MATRICES.

Definition 10.2: Dirac matrices.

The Dirac matrices $\gamma^\mu, \mu \in \{0, 1, 2, 3\}$ are matrices that satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$

that is

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu},$$

³ A slightly different derivation was done in class, but this one makes more sense to me.

We will use the explicit 4×4 matrix representation

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}.$$

The metric relations can also be written explicitly in the handy form

$$\begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^i)^2 &= -1. \end{aligned} \tag{10.49}$$

Written out explicitly, these matrices are

$$\begin{aligned} \gamma^0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, & \gamma^1 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ \gamma^2 &= \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, & \gamma^3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{10.50}$$

We will see in exercise 10.1 (Hw4) that Lorentz transformations take the form

$$x' \cdot \gamma = \Lambda_{1/2}^{-1} (x \cdot \gamma) \Lambda_{1/2}, \tag{10.51}$$

where

$$\Lambda_{1/2} = e^{-\frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}}, \tag{10.52}$$

where

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]. \tag{10.53}$$

In particular

$$\begin{aligned}
 S^{0k} &= \frac{i}{4} [\gamma^0, \gamma^k] \\
 &= \frac{i}{2} \gamma^0 \gamma^k \\
 &= \frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \\
 &= \frac{i}{2} \begin{bmatrix} -\sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix}
 \end{aligned} \tag{10.54}$$

will generate boosts, whereas (for $j \neq k$)

$$\begin{aligned}
 S^{jk} &= \frac{i}{4} [\gamma^j, \gamma^k] \\
 &= \frac{i}{2} \gamma^j \gamma^k \\
 &= \frac{i}{2} \begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \\
 &= -\frac{i}{2} \begin{bmatrix} \sigma^k \sigma^j & 0 \\ 0 & \sigma^k \sigma^j \end{bmatrix} \\
 &= \frac{1}{2} \epsilon^{jkl} \begin{bmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{bmatrix},
 \end{aligned} \tag{10.55}$$

are rotations (and in this case, are Hermitian).

The explicit expansion of the half Lorentz transformation operator is

$$\begin{aligned}
 \Lambda_{1/2} &= e^{-\frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}} \\
 &= e^{-i\omega_{0k} S^{0k} - \frac{i}{2} \omega_{jk} S^{jk}} \\
 &= \exp \left(-\frac{1}{2} \begin{bmatrix} \omega_{0k} \sigma^k & 0 \\ 0 & -\omega_{0k} \sigma^k \end{bmatrix} - \frac{i}{4} \begin{bmatrix} \omega_{jk} \epsilon^{jkl} \sigma^l & 0 \\ 0 & \omega_{jk} \epsilon^{jkl} \sigma^l \end{bmatrix} \right) \\
 &= \begin{bmatrix} e^{-\left(\frac{1}{2} \omega_{0k} \sigma^k + \frac{i}{4} \omega_{jk} \epsilon^{jkl} \sigma^l\right)} & 0 \\ 0 & e^{-\left(-\frac{1}{2} \omega_{0k} \sigma^k + \frac{i}{4} \omega_{jk} \epsilon^{jkl} \sigma^l\right)} \end{bmatrix}
 \end{aligned} \tag{10.56}$$

where the $1/2$ factor of ω_{0i} vanished because we had a sum over $0i$ and $i0$ which have been grouped.

Lemma 10.2: Some Dirac matrix identities.

$$\begin{aligned}
(\gamma^0)^\dagger &= \gamma^0 \\
(\gamma^k)^\dagger &= -\gamma^k \\
\gamma^0(\gamma^\mu)^\dagger\gamma^0 &= \gamma^\mu.
\end{aligned}$$

The first two are clear from inspection of eq. (10.50). For the last, for $\mu = 0$

$$\begin{aligned}
\gamma^0(\gamma^0)^\dagger\gamma^0 &= \gamma^0\gamma^0\gamma^0 \\
&= \gamma^0,
\end{aligned} \tag{10.57}$$

and for $\mu = k \neq 0$

$$\begin{aligned}
\gamma^0(\gamma^k)^\dagger\gamma^0 &= \gamma^0(-\gamma^k)\gamma^0 \\
&= -\gamma^0\gamma^k\gamma^0 \\
&= +\gamma^0\gamma^0\gamma^k \\
&= \gamma^k,
\end{aligned} \tag{10.58}$$

which completes the proof.

10.6 DIRAC LAGRANGIAN.

We postulate that there is a four-component object

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \quad \Psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*), \tag{10.59}$$

where ψ_μ 's are all complex fields, and assume that the fields transform as

$$\Psi(x) \rightarrow \Psi'(x') = \Lambda_{1/2} \Psi(x), \tag{10.60}$$

where our vectors transform in the usual $x \rightarrow x' = \Lambda x$ fashion, where the incremental form of the Lorentz transformation is the usual

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + O(\omega^2). \tag{10.61}$$

Definition 10.3: Overbar operator (name?).

$$\bar{\Psi} = \Psi^\dagger \gamma^0.$$

Definition 10.4: Dirac Lagrangian.

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x).$$

Armed with lemma 10.2 we can now show the following.

Theorem 10.7: The Dirac action is a real Lorentz scalar.

The action

$$S = \int d^4x \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi,$$

is a real scalar and is Lorentz invariant.

Real: To show that the action is real, we compute it's Hermitian conjugate, apply lemma 10.2 and integrate by parts

$$\begin{aligned}
 S^\dagger &= \int d^4x \Psi^\dagger \left(-i(\gamma^\mu)^\dagger \overleftarrow{\partial}_\mu - m \right) (\gamma^0)^\dagger \Psi \\
 &= \int d^4x \Psi^\dagger \left((i\gamma^\mu)^\dagger \overleftarrow{\partial}_\mu - m \right) \gamma^0 \Psi \\
 &= \int d^4x \Psi^\dagger \left(-\gamma^0 (i\gamma^\mu) \gamma^0 \overleftarrow{\partial}_\mu - m \right) \gamma^0 \Psi \\
 &= \int d^4x \bar{\Psi} \left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m \right) \Psi \\
 &= - \int d^4x \partial_\mu (\bar{\Psi} i\gamma^\mu \Psi) + \int d^4x \bar{\Psi} i\gamma^\mu \partial_\mu \Psi - \int d^4x \bar{\Psi} m \Psi \\
 &= \int d^4x \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi \\
 &= S,
 \end{aligned} \tag{10.62}$$

where ∂_μ without an overarrow means the traditional right acting operator, and assuming that the boundary terms vanish.

To show the Lorentz invariance, we will consider just the transformation of the Dirac Lagrangian density. We need a couple additional pieces of information to do so, the first of which is the transformation property⁴

$$\bar{\Psi} \rightarrow \Psi \Lambda_{1/2}^{-1}, \quad (10.63)$$

and (from exercise 10.1 (Hw4))

$$\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\alpha \gamma^\alpha. \quad (10.64)$$

The Lagrangian transforms as

$$\begin{aligned} \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x) &\rightarrow \bar{\Psi}'(x') \gamma^0 \left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m \right) \Psi'(x') \\ &= \bar{\Psi}(x) \Lambda_{1/2}^{-1} \left(i\gamma^\mu (\Lambda^{-1})^\alpha{}_\mu \partial_\alpha - m \right) \Lambda_{1/2} \Psi(x) \\ &= \bar{\Psi}(x) \left(i\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} (\Lambda^{-1})^\alpha{}_\mu \partial_\alpha - m \right) \Psi(x) \\ &= \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi \end{aligned} \quad (10.65)$$

We find that $\bar{\Psi}\Psi = \Psi^\dagger \gamma^0 \Psi$ is a Lorentz scalar, whereas $\bar{\Psi} \gamma^\mu \Psi$ is a 4 vector.

10.7 REVIEW.

Last time we

- introduced the Clifford algebra Dirac matrix (gamma matrices) elements satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (10.66)$$

where we use the Weyl representation

$$\begin{aligned} \gamma^0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \gamma^k &= \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}. \end{aligned} \quad (10.67)$$

In particular $(\gamma^0)^2 = 1, (\gamma^0)^\dagger = \gamma^0$.

⁴ Not proven here, but there's an argument for that in [14] (eq. 3.33).

- and left off after showing that the Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi, \quad (10.68)$$

where $\bar{\Psi} = \Psi^\dagger \gamma^0$, is Lorentz invariant. We argued that a single spinor can only describe a massless field, and that a two spinor construction can be used for a massive field. We skipped from there to the Dirac Lagrangian above.

10.8 DIRAC EQUATION.

Varying the Dirac action (exercise 10.4), we find the Dirac equation

$$(i\gamma^\mu \partial_\mu - m) \Psi = 0. \quad (10.69)$$

Theorem 10.8: Dirac equations solutions satisfy the Klein-Gordon equation

If Ψ obeys eq. (10.69), the Dirac equation, then Ψ obeys the Klein-Gordon equation.

Theorem 10.8 follows by pre-multiplying by a sort of “conjugate” operator⁵ $i\gamma^\mu + m$ to find

$$\begin{aligned} 0 &= (i\gamma^\mu \partial_\mu + m)(i\gamma^\mu \partial_\mu - m) \Psi \\ &= (-\gamma^\mu \gamma_\nu \partial_\mu \partial_\nu - m^2) \Psi \\ &= \left(-\frac{1}{2}(\gamma^\mu \gamma_\nu + \gamma^\nu \gamma_\mu) \partial_\mu \partial_\nu - m^2\right) \Psi \\ &= (-g^{\mu\nu} \partial_\mu \partial_\nu - m^2) \Psi \\ &= -(\partial_\mu \partial^\mu + m^2) \Psi \\ &= -(\partial_{00} - \nabla^2 + m^2) \Psi, \end{aligned} \quad (10.70)$$

which is a Klein-Gordon equation for Ψ .

Goal: Expand $\Psi(\mathbf{x}, t)$ in a basis of solutions of the Dirac equation. Call the coefficients a, b, \dots . This will be like

$$\phi \sim \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} (e^{ip \cdot x} a_{\mathbf{p}}^\dagger + e^{-ip \cdot x} a_{\mathbf{p}}) \quad (10.71)$$

⁵ Q: Is there a name for such a conjugation operation?

As with the scalar field, let's look for plane wave solutions. We'll first look for solutions of the form

$$\Psi(x) = u(p)e^{-ip \cdot x}, \quad (10.72)$$

where $p^2 = m^2$, $p^0 > 0 \forall p$. Plugging into eq. (10.69) we find

$$(\gamma^\mu p_\mu - m)u(p) = 0. \quad (10.73)$$

Let's write this out explicitly for exposition, first noting that

$$\begin{aligned} \gamma^\mu p_\mu &= \gamma^0 p_0 + \gamma^k p_k \\ &= p_0 \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix} + p_k \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & p^0 \sigma^0 - \boldsymbol{\sigma} \cdot \mathbf{p} \\ p^0 \sigma^0 + \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \end{bmatrix}, \end{aligned} \quad (10.74)$$

so eq. (10.73) becomes

$$\begin{aligned} 0 &= \begin{bmatrix} -m & p^0 \sigma^0 - \boldsymbol{\sigma} \cdot \mathbf{p} \\ p^0 \sigma^0 + \boldsymbol{\sigma} \cdot \mathbf{p} & -m \end{bmatrix} \begin{bmatrix} u_1(p) \\ u_2(p) \end{bmatrix} \\ &= \begin{bmatrix} -m & p \cdot \boldsymbol{\sigma} \\ p \cdot \bar{\boldsymbol{\sigma}} & -m \end{bmatrix} \begin{bmatrix} u_1(p) \\ u_2(p) \end{bmatrix}, \end{aligned} \quad (10.75)$$

where the following handy shorthand⁶ has been used to group the momentum related block matrices

$$\begin{aligned} p \cdot \boldsymbol{\sigma} &= p^0 \sigma^0 - \mathbf{p} \cdot \boldsymbol{\sigma} \\ p \cdot \bar{\boldsymbol{\sigma}} &= p^0 \sigma^0 + \mathbf{p} \cdot \boldsymbol{\sigma}. \end{aligned} \quad (10.76)$$

Note that these $p \cdot \boldsymbol{\sigma}$, $p \cdot \bar{\boldsymbol{\sigma}}$'s are both block matrices. In particular

$$\begin{aligned} p \cdot \boldsymbol{\sigma} &= p_\mu \sigma^\mu \\ &= \begin{bmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{bmatrix}. \end{aligned} \quad (10.77)$$

⁶ Assuming I wrote this down correctly, this follows the usual convention $x \cdot p = x^\mu p_\mu = x^0 p^0 - \mathbf{x} \cdot \mathbf{p}$. I had some doubt that I got the signs right in my notes from class, since a peek at [14] seemingly showed the opposite sign convention where $\sigma \cdot x$ was first defined, namely eq. 3.41/3.43. There they write $\sigma \cdot \partial = \partial_0 + \boldsymbol{\sigma} \cdot \nabla$, not $\sigma \cdot \partial = \partial_0 - \boldsymbol{\sigma} \cdot \nabla$. What explains this is the fact that the four-gradient in coordinate form should really be considered a lower index quantity (∂_μ), so in the scalar+vector tuple form, we should write $\partial^\mu = (\partial^0, -\nabla)$, or $\partial_\mu = (\partial_0, \nabla)$. This means that $\sigma \cdot \partial = \sigma^0 \partial^0 - \boldsymbol{\sigma} \cdot (-\nabla) = \partial_0 + \boldsymbol{\sigma} \cdot \nabla$. Having a tuple notation that can be used to represent either lower or upper index quantities is very confusing, and probably justifies avoiding that notation for any lower index quantity whenever possible for clarity!

The question is what u 's obey such an equation.

We can gain some insight by first considering the rest frame, where $\mathbf{p} = 0, p^0 = m$. Going back to eq. (10.73), the rest frame Dirac equation becomes

$$\begin{aligned} 0 &= (\gamma^0 p_0 - m)u \\ &= m(\gamma^0 - 1)u. \end{aligned} \quad (10.78)$$

Our block matrix equation is now reduced to a set of 2×2 identity matrices

$$0 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} u(\mathbf{p} = 0). \quad (10.79)$$

The solution space is given by

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \zeta \\ \zeta \end{bmatrix} = 0, \quad (10.80)$$

where ζ is itself a 2×1 column matrix, say

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}. \quad (10.81)$$

so our solutions are all proportional to column

$$u(\mathbf{p} = 0) \sim \sqrt{m} \begin{bmatrix} \zeta \\ \zeta \end{bmatrix}. \quad (10.82)$$

We'll figure out the desired normalization later⁷, and have added a \sqrt{m} factor into the mix for later convenience. Equation (10.82) is a solution of the Dirac equation in the rest frame where $\mathbf{p} = 0$. A solution in a frame where $\mathbf{p} \neq 0$ can be found using a boost. We won't work that out explicitly here, but instead show the answer and argue that it must be valid, but the interested student can find that boost calculated explicitly in [14]. A nice treatment of such a boost can also be found in [11] supplemented by exercise 10.5.

Claim: in a boosted frame where $\mathbf{p} \neq 0$ solution is

$$u(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta \\ \sqrt{p \cdot \bar{\sigma}} \zeta \end{bmatrix} \quad (10.83)$$

What do we mean by these square roots? Since $p \cdot \sigma, p \cdot \bar{\sigma}$ are both Hermitian 2×2 matrices, we can define the square root as the matrix of the square roots of the eigenvalues.

⁷ [14] says of this that we pick a normalization with $\zeta^\dagger \zeta = 1$.

Check: In the rest frame

$$\begin{aligned}
 \sqrt{p \cdot \sigma}|_{\mathbf{p}=0} &= \sqrt{p \cdot \bar{\sigma}}|_{\mathbf{p}=0} \\
 &= \begin{bmatrix} \sqrt{p_0} & 0 \\ 0 & \sqrt{p_0} \end{bmatrix} \\
 &= \begin{bmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{m} \end{bmatrix},
 \end{aligned} \tag{10.84}$$

so

$$\begin{aligned}
 u(p) &= \sqrt{m} \begin{bmatrix} \sigma^0 \zeta \\ \sigma^0 \zeta \end{bmatrix} \\
 &= \sqrt{m} \begin{bmatrix} \zeta \\ \zeta \end{bmatrix},
 \end{aligned} \tag{10.85}$$

as we already found.

We claim that the structure of the boost is

$$u(\mathbf{p}) = \begin{bmatrix} \frac{\sqrt{p \cdot \sigma}}{m} & 0 \\ 0 & \frac{\sqrt{p \cdot \bar{\sigma}}}{m} \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{m} \zeta \\ \sqrt{m} \zeta \end{bmatrix}}_{u(\mathbf{p}=0)} \tag{10.86}$$

We'd like to check that this is an element of $SL(2, \mathbb{C})$. We'll also see in the end that we don't have to calculate these square roots, since we always end up with two spinors and when all is said we end up with products of these roots.

Lemma 10.3: Determinant of square root.

If matrix A is diagonalizable, then $\det \sqrt{A} = \sqrt{\det A}$.

Proof: Suppose that

$$A = U \operatorname{diag}(\lambda_1, \dots, \lambda_n) U^\dagger, \tag{10.87}$$

then

$$\begin{aligned}
 \det \sqrt{A} &= \det \left(U \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) U^\dagger \right) \\
 &= \prod_j \sqrt{\lambda_j} \\
 &= \sqrt{\prod_j \lambda_j} \\
 &= \sqrt{\det A}.
 \end{aligned} \tag{10.88}$$

Lemma 10.4: Determinant of $p \cdot \sigma$.

$$\det \frac{\sqrt{p \cdot \sigma}}{\sqrt{m}} = 1$$

Proof:

$$\begin{aligned}
 \det \frac{\sqrt{p \cdot \sigma}}{\sqrt{m}} &= \sqrt{\det \frac{(p \cdot \sigma)}{m}}, \\
 &= \sqrt{\det \frac{1}{m} \begin{bmatrix} p^0 + p^3 & -p_1 + ip_2 \\ -p_1 - ip_2 & p^0 + p^3 \end{bmatrix}} \\
 &= \sqrt{\frac{1}{m^2} ((p^0)^2 - \mathbf{p}^2)} \\
 &= \sqrt{\frac{m^2}{m^2}} \\
 &= 1.
 \end{aligned} \tag{10.89}$$

Lemma 10.5: $(p \cdot \sigma)(p \cdot \bar{\sigma})$

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = m^2.$$

Proof:

$$\begin{aligned}
 (p \cdot \sigma)(p \cdot \bar{\sigma}) &= (p^0 - \mathbf{p} \cdot \boldsymbol{\sigma})(p^0 + \mathbf{p} \cdot \boldsymbol{\sigma}) \\
 &= (p^0)^2 - (\mathbf{p} \cdot \boldsymbol{\sigma})^2 \\
 &= (p^0)^2 - \mathbf{p}^2 \\
 &= m^2.
 \end{aligned} \tag{10.90}$$

Theorem 10.9: $u(p)$ is a solution to the Dirac equation.

Equation (10.83) is a solution of eq. (10.69), the Dirac equation.

Proof:

$$\begin{aligned}
 (\gamma^\mu p_\mu - m)u(p) &= \begin{bmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta \\ \sqrt{p \cdot \bar{\sigma}} \zeta \end{bmatrix} \\
 &= \begin{bmatrix} (-m \sqrt{p \cdot \sigma} + p \cdot \sigma \sqrt{p \cdot \bar{\sigma}}) \zeta \\ (p \cdot \bar{\sigma} \sqrt{p \cdot \sigma} - m \sqrt{p \cdot \bar{\sigma}}) \zeta \end{bmatrix} \\
 &= \begin{bmatrix} \sqrt{p \cdot \sigma} (-m + \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}}) \zeta \\ \sqrt{p \cdot \bar{\sigma}} (p \sqrt{\bar{\sigma}} \sqrt{p \cdot \sigma} - m) \zeta \end{bmatrix} \\
 &= \begin{bmatrix} \sqrt{p \cdot \sigma} (-m + \sqrt{m^2}) \zeta \\ \sqrt{p \cdot \bar{\sigma}} (p \sqrt{m^2} - m) \zeta \end{bmatrix} \\
 &= 0.
 \end{aligned} \tag{10.91}$$

Summary: For $p^0 > 0$, $p^2 = m^2$

$$\Psi(x) = e^{-ip \cdot x} u(p) \tag{10.92}$$

$$u(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta \\ \sqrt{p \cdot \bar{\sigma}} \zeta \end{bmatrix} \tag{10.93}$$

Example:

$$p = (E, 0, 0, p^3) \tag{10.94}$$

We have

$$\begin{aligned}
 \sqrt{\vec{\sigma} \cdot \vec{p}} &= \sqrt{E - p^3 \sigma^3} \\
 &= \sqrt{\begin{bmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{bmatrix}} \\
 &= \begin{bmatrix} \sqrt{E - p^3} & 0 \\ 0 & \sqrt{E + p^3} \end{bmatrix}.
 \end{aligned} \tag{10.95}$$

Similarly

$$\sqrt{\vec{\sigma} \cdot \vec{p}} = \begin{bmatrix} \sqrt{E + p^3} & 0 \\ 0 & \sqrt{E - p^3} \end{bmatrix}, \tag{10.96}$$

so

$$u(p) = \begin{bmatrix} \begin{bmatrix} \sqrt{E - p^3} & 0 \\ 0 & \sqrt{E + p^3} \end{bmatrix} \zeta \\ \begin{bmatrix} \sqrt{E + p^3} & 0 \\ 0 & \sqrt{E - p^3} \end{bmatrix} \zeta \end{bmatrix} \tag{10.97}$$

Suppose we let

$$\zeta = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10.98}$$

we are left with

$$u(p) = \begin{bmatrix} \sqrt{E - p^3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \sqrt{E + p^3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} \tag{10.99}$$

Alternatively for $\zeta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$u(p) = \begin{bmatrix} \sqrt{E + p^3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \sqrt{E - p^3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \tag{10.100}$$

If we pick $p_3 = E^8$, then we find two solutions

$$u(p)|_{\zeta=(1,0)^T, p_3=E} = \begin{bmatrix} 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \sqrt{2E} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} \quad (10.101)$$

and

$$u(p)|_{\zeta=(0,1)^T, p_3=E} = \begin{bmatrix} \sqrt{2E} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \quad (10.102)$$

10.9 HELICITY.

Let h (the helicity) be

$$\begin{aligned} h &= \frac{1}{2} \begin{bmatrix} \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{bmatrix} \\ &= \hat{\mathbf{p}} \cdot \mathbf{S}, \end{aligned} \quad (10.103)$$

where

$$\mathbf{S} = \begin{bmatrix} \frac{\boldsymbol{\sigma}}{2} & 0 \\ 0 & \frac{\boldsymbol{\sigma}}{2} \end{bmatrix}. \quad (10.104)$$

h has eigenvalues $\pm 1/2$.

It turns out that eq. (10.101), and eq. (10.102) are both eigenstates of the helicity operator.

$$hu^{(1)} = \frac{1}{2}u^{(1)} \quad (10.105)$$

$$hu^{(2)} = -\frac{1}{2}u^{(2)}, \quad (10.106)$$

corresponding to momentum aligned with and opposing the spin directions as sketched in fig. 10.2.

8 we can boost a massive particle to be arbitrarily close to $p_3 = E$.

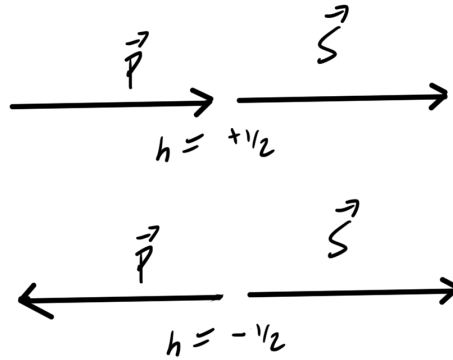


Figure 10.2: Helicity orientation.

10.10 NEXT TIME.

We found $\Psi = ue^{-ip \cdot x}$. Next time we will seek another solution $\Psi = ve^{+ip \cdot x}$, and we will also figure out how to normalize things.

10.11 REVIEW.

We were studying the Dirac Lagrangian

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi, \quad (10.107)$$

from which we find

$$(i\gamma^\mu \partial_\mu - m) \Psi = 0, \quad (10.108)$$

the Dirac equation, and saw that solutions to this equation satisfies the Klein-Gordon equation. We found solution

$$\Psi(x) = u(p)e^{-ip \cdot x}, \quad (10.109)$$

which is automatically a solution to the Klein-Gordon equation. There are actually two linearly independent solutions

$$u^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta^s \\ \sqrt{p \cdot \bar{\sigma}} \zeta^s \end{bmatrix}, \quad (10.110)$$

where $\zeta^1 = (1, 0)^T$, $\zeta^2 = (0, 1)^T$.

10.12 NORMALIZATION.

Theorem 10.10: $u^\dagger u$

$$u^{r\dagger} u^s = 2p_0 \delta^{rs}.$$

Proof:

$$\begin{aligned}
 u^{s\dagger} u^r &= \begin{bmatrix} \zeta^{s\dagger} \sqrt{p \cdot \sigma} & \zeta^{s\dagger} \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta^r \\ \sqrt{p \cdot \bar{\sigma}} \zeta^r \end{bmatrix} \\
 &= \zeta^{s\dagger} \left(\sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} + \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \bar{\sigma}} \right) \zeta^r \\
 &= \zeta^{s\dagger} (p \cdot \sigma + p \cdot \bar{\sigma}) \zeta^r \\
 &= \zeta^{s\dagger} (p_0 - \mathbf{p} \cdot \boldsymbol{\sigma} + p_0 + \mathbf{p} \cdot \boldsymbol{\sigma}) \zeta^r \\
 &= 2p_0 \zeta^{s\dagger} \zeta^r.
 \end{aligned} \tag{10.111}$$

We can easily see that $\zeta^{s\dagger} \zeta^r = \delta^{rs}$ by writing out those products

$$\begin{aligned}
 \zeta^{1\dagger} \zeta^1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \\
 \zeta^{1\dagger} \zeta^2 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \\
 \zeta^{2\dagger} \zeta^1 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \\
 \zeta^{2\dagger} \zeta^2 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1,
 \end{aligned} \tag{10.112}$$

which completes the proof.

We also want to compute $\bar{u}u$, but need a couple intermediate results.

Lemma 10.6: Products of $p \cdot \sigma, p \cdot \bar{\sigma}$.

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = (p \cdot \bar{\sigma})(p \cdot \sigma) = m^2.$$

Proof:

$$\begin{aligned}
 (p \cdot \sigma)(p \cdot \bar{\sigma}) &= (p^0 - \mathbf{p} \cdot \boldsymbol{\sigma})(p^0 + \mathbf{p} \cdot \boldsymbol{\sigma}) \\
 &= (p^0)^2 - (\mathbf{p} \cdot \boldsymbol{\sigma})^2 \\
 &= (p^0)^2 - \mathbf{p}^2 \\
 &= m^2
 \end{aligned} \tag{10.113}$$

and

$$\begin{aligned}
 (p \cdot \bar{\sigma})(p \cdot \sigma) &= (p^0 + \mathbf{p} \cdot \boldsymbol{\sigma})(p^0 - \mathbf{p} \cdot \boldsymbol{\sigma}) \\
 &= (p^0)^2 - (\mathbf{p} \cdot \boldsymbol{\sigma})^2 \\
 &= (p^0)^2 - \mathbf{p}^2 \\
 &= m^2.
 \end{aligned} \tag{10.114}$$

Theorem 10.11: $\bar{u}u$.

$$\bar{u}^r(\mathbf{p})u^s(\mathbf{p}) = 2m\delta^{rs}.$$

Proof:

$$\begin{aligned}
 \bar{u}^r u^s &= u^{r\dagger} \gamma^0 u^s \\
 &= \begin{bmatrix} \zeta^{r\dagger} \sqrt{p \cdot \sigma} & \zeta^{r\dagger} \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta^s \\ \sqrt{p \cdot \bar{\sigma}} \zeta^s \end{bmatrix} \\
 &= \zeta^{r\dagger} \left(\sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} + \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \right) \zeta^s \\
 &= 2m \zeta^{r\dagger} \zeta^s \\
 &= 2m \delta^{rs},
 \end{aligned} \tag{10.115}$$

which completes the proof.

10.13 OTHER SOLUTION.

Now we seek the other plane wave solution

$$\Psi(x) = v(p)e^{ip \cdot x}. \tag{10.116}$$

Theorem 10.12: v solution to the Dirac equation.

Equation (10.116) is a solution to the Dirac equation, provided

$$v^s(p) = \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix},$$

where $\eta^1 = (1, 0)^T, \eta^2 = (0, 1)^T$.

Proof is left to exercise 10.6.

Theorem 10.13: v normalization.

$$\begin{aligned} \bar{v}^r(p) v^s(p) &= -2m \delta^{rs} \\ v^{r\dagger}(p) v^s(p) &= 2p^0 \delta^{rs}. \end{aligned}$$

Theorem 10.13 is proven in exercise 10.7.

It will also be useful to restate the $2\delta^{rs} p_0$ normalization conditions as

$$\begin{aligned} u^{r\dagger}(\mathbf{p}) u^s(\mathbf{p}) &= 2\omega_{\mathbf{p}} \delta^{sr} \\ v^{r\dagger}(\mathbf{p}) v^s(\mathbf{p}) &= 2\omega_{\mathbf{p}} \delta^{sr}. \end{aligned} \tag{10.117}$$

Various orthogonality conditions exist between the u 's and v 's

Theorem 10.14: Dirac adjoint orthogonality conditions.

$$\begin{aligned} \bar{u}^r(p) v^s(p) &= 0 \\ \bar{v}^r(p) u^s(p) &= 0. \end{aligned}$$

Proof left to exercise 10.8.

Theorem 10.15: Dagger orthogonality conditions.

$$\begin{aligned} v^{r\dagger}(-\mathbf{p}) u^s(\mathbf{p}) &= 0 \\ u^{r\dagger}(\mathbf{p}) v^s(-\mathbf{p}) &= 0. \end{aligned}$$

Proof left to exercise 10.9.

Finally, there are a couple tensor products of interest.

Definition 10.5: Tensor product.

Given a pair of vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

the tensor product is the matrix of all elements $x_i y_j$

$$x \otimes y^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \otimes [y_1 \cdots y_n] = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ x_3 y_1 & & \ddots & \\ \vdots & & & \\ x_n y_1 & \cdots & & x_n y_n \end{bmatrix}.$$

Theorem 10.16: Direct product relations.

$$\sum_{s=1}^2 u^s(p) \otimes \bar{u}^s(p) = \gamma \cdot p + m$$

$$\sum_{s=1}^2 v^s(p) \otimes \bar{v}^s(p) = \gamma \cdot p - m$$

For the v 's

$$\begin{aligned}
& \sum_{s=1,2} \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix} \otimes \begin{bmatrix} (\eta^s)^T \sqrt{p \cdot \sigma} & -(\eta^s)^T \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \sum_{s=1,2} \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ \sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix} \otimes \begin{bmatrix} -(\eta^s)^T \sqrt{p \cdot \sigma} & (\eta^s)^T \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \\
&= \sum_{s=1,2} \begin{bmatrix} -\sqrt{p \cdot \sigma} \eta^s \otimes (\eta^s)^T \sqrt{p \cdot \sigma} & \sqrt{p \cdot \sigma} \eta^s \otimes (\eta^s)^T \sqrt{p \cdot \bar{\sigma}} \\ \sqrt{p \cdot \bar{\sigma}} \eta^s \otimes (\eta^s)^T \sqrt{p \cdot \sigma} & -\sqrt{p \cdot \bar{\sigma}} \eta^s \otimes (\eta^s)^T \sqrt{p \cdot \bar{\sigma}} \end{bmatrix},
\end{aligned} \tag{10.118}$$

but

$$\begin{aligned}
\eta^1 \otimes \eta^{1T} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned} \tag{10.119}$$

and

$$\begin{aligned}
\eta^2 \otimes \eta^{2T} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned} \tag{10.120}$$

so $\sum_{s=1,2} \eta^s \otimes \eta^{sT} = 1$, leaving

$$\begin{aligned}
\sum_{s=1}^2 v^s(p) \otimes \bar{v}^s(p) &= \begin{bmatrix} -\sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} \\ \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} & -\sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \end{bmatrix} \\
&= \begin{bmatrix} -\sqrt{p \cdot \sigma p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma p \cdot \bar{\sigma}} \\ \sqrt{p \cdot \bar{\sigma} p \cdot \sigma} & -\sqrt{p \cdot \bar{\sigma} p \cdot \sigma} \end{bmatrix} \\
&= \begin{bmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{bmatrix} \\
&= -m \mathbf{1} + p^0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} 0 & -\sigma \\ \sigma & 0 \end{bmatrix} \\
&= -m + p^\mu \gamma_\mu,
\end{aligned} \tag{10.121}$$

as stated. Proof for the u 's is left to exercise 10.10.

10.14 LAGRANGIAN.

Theorem 10.17: Dirac Hamiltonian.

The Dirac Hamiltonian is

$$H = \int d^3x \Psi^\dagger (-i\gamma^0 \gamma^j \partial_j \Psi + m\gamma^0) \Psi.$$

To prove theorem 10.17, we start with the spacetime expansion of the Dirac Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{Dirac}} &= \bar{\Psi} i\gamma^0 \partial_0 \Psi + i\bar{\Psi} \gamma^j \partial_j \Psi - m\bar{\Psi} \Psi \\ &= \Psi^\dagger \gamma^0 i\gamma^0 \partial_0 \Psi + i\Psi^\dagger \gamma^0 \gamma^j \partial_j \Psi - m\Psi^\dagger \gamma^0 \Psi \\ &= \Psi^\dagger i\dot{\Psi} + i\Psi^\dagger \gamma^0 \gamma^j \partial_j \Psi - m\Psi^\dagger \gamma^0 \Psi. \end{aligned} \quad (10.122)$$

We see that the momentum conjugate to Ψ is

$$\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i\Psi^\dagger. \quad (10.123)$$

Computing the Hamiltonian density in the usual way, we have

$$\begin{aligned} \mathcal{H}_{\text{Dirac}} &= \pi_\Psi \dot{\Psi} - \mathcal{L} \\ &= i\Psi^\dagger \dot{\Psi} - (\Psi^\dagger i\dot{\Psi} + i\Psi^\dagger \gamma^0 \gamma^j \partial_j \Psi - m\Psi^\dagger \gamma^0 \Psi) \\ &= -i\Psi^\dagger \gamma^0 \gamma^j \partial_j \Psi + m\Psi^\dagger \gamma^0 \Psi. \end{aligned} \quad (10.124)$$

Integrating over a 3-volume provides the Dirac Hamiltonian of theorem 10.17.

Now we want to examine the action of $-i\gamma^0 \gamma^j \partial_j + m\gamma^0 = \gamma^0 (-i\gamma^j \partial_j + m)$ on the plane wave solutions we have found.

Theorem 10.18: Hamiltonian action on Dirac plane wave solutions.

For $\Psi_u = u(p)e^{-ip \cdot x}$, and $\Psi_v = v(p)e^{ip \cdot x}$, we have

$$\begin{aligned} -\gamma^0 (i\gamma^j \partial_j - m) \Psi_u &= p_0 \Psi_u \\ -\gamma^0 (i\gamma^j \partial_j - m) \Psi_v &= -p_0 \Psi_v. \end{aligned}$$

Theorem 10.18, which shows that Ψ_u, Ψ_v are eigenvectors of the operator $\gamma^0(-i\gamma^j\partial_j + m)$ with eigenvalues $\pm\omega_{\mathbf{p}}$. These eigenvalue equations follow from the Dirac equation for Ψ_u, Ψ_v . These are

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)ue^{-ip\cdot x} &= (i\gamma^j\partial_j + i\gamma^0\partial_0 - m)ue^{-ip\cdot x} \\ &= (i\gamma^j\partial_j + i(-i)\gamma^0 p_0 - m)ue^{-ip\cdot x}\end{aligned}\tag{10.125}$$

and

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)ve^{ip\cdot x} &= (i\gamma^j\partial_j + i\gamma^0\partial_0 - m)ve^{ip\cdot x} \\ &= (i\gamma^j\partial_j + i(i)\gamma^0 p_0 - m)ve^{ip\cdot x}.\end{aligned}\tag{10.126}$$

Rearranging gives

$$\begin{aligned}(i\gamma^j\partial_j - m)ue^{-ip\cdot x} &= -\gamma^0 p_0 ue^{-ip\cdot x} \\ (i\gamma^j\partial_j - m)ve^{ip\cdot x} &= +\gamma^0 p_0 ve^{ip\cdot x},\end{aligned}\tag{10.127}$$

and theorem 10.18 follows immediately.

10.15 GENERAL SOLUTION AND HAMILTONIAN.

As with the Klein-Gordon equation, let's introduce a generic solution formed from linear combinations of our specific $u^s(p) = u_{\mathbf{p}}^s, v^s(p) = v_{\mathbf{p}}^s$ solutions

$$\Psi(\mathbf{x}, t) = \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip\cdot x} u_{\mathbf{p}}^s a_{\mathbf{p}}^s + e^{ip\cdot x} v_{\mathbf{p}}^s b_{\mathbf{p}}^s \right).\tag{10.128}$$

Theorem 10.19: Dirac Hamiltonian in terms of creation and annihilation operators.

Substitution of the superposition eq. (10.128) into the Dirac Hamiltonian of theorem 10.17 results in

$$H_{\text{Dirac}} = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^r - b_{-\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^r \right).$$

Deferring interpretation slightly, we first prove theorem 10.19, making the somewhat lazy guess that all the time dependent terms will be wiped out. This assumption allows us to use the zero time fields of our superposition solution

$$\Psi(\mathbf{x}, 0) = \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} (u_{\mathbf{p}}^s a_{\mathbf{p}}^s + v_{-\mathbf{p}}^s b_{-\mathbf{p}}^s) \quad (10.129a)$$

$$\Psi^\dagger(\mathbf{x}, 0) = \sum_{r=1}^2 \int \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} e^{-i\mathbf{q}\cdot\mathbf{x}} (u_{\mathbf{q}}^{r\dagger} a_{\mathbf{q}}^{r\dagger} + v_{-\mathbf{q}}^{r\dagger} b_{-\mathbf{q}}^{r\dagger}). \quad (10.129b)$$

Making use of the eigenvalue equations theorem 10.18 the Hamiltonian is reduced to

$$\begin{aligned} H_{\text{Dirac}} &= \sum_{r,s=1}^2 \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 2 \sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{q}}}} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} (u_{\mathbf{q}}^{r\dagger} a_{\mathbf{q}}^{r\dagger} + v_{-\mathbf{q}}^{r\dagger} b_{-\mathbf{q}}^{r\dagger}) \omega_{\mathbf{p}} (u_{\mathbf{p}}^s a_{\mathbf{p}}^s - v_{-\mathbf{p}}^s b_{-\mathbf{p}}^s) \\ &= \sum_{r,s=1}^2 \int \frac{d^3 p}{(2\pi)^3 2 \cancel{\omega_{\mathbf{p}}}} (u_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^{r\dagger} + v_{-\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^{r\dagger}) \cancel{\omega_{\mathbf{p}}} (u_{\mathbf{p}}^s a_{\mathbf{p}}^s - v_{-\mathbf{p}}^s b_{-\mathbf{p}}^s) \\ &= \frac{1}{2} \sum_{r,s=1}^2 \int \frac{d^3 p}{(2\pi)^3} (u_{\mathbf{p}}^{r\dagger} u_{\mathbf{p}}^s a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s - \cancel{u_{\mathbf{p}}^{r\dagger} v_{-\mathbf{p}}^s a_{\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^s} + \cancel{v_{-\mathbf{p}}^{r\dagger} u_{\mathbf{p}}^s b_{-\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s} - v_{-\mathbf{p}}^{r\dagger} v_{-\mathbf{p}}^s b_{-\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^s), \end{aligned} \quad (10.130)$$

where care was taken not to commute any a, b 's. Recall that

$$u_{\mathbf{p}}^{r\dagger} u_{\mathbf{p}}^s = v_{\mathbf{p}}^{r\dagger} v_{\mathbf{p}}^s = 2\omega_{\mathbf{p}} \delta^{rs} \quad (10.131a)$$

$$u_{\mathbf{p}}^{r\dagger} v_{-\mathbf{p}}^s = v_{-\mathbf{p}}^{r\dagger} u_{\mathbf{p}}^s = 0. \quad (10.131b)$$

Equation (10.131b) kills off our cross terms, and eq. (10.131a) wipes out one of the summation indexes

$$\begin{aligned} H_{\text{Dirac}} &= \frac{1}{2} \sum_{r,s=1}^2 \int \frac{d^3 p}{(2\pi)^3} (u_{\mathbf{p}}^{r\dagger} u_{\mathbf{p}}^s a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s - \cancel{u_{\mathbf{p}}^{r\dagger} v_{-\mathbf{p}}^s a_{\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^s} + \cancel{v_{-\mathbf{p}}^{r\dagger} u_{\mathbf{p}}^s b_{-\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s} - v_{-\mathbf{p}}^{r\dagger} v_{-\mathbf{p}}^s b_{-\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^s) \\ &= \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} (a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^r - b_{-\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^r). \end{aligned} \quad (10.132)$$

We see above how the mixed terms were killed off nicely by eq. (10.131b). That also justifies the use of the zero-time fields in this derivation, which can also be seen explicitly without use of the zero-time fields exercise 10.11.

Interpretation. With a minus sign in the Hamiltonian, there is no bound to the energy from below! This makes it troublesome to interpret the a_p 's and b_p 's as the familiar raising and lowering operators that we know.

We can save the day, making the “Dirac sea” argument, roughly speaking that we can consider a set of completely full negative energy states, where creation of a particle makes a hole in one of those states⁹, as sketched roughly in fig. 10.3. Such an argument does not work for bosons

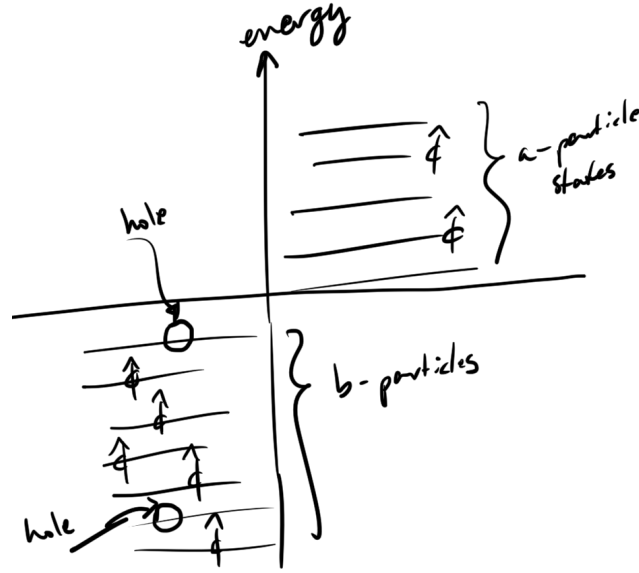


Figure 10.3: Dirac Sea.

(photons, ...) since an arbitrary number of such particles can be stuffed into any given state. It will turn out that our operators are fermions, which gets us out of this trouble.

We can also get out of this hole algebraically. For $X = a, b$, let

$$\begin{aligned} X_p^{s\dagger} &= \tilde{X}_p^s \\ X_p^s &= \tilde{X}_p^{s\dagger}. \end{aligned} \tag{10.133}$$

It turns out that some properties of our creation and annihilation operators are

$$\begin{aligned} (a_p^s)^2 &= 0 \\ (a_p^{s\dagger})^2 &= 0 \\ (b_p^s)^2 &= 0 \\ (b_p^{s\dagger})^2 &= 0, \end{aligned} \tag{10.134}$$

⁹ There was a long discussion of this topic in class that I was not able to capture in my notes.

and

$$\begin{aligned}\{a_{\mathbf{p}}^s, a_{\mathbf{q}}^{r\dagger}\} &= \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ \{b_{\mathbf{p}}^s, b_{\mathbf{q}}^{r\dagger}\} &= \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}),\end{aligned}\tag{10.135}$$

where all other anticommutators are zero

$$\begin{aligned}\{a^r, b^s\} &= \{a^r, b^{s\dagger}\} \\ &= \{a^{r\dagger}, b^s\} \\ &= \{a^{r\dagger}, b^{s\dagger}\} \\ &= 0.\end{aligned}\tag{10.136}$$

Such a substitution gives

$$\begin{aligned}H_{\text{Dirac}} &= \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} (\tilde{a}_{\mathbf{p}}^s a_{\mathbf{p}}^s - \tilde{b}_{\mathbf{p}}^s b_{\mathbf{p}}^s) \\ &= \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} (\tilde{a}_{\mathbf{p}}^s a_{\mathbf{p}}^s + b_{\mathbf{p}}^s \tilde{b}_{\mathbf{p}}^s + \delta^{ss} \delta^{(3)}(\mathbf{p} - \mathbf{p})) \\ &= \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3} \left(\omega_{\mathbf{p}} (\tilde{a}_{\mathbf{p}}^s a_{\mathbf{p}}^s + b_{\mathbf{p}}^s \tilde{b}_{\mathbf{p}}^s) - 4V_3 \frac{\omega_{\mathbf{p}}}{2} \right).\end{aligned}\tag{10.137}$$

We'll end up dropping the vacuum energy term. We'll end up labelling the a 's as the operators associated with electrons, and the b 's with anti-electrons.

10.16 REVIEW.

From the Dirac Lagrangian density

$$\mathcal{L} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi,\tag{10.138}$$

we found that the energy can be expressed using Hamiltonian

$$H = \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1}^2 \omega_{\mathbf{p}} (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s).\tag{10.139}$$

This appears to be an energy with no bottom. Dirac prescribes: assume Pauli exclusion for b and fill all the negative energy levels. If we treat a, b as bosonic (commuting), then energy is unbounded from below. This is a problem, because once you add interactions the system falls

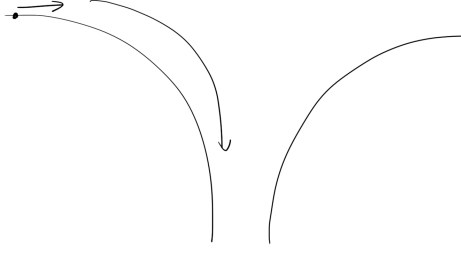


Figure 10.4: Unbounded potential well.

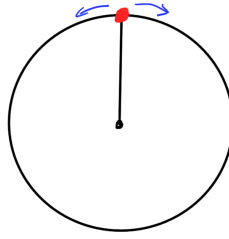


Figure 10.5: Unstable configuration (inverted pendulum).

into the abyss, something that we can represent as sketched in fig. 10.4. Another representation of such unstable system that comes to mind is the inverted pendulum sketched in fig. 10.5.

Dirac fixed this by imagining that all negative energy states are “full”. This doesn’t quite fix it, unless the particles obey the Pauli Principle. Creating a particle of negative energy b^\dagger is like destroying a hole.

Mathematically, we postulate that our operators

1. Obey Fermi statistics, behaving like “Grassman numbers”

$$(b^\dagger)^2 = 0 = b^2 = a^2 = (a^\dagger)^2. \quad (10.140)$$

All the a, b, a^\dagger, \dots ’s square to zero¹⁰

2. Our creation and annihilation operators are presumed to have non-trivial anti-commutation relations (unlike the scalar theory where we had the same sort of commutation relations)

$$\begin{aligned} \{a_{\mathbf{p}}^s, a_{\mathbf{q}}^{r\dagger}\} &= (2\pi)^3 \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ \{\tilde{b}_{\mathbf{p}}^s, \tilde{b}_{\mathbf{q}}^{r\dagger}\} &= (2\pi)^3 \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (10.141)$$

¹⁰ Is it a coincidence that these look like lightlike four-vectors $x^2 = x^\mu x_\mu = 0$?

The relations were used to cast the Hamiltonian in a more familiar form

$$\begin{aligned}
 H &= \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1}^2 \omega_{\mathbf{p}} (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + \tilde{b}_{\mathbf{p}}^{s\dagger} \tilde{b}_{\mathbf{p}}^s - \underbrace{(2\pi)^3 \delta^{(3)}(0)}_{\text{zero point energy}}) \\
 &= \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1}^2 \omega_{\mathbf{p}} (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + \tilde{b}_{\mathbf{p}}^{s\dagger} \tilde{b}_{\mathbf{p}}^s) - V_3 \int \frac{d^3 p}{(2\pi)^3} 2\omega_{\mathbf{p}}.
 \end{aligned} \tag{10.142}$$

Fermions have negative zero-point energy $-4\times$ that of real massive scalar.

Digression

Supersymmetry transforms these into one another, and was thought to solve the cosmic constant problem.

10.17 HAMILTONIAN ACTION ON SINGLE PARTICLE STATES.

We now switch notations, drop the tildes, and ignore the zero point energy

$$H = \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1}^2 \omega_{\mathbf{p}} (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s) \tag{10.143}$$

We define the Fock vacuum by

$$\begin{aligned}
 a_{\mathbf{p}}^s |0\rangle &= 0 \\
 b_{\mathbf{p}}^s |0\rangle &= 0,
 \end{aligned} \tag{10.144}$$

and presume that we have relativistically normalized creation operators

$$\begin{aligned}
 a^s(p) |0\rangle &= \sqrt{2\omega_{\mathbf{p}}} a_{\mathbf{p}}^{s\dagger} |0\rangle \\
 b^s(p) |0\rangle &= \sqrt{2\omega_{\mathbf{p}}} b_{\mathbf{p}}^{s\dagger} |0\rangle.
 \end{aligned} \tag{10.145}$$

Let's see how the Hamiltonian acts on each of our possible a single particle states (with momentum \mathbf{p} and spin r)

$$\begin{aligned}
H|\mathbf{p}, r\rangle &= H\sqrt{2\omega_{\mathbf{p}}}a_{\mathbf{p}}^{r\dagger}|0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3} \sum_{s=1}^2 \omega_{\mathbf{q}} (a_{\mathbf{q}}^{s\dagger} a_{\mathbf{q}}^s + b_{\mathbf{q}}^{s\dagger} b_{\mathbf{q}}^s) \sqrt{2\omega_{\mathbf{p}}}a_{\mathbf{p}}^{r\dagger}|0\rangle \\
&= \int \frac{d^3q}{(2\pi)^3} \sum_{s=1}^2 \omega_{\mathbf{q}} a_{\mathbf{q}}^{s\dagger} a_{\mathbf{q}}^s a_{\mathbf{p}}^{r\dagger}|0\rangle \sqrt{2\omega_{\mathbf{p}}} \\
&= \int \frac{d^3q}{(2\pi)^3} \sum_{s=1}^2 \omega_{\mathbf{q}} a_{\mathbf{q}}^{s\dagger} (-a_{\mathbf{p}}^{r\dagger} a_{\mathbf{q}}^s + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})) |0\rangle \sqrt{2\omega_{\mathbf{p}}} \\
&= \int \frac{d^3q}{(2\pi)^3} \sum_{s=1}^2 \omega_{\mathbf{q}} a_{\mathbf{q}}^{s\dagger} (-a_{\mathbf{p}}^{r\dagger} \underbrace{a_{\mathbf{q}}^s |0\rangle}_{=0} + (2\pi)^3 \delta^{rs} \delta^{(3)}(\mathbf{p} - \mathbf{q}) |0\rangle) \sqrt{2\omega_{\mathbf{p}}} \\
&= \omega_{\mathbf{p}} (a_{\mathbf{p}}^{r\dagger} |0\rangle \sqrt{2\omega_{\mathbf{p}}}) \\
&= \omega_{\mathbf{p}} |\mathbf{p}, r\rangle.
\end{aligned} \tag{10.146}$$

The Hamiltonian has the expected energy operator characteristics. This is also clearly the case for our b operators too.

10.18 SPACETIME TRANSLATION SYMMETRIES.

For the scalar field, using Noether's theorem, we identified the conserved charge of a spatial translation as the momentum operator

$$\begin{aligned}
P^i &= \int d^3x T^{0i} \\
&= - \int d^3x \pi(x) \nabla \phi(x),
\end{aligned} \tag{10.147}$$

and if we plugged in the creation and annihilation operator representation of π, ϕ , out comes

$$\mathbf{P} = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger), \tag{10.148}$$

(plus $e^{\pm 2i\omega_{\mathbf{p}}t}$ terms that we can argue away.)

For the Dirac field, this works the same way if we systematically apply Noether's theorem. In particular, for a spacetime translation

$$x^\mu \rightarrow x^\mu + a^\mu, \quad (10.149)$$

we find

$$\delta\Psi = -a^\mu \partial_\mu \Psi, \quad (10.150)$$

so for the Dirac Lagrangian, we have

$$\begin{aligned} \delta\mathcal{L} &= \delta\left(\bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi\right) \\ &= (\delta\bar{\Psi})\left(i\gamma^\mu\partial_\mu - m\right)\Psi + \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\delta\Psi \\ &= (-a^\sigma\partial_\sigma\bar{\Psi})\left(i\gamma^\mu\partial_\mu - m\right)\Psi + \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)(-a^\sigma\partial_\sigma\Psi) \\ &= -a^\sigma\partial_\sigma\mathcal{L} \\ &= \partial_\sigma(-a^\sigma\mathcal{L}), \end{aligned} \quad (10.151)$$

i.e. $J^\mu = -a^\mu\mathcal{L}$. To plugging this into the Noether current calculating machine, we have

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} &= \frac{\partial}{\partial(\partial_\mu\Psi)}\left(\bar{\Psi}i\gamma^\sigma\partial_\sigma\Psi - m\bar{\Psi}\Psi\right) \\ &= \bar{\Psi}i\gamma^\mu, \end{aligned} \quad (10.152)$$

and

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})} = 0, \quad (10.153)$$

so

$$\begin{aligned} j^\mu &= (\delta\bar{\Psi})\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}(\delta\Psi) - a^\mu\mathcal{L} \\ &= \bar{\Psi}i\gamma^\mu(-a^\sigma\partial_\sigma\Psi) - a^\sigma\delta^\mu_{\sigma}\mathcal{L} \\ &= -a^\sigma\left(\bar{\Psi}i\gamma^\mu\partial_\sigma\Psi + \delta^\mu_{\sigma}\mathcal{L}\right) \\ &= -a_\nu\left(\bar{\Psi}i\gamma^\mu\partial^\nu\Psi + g^{\mu\nu}\mathcal{L}\right). \end{aligned} \quad (10.154)$$

We can now define an energy-momentum tensor

$$T^{\mu\nu} = \bar{\Psi}i\gamma^\mu\partial^\nu\Psi + g^{\mu\nu}\mathcal{L}. \quad (10.155)$$

A couple things are of notable in this tensor. One is that it is not symmetric, and there's doesn't appear to be any hope of making it so. For example, the space+time components are way different

$$\begin{aligned} T^{0k} &= \bar{\Psi} i \gamma^0 \partial^k \Psi \\ T^{k0} &= \bar{\Psi} i \gamma^k \partial^0 \Psi, \end{aligned} \tag{10.156}$$

so if we want a momentum like creature, we have to use T^{0k} , not T^{k0} . The charge associated with that current is

$$\begin{aligned} Q^k &= \int d^3x \bar{\Psi} i \gamma^0 \partial^k \Psi \\ &= \int d^3x \Psi^\dagger (-i \partial_k) \Psi, \end{aligned} \tag{10.157}$$

or translating from component to vector form

$$\mathbf{P} = \int d^3x \Psi^\dagger (-i \nabla) \Psi, \tag{10.158}$$

which is the how the momentum operator is first stated in [14]. Here the vector notation doesn't have any specific representation, but it is interesting to observe how this is directly related to the massless Dirac Lagrangian

$$\begin{aligned} \mathcal{L}(m=0) &= \bar{\Psi} i \gamma^\mu \partial_\mu \Psi \\ &= \Psi^\dagger i \gamma^\mu \partial_\mu \Psi \\ &= \Psi^\dagger i (\partial_0 + \gamma_0 \gamma^k \partial_k) \Psi \\ &= \Psi^\dagger i (\partial_0 - \gamma_0 \gamma_k \partial_k) \Psi, \end{aligned} \tag{10.159}$$

but since $\gamma_0 \gamma_k$ is a 4×4 representation of the Pauli matrix σ_k ¹¹ Lagrangian itself breaks down into

$$\mathcal{L}(m=0) = \Psi^\dagger i \partial_0 \Psi + \boldsymbol{\sigma} \cdot (\Psi^\dagger (-i \nabla) \Psi), \tag{10.160}$$

components, and lo and behold, out pops the momentum operator density! Some part of this should be expected this since the Dirac equation in momentum space is just $(\not{p} - m)e^{-ip \cdot x} = 0$, so there is an intimate connection with the operator portion and momentum.

¹¹ There is ambiguity as to what order of products $\gamma_0 \gamma_k$, or $\gamma_k \gamma_0$ to pick to represent the Pauli basis ([4] uses $\gamma_k \gamma_0$), but we also have sign ambiguity in assembling a Noether charge from the conserved current, so I don't think that matters.

Theorem 10.20: Creation and annihilation form of the momentum operator.

The momentum operator can be written as

$$\mathbf{P} = \sum_{s=1}^2 \int \frac{d^3 q}{(2\pi)^3} \mathbf{p} (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s).$$

To derive this, we have to assign meaning to $\mathbf{P} = \int \Psi^\dagger (-i\nabla) \Psi$. There is an implied basis for these vectors that presumably commutes with Ψ, Ψ^\dagger . Let's suppose that we have a standard orthonormal basis for \mathbb{R}^3 $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, so that our two vectors can be written.

$$\begin{aligned} \mathbf{P} &= \sum_{k=1}^3 \mathbf{e}_k P^k \\ \nabla &= \sum_{k=1}^3 \mathbf{e}_k \partial_k = - \sum_{k=1}^3 \mathbf{e}_k \partial^k. \end{aligned} \tag{10.161}$$

Now we can express the momentum operator in coordinate form (sums implied), as

$$\mathbf{e}_k P^k = \int d^3 x \Psi^\dagger (i \mathbf{e}_k \partial^k) \Psi, \tag{10.162}$$

so if we can commute these assumed basis elements \mathbf{e}_k with Ψ^\dagger we have

$$P^k = i \int d^3 x \Psi^\dagger \partial^k \Psi. \tag{10.163}$$

Note that this disagrees with [8], but I believe it is correct (and it works).

Inserting the field representations ([14] eq. 3.99, 3.100)

$$\begin{aligned} \Psi(x) &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \sum_{s=1}^2 (a_{\mathbf{p}}^s u^s(p) e^{-ip \cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip \cdot x}) \\ \bar{\Psi}(x) &= \int \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} \sum_{r=1}^2 (b_{\mathbf{q}}^r \bar{v}^r(q) e^{-iq \cdot x} + a_{\mathbf{q}}^{r\dagger} \bar{u}^r(q) e^{iq \cdot x}), \end{aligned} \tag{10.164}$$

we can now compute

$$\begin{aligned}
 P^k &= i \int \Psi^\dagger \partial^k \Psi \\
 &= i \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 \sqrt{2\omega_{\mathbf{p}} 2\omega_{\mathbf{q}}}} \sum_{r,s=1}^2 \left(b_{\mathbf{q}}^r \bar{v}^r(q) e^{-iq \cdot x} + a_{\mathbf{q}}^{r\dagger} \bar{u}^r(q) e^{iq \cdot x} \right) \gamma^0(i p^k) \left(-a_{\mathbf{p}}^s u^s(p) e^{-ip \cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip \cdot x} \right) \\
 &= \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} p^k \sum_{r,s=1}^2 \left(+b_{-\mathbf{q}}^r a_{\mathbf{p}}^s \bar{v}^r(-\mathbf{p}) \gamma^0 u^s(\mathbf{p}) e^{-2i\omega_{\mathbf{p}} t} - a_{-\mathbf{q}}^{r\dagger} b_{\mathbf{p}}^{s\dagger} \bar{u}^r(-\mathbf{p}) \gamma^0 v^s(\mathbf{p}) e^{2i\omega_{\mathbf{p}} t} \right. \\
 &\quad \left. - b_{\mathbf{p}}^r b_{\mathbf{p}}^{s\dagger} \bar{v}^r(\mathbf{p}) \gamma^0 v^s(\mathbf{p}) + a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s \bar{u}^r(\mathbf{p}) \gamma^0 u^s(\mathbf{p}) \right).
 \end{aligned} \tag{10.165}$$

Using

$$\begin{aligned}
 v^{r\dagger}(-\mathbf{p}) u^s(\mathbf{p}) &= u^{r\dagger}(\mathbf{p}) v^s(-\mathbf{p}) = 0 \\
 u^{r\dagger}(\mathbf{p}) u^s(\mathbf{p}) &= v^{r\dagger}(\mathbf{p}) v^s(\mathbf{p}) = 2\omega_{\mathbf{p}} \delta^{sr},
 \end{aligned} \tag{10.166}$$

(theorem 10.15, eq. (10.117)) the frequency dependent cross terms are killed¹², and the rest simplify to give

$$P^k = \int \frac{d^3 p}{(2\pi)^3} p^k \sum_{r=1}^2 \left(\left(+b_{\mathbf{p}}^{r\dagger} b_{\mathbf{p}}^r + (2\pi)^3 \delta^{(3)}(0) \right) + a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^r \right). \tag{10.167}$$

This has a vacuum term that can be ignored, so a final multiplication with and sum over the basis vectors \mathbf{e}_k , completes the proof.

Theorem 10.21: Momentum operator eigenvalues.

The eigenvalues of the momentum operator with respect to single particle momentum states are just those momenta

$$\mathbf{P} a_{\mathbf{q}}^{s\dagger} |0\rangle = \mathbf{q} (a_{\mathbf{q}}^{s\dagger} |0\rangle).$$

Here's a partial proof, introducing a state associated with a fermion creation operator

$$|\mathbf{q}, r\rangle = \sqrt{2\omega_{\mathbf{q}}} a_{\mathbf{q}}^{r\dagger} |0\rangle. \tag{10.168}$$

¹² For the Klein-Gordon scalar field we had to work much harder to argue those cross terms away in the momentum operator.

The action of the momentum operator on such a state is

$$\begin{aligned}
 \mathbf{P}|\mathbf{q}, r\rangle &= \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} (b_{\mathbf{p}}^{r\dagger} b_{\mathbf{p}}^r + a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^r) \sqrt{2\omega_{\mathbf{q}}} a_{\mathbf{q}}^{r\dagger} |0\rangle \\
 &= \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^{r\dagger} (-a_{\mathbf{q}}^{r\dagger} a_{\mathbf{p}}^r + \delta^{rs} (2\pi)^3 \delta^{(3)}(\mathbf{q} - \mathbf{p})) \sqrt{2\omega_{\mathbf{q}}} |0\rangle \\
 &= \mathbf{q} a_{\mathbf{q}}^{r\dagger} \sqrt{2\omega_{\mathbf{q}}} |0\rangle \\
 &= \mathbf{q} |\mathbf{q}, r\rangle.
 \end{aligned} \tag{10.169}$$

Clearly the same argument holds for anti-fermion states.

10.19 ROTATION SYMMETRIES: ANGULAR MOMENTUM OPERATOR.

Under Lorentz transformation, including rotations:

$$\begin{aligned}
 \Psi(x) &\rightarrow \Psi'(x') = \Lambda_{1/2} \Psi(x) \\
 \delta\Psi(x) &= \Psi'(x) - \Psi(x) \\
 \Lambda_{1/2} &= e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}} \approx 1 - \frac{i\omega_{\mu\nu}}{2} S^{\mu\nu} \\
 \Psi'(x) &= \Lambda_{1/2} \Psi(\Lambda^{-1}x).
 \end{aligned} \tag{10.170}$$

For a rotation around $\hat{\mathbf{z}}$ only ω_{12} is non-zero. We also have $S^{12} = S^{21}$ and

$$\begin{aligned}
 S^{12} &= \frac{i}{4} [\gamma^1, \gamma^2] \\
 &= \frac{i}{2} \gamma^1 \gamma^2 \\
 &= \frac{i}{2} \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \\
 &= \frac{i}{2} \begin{bmatrix} -\sigma^1 \sigma^2 & 0 \\ 0 & -\sigma^1 \sigma^2 \end{bmatrix} \\
 &= \frac{i}{2} \begin{bmatrix} -i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix}
 \end{aligned} \tag{10.171}$$

so

$$-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = -i\omega_{12}S^{12} = -\frac{i}{2}\omega_{12}\begin{bmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{bmatrix}. \quad (10.172)$$

If we let $\omega_{12} = \alpha$, we have

$$\Lambda_{1/2} = 1 - \frac{i\alpha}{2}\begin{bmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{bmatrix}. \quad (10.173)$$

To compute the in the field we also need

$$\begin{aligned} \Psi(\Lambda^{-1}x) &= \Psi(x^\mu - \omega^\mu{}_\nu x^\nu) \\ &= \Psi(x) - a^\sigma \partial_\sigma \Psi(x) \Big|_{a^\sigma = \omega^\sigma{}_\nu x^\nu} \\ &= \Psi(x) - \omega^\sigma{}_\nu x^\nu \partial_\sigma \Psi(x) \\ &= \Psi(x) - \omega^{\sigma\nu} x_\nu \partial_\sigma \Psi(x) \\ &= \Psi(x) - \omega^{\sigma\nu} x_\nu \partial_\sigma \Psi(x) \\ &= \Psi(x) - \sum_{\sigma < \nu} \omega^{\sigma\nu} (x_\nu \partial_\sigma \Psi(x) - x_\sigma \partial_\nu \Psi(x)) \\ &= \Psi(x) - \alpha (x_2 \partial_1 \Psi(x) - x_1 \partial_2 \Psi(x)) \\ &= \Psi(x) - \alpha (x^1 \partial_2 \Psi(x) - x^2 \partial_1 \Psi(x)) \\ &= \Psi(x) - \alpha (\mathbf{x} \times \nabla)_z \Psi(x). \end{aligned} \quad (10.174)$$

We can now compute the variation of the field

$$\begin{aligned} \delta\Psi &= \Lambda_{1/2}\Psi(\Lambda^{-1}x) - \Psi(x) \\ &= \left(1 - \frac{i\alpha}{2}\begin{bmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{bmatrix}\right)(\Psi(x) - \alpha(\mathbf{x} \times \nabla)_z \Psi(x)) - \Psi(x) \\ &= -\frac{i\alpha}{2}\begin{bmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{bmatrix}\Psi - \alpha(\mathbf{x} \times \nabla)_z \Psi(x) + O(\alpha^2). \end{aligned} \quad (10.175)$$

Because the Dirac Lagrangian is Lorentz invariant, the Noether current has no J^μ term, and is

$$\begin{aligned} j_\alpha^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \delta\Psi \\ &= (\bar{\Psi} i \gamma^\mu) \left(-\frac{i\alpha}{2}\begin{bmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{bmatrix} - \alpha(\mathbf{x} \times \nabla)_z \right) \Psi. \end{aligned} \quad (10.176)$$

In particular, the conserved charge (setting $\alpha = 1$) is

$$J^0 = \int d^3x (\Psi^\dagger) \left(\frac{1}{2} \begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix} - i(\mathbf{x} \times \nabla)_z \right) \Psi. \quad (10.177)$$

Generalizing to arbitrary rotation orientation, this can be written out as

$$\mathbf{J} = \int d^3x \Psi^\dagger(x) \left(\underbrace{\mathbf{x} \times (-i\nabla)}_{\text{orbital angular momentum}} + \frac{1}{2} \underbrace{\mathbf{1} \otimes \boldsymbol{\sigma}}_{\text{spin angular momentum}} \right) \Psi, \quad (10.178)$$

where

$$\mathbf{1} \otimes \boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix}, \quad (10.179)$$

and where the orbital and spin angular momenta have been called out. A nice video treatment of this topic can be found in [13].

For the rest frame of a particle (zero momentum), [14] makes an argument that

$$\begin{aligned} J^3 a_{\mathbf{p}}^{s\dagger} |0\rangle &= \pm \frac{1}{2} a_{\mathbf{p}}^{s\dagger} |0\rangle \Big|_{\mathbf{p}=0} \\ J^3 b_{\mathbf{p}}^{s\dagger} |0\rangle &= \mp \frac{1}{2} b_{\mathbf{p}}^{s\dagger} |0\rangle \Big|_{\mathbf{p}=0}. \end{aligned} \quad (10.180)$$

where the + is for $s = 1$ and the – is for $s = 2$. The eigenvectors of the angular momentum operator are the single particle states, with eigenvalues $\pm 1/2$, where the sign of the eigenvalues toggles for anti-fermions.

10.20 $U(1)_V$ SYMMETRY: CHARGE!

We also have a $U(1)$ global symmetry which implies charge. If we let

$$\begin{aligned} \Psi &\rightarrow e^{i\alpha} \Psi \\ \bar{\Psi} &\rightarrow e^{-i\alpha} \bar{\Psi}, \end{aligned} \quad (10.181)$$

then

$$\begin{aligned}
 j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \delta \Psi \\
 &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} i \alpha \Psi \\
 &= \bar{\Psi} i \gamma^\mu i \alpha \Psi \\
 &= -\bar{\Psi} \gamma^\mu \alpha \Psi \\
 &\equiv -\alpha J^\mu,
 \end{aligned} \tag{10.182}$$

that is

$$J^\mu = \bar{\Psi} \gamma^\mu \Psi \tag{10.183}$$

Define the charge as

$$\begin{aligned}
 Q &= \int d^3x J^0 \\
 &= \int d^3x \bar{\Psi} \gamma^0 \Psi \\
 &= \int d^3x \Psi^\dagger \Psi \\
 &= \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} \sum_{r,s=1}^2 \left(+b_{-\mathbf{q}}^r a_{\mathbf{p}}^s \bar{v}^r(-\mathbf{p}) \gamma^0 u^s(\mathbf{p}) e^{-2i\omega_{\mathbf{p}}t} + a_{-\mathbf{q}}^{r\dagger} b_{\mathbf{p}}^{s\dagger} \bar{u}^r(-\mathbf{p}) \gamma^0 v^s(\mathbf{p}) e^{2i\omega_{\mathbf{p}}t} \right. \\
 &\quad \left. + b_{\mathbf{p}}^r b_{\mathbf{p}}^{s\dagger} \bar{v}^r(\mathbf{p}) \gamma^0 v^s(\mathbf{p}) + a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s \bar{u}^r(\mathbf{p}) \gamma^0 u^s(\mathbf{p}) \right) \\
 &= \int \frac{d^3q}{(2\pi)^3} \sum_{s=1}^2 \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s \right),
 \end{aligned} \tag{10.184}$$

where the expansion of $\Psi^\dagger \Psi$ was lifted from eq. (10.165) (removing the p^k 's and flipping all signs positive), and where any charge associated with the Dirac sea has been dropped.

This charge operator characterizes the a, b operators. a particles have charge +1, and b particles have charge -1, or vice-versa depending on convention.

- a : call it an electron.
- b : call it an positron.

Each come with spin up and down variations.

10.21 $U(1)_A$ SYMMETRY: WHAT WAS THE CHARGE FOR THIS ONE CALLED?

There are two sets of $U(1)$ symmetries, the first called a vector symmetry (above)

$$U(1)_V : \Psi \rightarrow e^{i\alpha} \Psi, \quad (10.185)$$

where α is scalar valued. The other $U(1)$ symmetry is called an axial symmetry¹³

$$U(1)_A : \Psi \rightarrow e^{i\alpha\gamma_5} \Psi, \quad (10.186)$$

where

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (10.187)$$

See [uvspinors.nb](#) for a proof.

Observe that $\gamma_5^\dagger = \gamma_5$

$$\begin{aligned} \gamma_5^\dagger &= -i\gamma_3^\dagger\gamma_2^\dagger\gamma_1^\dagger\gamma_0^\dagger \\ &= -i\gamma^0\gamma_3\gamma^0\gamma^0\gamma_2\gamma^0\gamma^0\gamma_1\gamma^0\gamma_0 \\ &= -i\gamma^0\gamma_3\gamma_2\gamma_1 \\ &= \gamma^5. \end{aligned} \quad (10.188)$$

It can also be shown that

$$\{\gamma_5, \gamma^\mu\} = 0. \quad (10.189)$$

See [uvspinors.nb](#) for a proof.

Under this transformation

$$\begin{aligned} \bar{\Psi} i\gamma^\mu \partial_\mu \Psi &\rightarrow (\Psi^\dagger e^{-i\alpha\gamma_5} \gamma^0) i\gamma^\mu \partial_\mu (e^{i\alpha\gamma_5} \Psi) \\ &= (\Psi^\dagger \gamma^0 e^{i\alpha\gamma_5}) i e^{-i\alpha\gamma_5} \gamma^\mu \partial_\mu \Psi \\ &= \Psi^\dagger \gamma^0 i\gamma^\mu \partial_\mu \Psi, \end{aligned} \quad (10.190)$$

since the anticommutator property eq. (10.189) implies $e^{i\alpha\gamma_5} \gamma^\mu = \gamma^\mu e^{-i\alpha\gamma_5}$.

Also

$$\begin{aligned} m \bar{\Psi} \Psi &\rightarrow m (\Psi^\dagger e^{-i\alpha\gamma_5}) \gamma^0 (e^{i\alpha\gamma_5} \Psi) \\ &= m \bar{\Psi} e^{2i\alpha\gamma_5} \Psi, \end{aligned} \quad (10.191)$$

We see that for $m \neq 0$ the axial $U(1)$ transformation is only a symmetry when $\alpha = \pi$. This is called the Z_2 subgroup.

¹³ It was pointed out that we should recall that for $m = 0$ electrons and positrons separate, obeying separate equations. A nice presentation of that can be found in [11].

10.22 CPT SYMMETRIES.

Left to us to study up on the interesting stories of

- time reversal
- parity
- charge conjugation

Each of these can be studied as separate symmetries. References include [14], [16], and [8].

10.23 REVIEW.

The following notes follow ch. 1 [14] §5.1 fairly closely (filling in some details, leaving out some others.)

Our Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi, \quad (10.192)$$

which can be consider solved by fields $\Psi(x)$, $\bar{\Psi}(x) = \Psi^\dagger(x)\gamma^0$

$$\Psi(x) = \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip \cdot x} u^s(p) a_{\mathbf{p}}^s + e^{ip \cdot x} v^s(p) a_{\mathbf{p}}^{s\dagger} \right) \quad (10.193a)$$

$$\bar{\Psi}(x) = \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{ip \cdot x} \bar{u}^s(p) a_{\mathbf{p}}^{s\dagger} + e^{-ip \cdot x} \bar{v}^s(p) a_{\mathbf{p}}^s \right) \quad (10.193b)$$

where the creation and annihilation operators satisfy

$$\{a_{\mathbf{p}}^s, a_{\mathbf{q}}^{r\dagger}\} = (2\pi)^3 \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad (10.194a)$$

$$\{b_{\mathbf{p}}^s, b_{\mathbf{q}}^{r\dagger}\} = (2\pi)^3 \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \quad (10.194b)$$

(plus various relations for the u, v 's.)

10.24 PHOTON.

Recall that we identified a number of symmetries

- $SO(1, 3)$
- P, C, T : DIY
- $U(1)_V : \Psi \rightarrow e^{i\alpha} \Psi$
- $U(1)_A$: If $m = 0$, then $U(1)_A : \Psi \rightarrow e^{i\alpha\gamma_5} \Psi$. If $m \neq 0$ only for $\alpha = \pi : \Psi \rightarrow -\Psi$.

Photon interaction can be introduced by utilizing a $U(1)$ gauge field, demanding invariance under $U(1)_V$ with $\alpha = \alpha(x)$. That is

$$\Psi(x) \rightarrow e^{i\alpha(x)} \Psi(x), \quad (10.195)$$

which has derivatives

$$\partial_\mu \Psi(x) \rightarrow e^{i\alpha(x)} \left(\partial_\mu \Psi(x) + i\partial_\mu \alpha(x) \Psi(x) \right). \quad (10.196)$$

Solution. Introduce $A_\mu(x)$, such that under $U(1)_V$ we have

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \quad (10.197)$$

where “e” is a dimensionless coupling constant

$$\begin{aligned} \partial_\mu \Psi(x) &\rightarrow (\partial_\mu + ieA_\mu) \Psi \\ &\rightarrow e^{i\alpha(x)} \left(\partial_\mu \Psi + i\cancel{\partial_\mu \alpha} \Psi - i\cancel{\partial_\mu \alpha} \Psi \right) \end{aligned} \quad (10.198)$$

We’ve now constructed the QED Lagrangian density

$$\mathcal{L}_{\text{QED}} = \bar{\Psi} \left(i\gamma^\mu (\partial_\mu + ieA_\mu) - m \right) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (10.199)$$

We may write this as

$$\mathcal{L}_{\text{QED}} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi}_{\text{Free Lagrangian}} - \underbrace{e \bar{\Psi} \gamma_\mu \Psi A^\mu}_{\text{interaction Lagrangian}} \quad (10.200)$$

We introduce spinor fields Ψ_e and muon fields Ψ_μ , so that the total Lagrangian is now

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}_e (i\gamma^\mu \partial_\mu - m) \Psi_e - e \bar{\Psi}_e \gamma_\mu \Psi_e A^\mu + \bar{\Psi}_\mu (i\gamma^\mu \partial_\mu - m) \Psi_\mu - e \bar{\Psi}_\mu \gamma_\mu \Psi_\mu A^\mu \quad (10.201)$$

- $m_e \sim 0.5 \text{ MeV}$
- $m_\mu \sim 105 \text{ MeV}$

There are also quark fields that we can add into the mix

$$\mathcal{L}_{\text{quarks}} = \sum_q \bar{\Psi}_q (i\gamma^\mu - m_q) \Psi_q + e Q_q \bar{\Psi}_q \gamma^\nu \Psi_q A_\nu \quad (10.202)$$

Quark charges are $Q_q = (2/3, -1/3)$. It turns out that the only way to produce quarks is through (electron?) interaction?

Can also introduce a Fermi interaction

$$\mathcal{L}_{4\text{-Fermi}} = \frac{c}{v^2} \bar{\Psi}_\mu \gamma^\nu (1 - \gamma_5) \Psi_{\nu,\mu} - \bar{\Psi}_e (1 - \gamma_5) \dots \quad (10.203)$$

We now want to do some calculations with the photon interactions from eq. (10.201). In particular, we will study the effects of the $-e \bar{\Psi}_e \gamma_\mu \Psi_e A^\mu$ interaction Lagrangian.

10.25 PROPAGATOR.

Before we can study the interaction, we need to determine the structure of the propagator. For Grassman (anti-commuting) operators

$$T(O_f(x) O'_f(x)) = \Theta(x_0 - x'_0) O_f(x) O_f(x') + \Theta(x'_0 - x_0) O_f(x') O_f(x) \quad (10.204)$$

The propagator can be determined from

$$\langle T(\Psi_\alpha(x) \Psi_\beta(x)) \rangle_0 = D_{F_{\alpha\beta}}(x - y), \quad (10.205)$$

where $\alpha, \beta = 1, 2, 3, 4$.

Referring back to eq. (10.193a), eq. (10.193b), that propagator is

$$\begin{aligned} \langle T(\Psi_\alpha(x) \Psi_\beta(x)) \rangle_0 &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \int \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} (e^{-ip \cdot x} e^{iq \cdot y} \Theta(x_0 - y_0) u_\alpha^s(p) \bar{u}_\beta^r(q) \\ &\quad + e^{ip \cdot x} e^{-iq \cdot y} \Theta(y_0 - x_0) \bar{v}_\beta^s(p) v_\beta^r(q) \langle b_{\mathbf{q}}^s a_{\mathbf{p}}^{r\dagger} \rangle) \\ &= \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} (e^{-ip \cdot (x-y)} \Theta(x_0 - y_0) u_\alpha^s(p) \bar{u}_\beta^r(p) + e^{ip \cdot (x-y)} \Theta(y_0 - x_0) \bar{v}_\beta^s(p) v_\beta^r(p)) \\ &= \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} (e^{-ip \cdot x} \Theta(x_0 - y_0) (\gamma_{\alpha\beta}^\mu p_\mu + m) + e^{ip \cdot x} \Theta(y_0 - x_0) (\gamma_{\alpha\beta}^\mu p_\mu - m)) \end{aligned} \quad (10.206)$$

where $\gamma_{\alpha\beta}^\mu$ are the α, β components of the gamma matrices. Now we can replace the p_μ 's with derivatives acting on the exponentials

$$\begin{aligned}
\langle T(\Psi_\alpha(x)\Psi_\beta(x)) \rangle_0 &= \Theta(x_0 - y_0)(i\gamma_{\alpha\beta}^\mu \partial_\mu + m) \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip \cdot (x-y)} \\
&\quad - \Theta(y_0 - x_0)(-i\gamma_{\alpha\beta}^\mu \partial_\mu - m) \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip \cdot (x-y)} \\
&= \Theta(x_0 - y_0)(i\gamma_{\alpha\beta}^\mu \partial_\mu + m) D(x-y) - \Theta(y_0 - x_0)(-i\gamma_{\alpha\beta}^\mu \partial_\mu - m) D(y-x) \\
&= (\gamma_{\alpha\beta}^\mu \partial_\mu^{(x)} + m)(\Theta(x_0 - y_0)D(x-y) + \Theta(y_0 - x_0)D(y-x)) \\
&\quad - i\gamma^0 \delta(x^0 - y^0) \overline{D(x-y) - D(y-x)},
\end{aligned} \tag{10.207}$$

where we've killed off a factor that is zero (off the light cone?)

We are left with just an action on the Feynman propagator

$$\langle T(\Psi_\alpha(x)\Psi_\beta(x)) \rangle_0 = (\gamma_{\alpha\beta}^\mu \partial_\mu^{(x)} + m) D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i(\gamma_{\alpha\beta}^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} \tag{10.208}$$

Now that we have a propagator, let's try

$$\mathcal{L}_{\text{int}} = \int dt d^3 x (e \bar{\Psi} \gamma_\mu \Psi A^\mu). \tag{10.209}$$

10.26 FEYNMAN RULES.

We can consider various scattering processes, such as $e^+ e^- \rightarrow \mu^+ \mu^-$ as sketched in fig. 10.6, or $e^+ e^- \rightarrow e^+ e^-$ as sketched in fig. 10.7, or Compton scattering $e^- \gamma \rightarrow e^- \gamma$ as sketched in fig. 10.8.

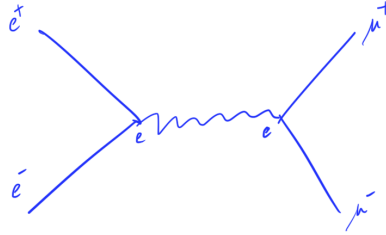


Figure 10.6: Electron, positron decay to muon pairs.

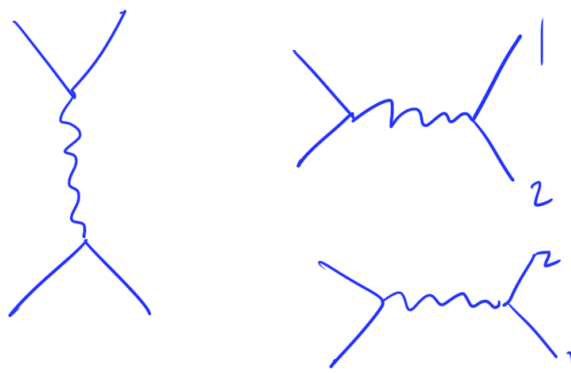


Figure 10.7: Electron, positron collision.

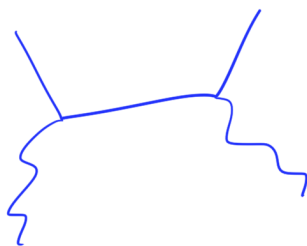


Figure 10.8: Compton scattering.

To do so we need to determine the Feynman rules for fermions. For fermions Ψ and anti-fermions $\bar{\Psi}$ we have

$$\begin{aligned}\overline{\Psi} | \mathbf{p}, s \rangle &= u^s(p) \\ \bar{\Psi} | \mathbf{p}, s \rangle &= \bar{v}^s(p) \\ \langle \mathbf{p}, s | \Psi &= \bar{u}^s(p) \\ \langle \mathbf{p}, s | \bar{\Psi} &= v^s(p),\end{aligned}\tag{10.210}$$

where we mean

$$| \mathbf{p}, s \rangle = a_{\mathbf{p}}^{s\dagger} | 0 \rangle \sqrt{2\omega_{\mathbf{p}}},\tag{10.211}$$

for fermions, and

$$| \mathbf{p}, s \rangle = b_{\mathbf{p}}^{s\dagger} | 0 \rangle \sqrt{2\omega_{\mathbf{p}}},\tag{10.212}$$

for anti-fermions.

The flow of fermion and anti-fermion number charge is designated by arrow direction in the diagram, as in the respective diagrams of fig. 10.9.

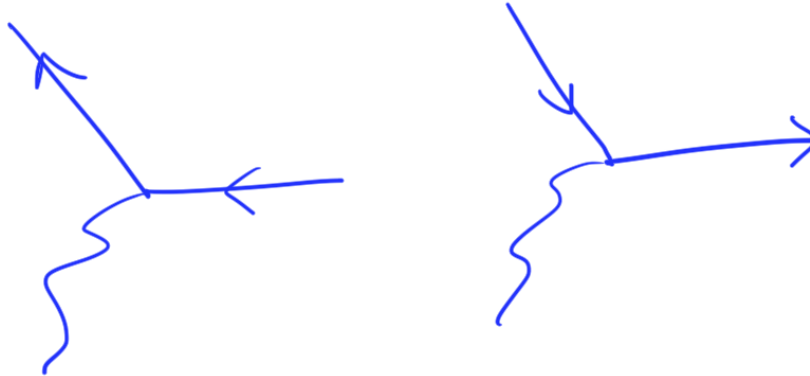


Figure 10.9: Flow of # charge.

The Feynman propagator for fermions is

$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon},\tag{10.213}$$

whereas the photon propagator is

$$\langle A_{\mu} A_{\nu} \rangle = -i \frac{g_{\mu\nu}}{q^2 + i\epsilon}.\tag{10.214}$$

10.27 EXAMPLE: $e^-e^+ \rightarrow \mu^-\mu^+$.

As an example, consider the process sketched in fig. 10.10. Such a process is “ultra-relativistic”,

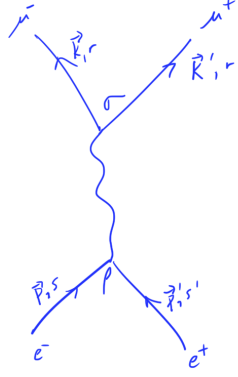


Figure 10.10: $e^-e^+ \rightarrow \mu^-\mu^+$ process.

in that the electron and positron pair must be moving very fast to create muons.

The matrix element is

$$\begin{aligned}
 \langle \mu^+ \mu^- | \bar{\Psi} \gamma^\sigma \Psi A^\sigma A^\rho \bar{\Psi} \gamma^\rho \Psi | e^+ e^- \rangle &= \overbrace{\bar{v}^{s'}(p')}^{\text{incoming anti-electron}} (-ie\gamma^\rho) \underbrace{u^s(p)}_{\text{incoming electron}} \overbrace{\left(\frac{-ig_{\rho\sigma}}{q^2} \right)}^{\text{ignoring } i\epsilon} \bar{u}^r(k) (-ie\gamma^\sigma) v^{r'}(k') \\
 &\quad (10.215)
 \end{aligned}$$

Question: Why are we writing the factors of the matrix element from left to right, corresponding to the right to left reading of the matrix element?

Equation (10.215) reduces to

$$iM = i \frac{e^2}{q^2} \bar{v}^{s'}(p') \gamma^\rho u^s(p) \bar{u}^r(k) \gamma_\rho v^{r'}(k'), \quad (10.216)$$

where the $(2\pi)^4 \delta^{(4)}(\dots)$ term hasn't been made explicit.

We'd like to compute the absolute square of eq. (10.216), and use the following lemma to do so.

Lemma 10.7: Some conjugates.

$$\begin{aligned}
(\bar{v}\gamma^\mu u)^\dagger &= \bar{u}\gamma^\mu v \\
(\bar{u}\gamma^\mu v)^\dagger &= \bar{v}\gamma^\mu u.
\end{aligned}$$

The proof is left to exercise 10.12. Employing this, we have

$$|M|^2 = \frac{e^4}{q^4} \left(\bar{v}^{s'}(k') \gamma_\rho u^r(k) \bar{u}^s(p) \gamma^\rho v^{s'}(p') \right) \times \left(\bar{v}^{s'}(p') \gamma^\mu u^s(p) \bar{u}^r(k) \gamma_\mu v^{r'}(k') \right). \quad (10.217)$$

The problem can be simplified by computing the cross section that sums over all spins, assuming that the states are not polarized (i.e. average over all the up, down states).

Digression

Such an average is related to the density matrix

$$\rho_{\text{in}} = \sum_{ss'} |ss'\rangle \frac{1}{4} \langle ss'|. \quad (10.218)$$

$$\text{tr} \left(e^{iHt} \rho_{\text{in}} e^{iHt} \rho_{\text{f}} |rr'\rangle \langle rr'| \right) \quad (10.219)$$

That is, We want to sum over all the initial and final state polarizations $\frac{1}{4} \sum_{ss'} \sum_{rr'} |M|^2$

$$\begin{aligned}
\frac{1}{4} \sum_{ss', rr'} |M|^2 &= \sum_{ss', rr'} \frac{e^4}{4q^4} \bar{v}^{s'}(k') \gamma_\rho u^r(k) \bar{u}^r(k) \gamma_\mu v^{r'}(k') \bar{u}^s(p) \gamma^\rho v^{s'}(p') \bar{v}^{s'}(p') \gamma^\mu u^s(p) \\
&= \frac{e^4}{4q^4} \sum_{r'} \bar{v}^{r'}(k') \gamma_\rho (\not{k} + m_\mu) \gamma_\mu v^{r'}(k') \times \sum_s \bar{u}^s(p) \gamma^\rho (\not{p}' - m_e) \gamma^\mu u^s(p),
\end{aligned} \quad (10.220)$$

where we first used the freedom to move the $\bar{u}\gamma v, \bar{v}\gamma u$ terms, which are scalars, and then used theorem 10.16 to eliminate the sum over s', r indexes.

Temporarily expressing the remaining factors in coordinates exposes a trace structure. For example

$$\begin{aligned}
 \sum_{r'} \bar{v}^{r'}(k') \gamma_\rho (\not{k} + m_\mu) \gamma_\mu v^{r'}(k') &= \sum_{r'} (\bar{v}^{r'}(k'))_a (\gamma_\rho)_{ab} (\not{k} + m_\mu)_{bc} (\gamma_\mu)_{cd} (v^{r'}(k'))_d \\
 &= \sum_{r'} (v^{r'}(k'))_d (\bar{v}^{r'}(k'))_a (\gamma_\rho)_{ab} (\not{k} + m_\mu)_{bc} (\gamma_\mu)_{cd} \quad (10.221) \\
 &= (\not{k}' - m_\mu)_{da} (\gamma_\rho)_{ab} (\not{k} + m_\mu)_{bc} (\gamma_\mu)_{cd} \\
 &= \text{tr}((\not{k}' - m_\mu) \gamma_\rho (\not{k} + m_\mu) \gamma_\mu),
 \end{aligned}$$

since the cyclic sum of matrix coordinates can be expressed as a trace, namely $\text{tr} ABC = A_{ab} B_{bc} C_{ca}$. We are left with

$$\frac{1}{4} \sum_{ss', rr'} |M|^2 = \frac{e^4}{4q^4} \text{tr}((\not{k}' - m_\mu) \gamma_\nu (\not{k} + m_\mu) \gamma_\mu) \times \text{tr}((\not{p} + m_e) \gamma^\nu (\not{p}' - m_e) \gamma^\mu). \quad (10.222)$$

Each trace is now a product of two, three, or four gamma matrices, which can be reduced using the identities:

Lemma 10.8: Dirac matrix product traces.

$$\begin{aligned}
 \text{tr}(\gamma_\mu \gamma_\nu) &= 4g_{\mu\nu} \\
 \text{tr}(\gamma_\mu \gamma_\nu \gamma_\alpha) &= 0 \\
 \text{tr}(\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta) &= 4(g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\alpha\nu})
 \end{aligned}$$

The proof is left to exercise 10.13.

Utilizing the above, and setting $m_e = 0$ (compared to m_μ) the p, p' dependent trace reduces to

$$\begin{aligned}
 \text{tr}((\not{p} + m_e) \gamma^\nu (\not{p}' - m_e) \gamma^\mu) &= \text{tr}(\not{p} \gamma^\nu \not{p}' \gamma^\mu) \\
 &= p_\alpha p'_\beta \text{tr}(\gamma^\alpha \gamma^\nu \gamma^\beta \gamma^\mu) \\
 &= 4p_\alpha p'_\beta (g^{\alpha\nu} g^{\beta\mu} - g^{\alpha\beta} g^{\nu\mu} + g^{\alpha\mu} g^{\nu\beta}) \\
 &= 4(-p \cdot p' g^{\nu\mu} + p^\nu p'^\mu + p^\mu p'^\nu), \quad (10.223)
 \end{aligned}$$

and the k, k' dependent trace reduces to

$$\begin{aligned}
 \text{tr}((\not{k}' - m_\mu) \gamma_\nu (\not{k} + m_\mu) \gamma_\mu) &= \text{tr}(\not{k}' \gamma_\nu \not{k} \gamma_\mu) - m_\mu^2 \text{tr}(\gamma_\nu \gamma_\mu) + m_\mu \text{tr}(\not{k}' \gamma_\nu \gamma_\mu) - m_\mu \text{tr}(\gamma_\nu \not{k} \gamma_\mu) \\
 &= 4(k'_\alpha k_\beta (g_{\alpha\nu} g_{\beta\mu} - g_{\alpha\beta} g_{\nu\mu} + g_{\alpha\mu} g_{\nu\beta}) - m_\mu^2 g_{\nu\mu}) \\
 &= 4(k'_\nu k_\mu + k'_\mu k_\nu - (k \cdot k' + m_\mu^2) g_{\nu\mu}). \quad (10.224)
 \end{aligned}$$

We can now multiply out the traces and simplify (exercise 10.14) to get

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{q^4} (p \cdot k' p' \cdot k + p \cdot k p' \cdot k' + p \cdot p' m_\mu^2). \quad (10.225)$$

The next task is to consider these four vector dot products from the center of mass frame for the electrons, as sketched in fig. 10.11. Let q represent the total rest frame four momentum

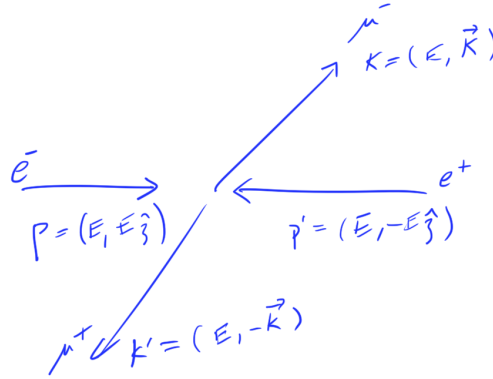


Figure 10.11: Electron center of mass frame.

$$\begin{aligned} q &= p + p' \\ &= (2E, \mathbf{0}), \end{aligned} \quad (10.226)$$

where $q^2 = 4E^2$. We also have

$$\begin{aligned} p \cdot p' &= (E, E\hat{\mathbf{z}}) \cdot (E, -E\hat{\mathbf{z}}) \\ &= E^2 - E^2(\hat{\mathbf{z}} \cdot (-\hat{\mathbf{z}})) \\ &= 2E^2. \end{aligned} \quad (10.227a)$$

$$\begin{aligned} p \cdot k &= (E, E\hat{\mathbf{z}}) \cdot (E, \mathbf{k}) \\ &= E^2 - E \|\mathbf{k}\| \cos \theta, \end{aligned} \quad (10.227b)$$

$$\begin{aligned} p \cdot k' &= (E, E\hat{\mathbf{z}}) \cdot (E, -\mathbf{k}) \\ &= E^2 - (E\hat{\mathbf{z}}) \cdot (-\mathbf{k}) \\ &= E^2 + E \|\mathbf{k}\| \cos \theta \end{aligned} \quad (10.227c)$$

$$\begin{aligned}
p' \cdot k' &= (E, -E\hat{\mathbf{z}}) \cdot (E, -\mathbf{k}) \\
&= E^2 - (-E\hat{\mathbf{z}}) \cdot (-\mathbf{k}) \\
&= E^2 - E \|\mathbf{k}\| \cos \theta
\end{aligned} \tag{10.227d}$$

$$\begin{aligned}
p' \cdot k &= (E, -E\hat{\mathbf{z}}) \cdot (E, \mathbf{k}) \\
&= E^2 - (-E\hat{\mathbf{z}}) \cdot \mathbf{k} \\
&= E^2 + E \|\mathbf{k}\| \cos \theta,
\end{aligned} \tag{10.227e}$$

but

$$\mathbf{k}^2 = E^2 - m_\mu^2, \tag{10.228}$$

or

$$\|\mathbf{k}\| = E \sqrt{1 - \frac{m_\mu^2}{E^2}}. \tag{10.229}$$

We can now put the pieces back together and almost have the non-polarized cross section

$$\begin{aligned}
\frac{1}{4} \sum_{\text{spins}} |M|^2 &= \frac{8e^4}{(4E^2)^2} \left((E^2 + E \|\mathbf{k}\| \cos \theta)^2 + (E^2 - E \|\mathbf{k}\| \cos \theta)^2 + m_\mu^2 2E^2 \right) \\
&= \frac{e^4}{2} \left(\left(1 + \sqrt{1 - \frac{m_\mu^2}{E^2}} \cos \theta \right)^2 + \left(1 - \sqrt{1 - \frac{m_\mu^2}{E^2}} \cos \theta \right)^2 + 2 \frac{m_\mu^2}{E^2} \right) \\
&= \frac{e^4}{2} \left(2 + 2 \left(1 - \frac{m_\mu^2}{E^2} \right) \cos^2 \theta + 2 \frac{m_\mu^2}{E^2} \right),
\end{aligned} \tag{10.230}$$

or

$$\boxed{\frac{1}{4} \sum_{\text{spins}} |M|^2 = e^4 \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) \cos^2 \theta \right).} \tag{10.231}$$

The total (average polarization) differential cross section ([14] eq. 4.84), is

$$\frac{d\sigma}{d\Omega_{\text{CM}}} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{CM}}} \frac{1}{4} \sum_{\text{spins}} |M|^2. \tag{10.232}$$

Plug in $E_A = E_B = 2E_{\text{CM}}$, $v_A - v_B \sim 2c = 2$, $e^2 = 4\pi\alpha$, and eq. (10.231) for

$$\begin{aligned} \frac{d\sigma}{d\Omega_{\text{CM}}} &= \frac{1}{E_{\text{CM}}^2(2)} \frac{1}{(4\pi)^2 E_{\text{CM}}} \frac{E_{\text{CM}}}{2} \sqrt{1 - \frac{m_\mu^2}{E^2}} (4\pi\alpha)^2 \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) \cos^2 \theta \right) \\ &= \frac{\alpha^2}{4E_{\text{CM}}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) \cos^2 \theta \right). \end{aligned} \quad (10.233)$$

Integrating to find the total cross section we have

$$\begin{aligned} \sigma_{\text{total}} &= \int d\Omega \frac{d\sigma}{d\Omega} \\ &= 2\pi \int_{-1}^1 d\cos\theta \frac{\alpha^2}{4E_{\text{CM}}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2} \right) \cos^2 \theta \right) \\ &= \frac{2\pi\alpha^2}{4E_{\text{CM}}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(2 \left(1 + \frac{m_\mu^2}{E^2} \right) + \left(1 - \frac{m_\mu^2}{E^2} \right) \int_{-1}^1 u^2 du \right) \\ &= \frac{4\pi\alpha^2}{4E_{\text{CM}}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{m_\mu^2}{E^2} + \frac{1}{3} \left(1 - \frac{m_\mu^2}{E^2} \right) \right), \end{aligned} \quad (10.234)$$

or

$$\sigma_{\text{total}} = \frac{4\pi\alpha^2}{3E_{\text{CM}}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right), \quad (10.235)$$

where $E_{\text{CM}} = 2E$.

At the start of the year dimensional analysis was used to state the total cross section, which was determined to have the form

$$\sigma_{\text{total}} \sim \frac{\alpha^2}{s}, \quad (10.236)$$

whereas for $E \gg m_\mu$ we've now found

$$\sigma_{\text{total}} = \frac{4\pi\alpha^2}{3E_{\text{CM}}^2}. \quad (10.237)$$

Three months of work has gained us an additional factor of 4/3!

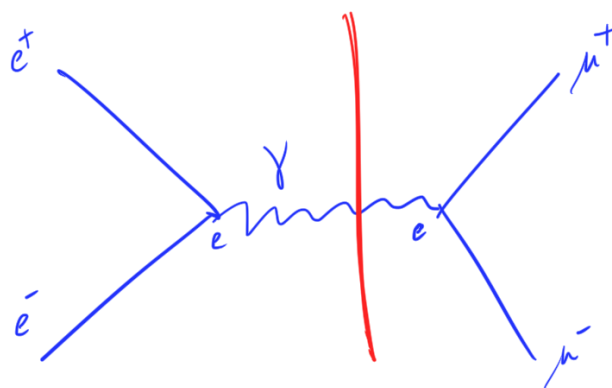


Figure 10.12: Electron and muon halves of the diagram

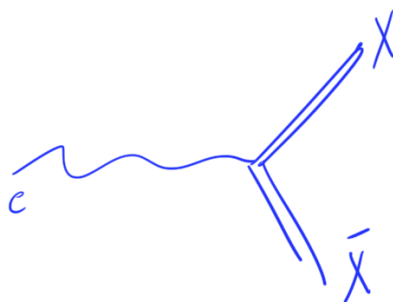


Figure 10.13: Alternate charged pair production.

10.28 MEASUREMENT OF INTERMEDIATE QUARK SCATTERING PROCESSES.

In the diagram that we are working from for the $e^-e^+ \rightarrow \mu^-\mu^+$ process, we can replace the muon half of the interaction (fig. 10.12) with anything else that is charged, as sketched in fig. 10.13. In particular, quark pairs from QCD are possible at high energies ($m_\mu \sim 105 \text{ MeV}$) and such products can be measured indirectly. Quarks were theorized to be strong force carriers, an intermediate stage similar to the photon propagators of QED, connecting two branches of a diagram, as sketched in fig. 10.14. If one hypothesizes a proportionality relationship between

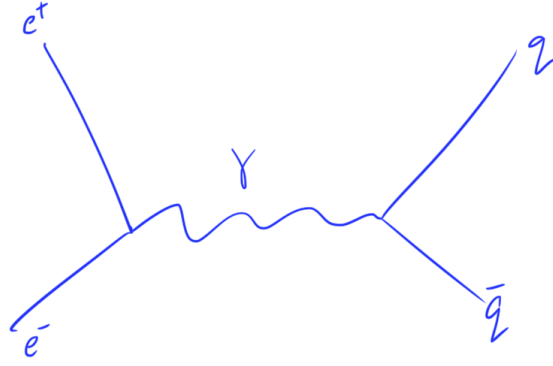


Figure 10.14: Quark pair production.

the hadron (i.e. muon) and quark scattering cross sections

$$\sigma_{\text{total}}(e^-e^+ \rightarrow \text{hadrons}) \propto \sigma_{\text{total}}(e^-e^+ \rightarrow \text{quarks}), \quad (10.238)$$

the ratio between the two

$$\begin{aligned} R &= \frac{\sigma_{\text{total}}(e^-e^+ \rightarrow \text{quarks})}{\sigma_{\text{total}}(e^-e^+ \rightarrow \text{hadrons})} \\ &= 3 \sum_q (Q_q)^4, \end{aligned} \quad (10.239)$$

can be measured, and such measurement was deemed to be one of the validations of the QCD theory. The $3 \sum_q (Q_q)^4$ expression includes a 3 that is related to quark “color”, and a sum over only the quark charges q that are light enough to be produced. [14] fig. 5.3 includes an experimental depiction of such a measurement, which has a step function form roughly like fig. 10.15, where the steps occur at the energy levels that are sufficient to produce new quarks.

END-REDACTION

Exercise 10.2 Show that $\bar{\Psi}\Psi$ is a Lorentz scalar.

Answer for Exercise 10.2

The Lorentz property follows from eq. (10.63)

$$\begin{aligned}\bar{\Psi}\Psi &\rightarrow (\bar{\Psi}\Lambda_{1/2}^{-1})(\Lambda_{1/2}\Psi) \\ &= \bar{\Psi}\Psi,\end{aligned}\tag{10.240}$$

The scalar nature of this product can be seen easily by expansion.

$$\begin{aligned}
\bar{\Psi}\Psi &= \Psi^\dagger \gamma^0 \Psi \\
&= \begin{bmatrix} \Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{bmatrix} \\
&= \begin{bmatrix} \Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^* \end{bmatrix} \begin{bmatrix} \Psi_3 \\ \Psi_4 \\ \Psi_1 \\ \Psi_2 \end{bmatrix} \\
&= \Psi_1^* \Psi_3 + \Psi_2^* \Psi_4 + \Psi_3^* \Psi_1 + \Psi_4^* \Psi_2 \\
&= 2 \operatorname{Re}(\Psi_1^* \Psi_3 + \Psi_2^* \Psi_4).
\end{aligned} \tag{10.241}$$

Clearly any individual $\Psi^\dagger \gamma^\mu \Psi$ product will also be a scalar.

Exercise 10.3 Show that $\bar{\Psi}\gamma^\mu\Psi$ transforms as a four vector.

Answer for Exercise 10.3

$$\begin{aligned}
\bar{\Psi} \gamma^\mu \Psi &\rightarrow (\bar{\Psi} \Lambda_{1/2}^{-1}) \gamma^\mu (\Lambda_{1/2} \Psi) \\
&= \bar{\Psi} (\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2}) \Psi \\
&= \bar{\Psi} (\Lambda^\mu{}_\nu \gamma^\nu) \Psi \\
&= \Lambda^\mu{}_\nu \bar{\Psi} \gamma^\nu \Psi.
\end{aligned}
\tag{10.242}$$

Exercise 10.4 Vary the Dirac action definition 10.4.

Answer for Exercise 10.4

From the action we find

$$\delta S = \int d^4x \delta \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi + \int d^4x \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \delta \Psi. \quad (10.243)$$

There are two ways to deal with this. One (somewhat unsatisfactory seeming to me) is to treat both $\delta \bar{\Psi}$ and $\delta \Psi$ as independent variations, requiring that $\delta S = 0$ for any such variations. In that case we find that $(i\gamma^\mu \partial_\mu - m) \Psi = 0$ if the total variation of the action is zero. That leaves the somewhat awkward question of what to do with the $0 = \int d^4x \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \delta \Psi$ constraint. However, that question can be resolved by observing that these two contributions to the variation are not independent. In particular

$$\begin{aligned} \left(\int d^4x \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \delta \Psi \right)^\dagger &= \int d^4x \delta \Psi^\dagger \left(-i(\gamma^\mu)^\dagger \partial_\mu - m \right) \gamma^0 \Psi \\ &= \int d^4x \delta \Psi^\dagger \gamma^0 \left(-i\gamma^0 (\gamma^\mu)^\dagger \gamma^0 \overleftarrow{\partial}_\mu - m \right) \Psi \\ &= \int d^4x \delta \bar{\Psi} \left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m \right) \Psi \\ &= \int d^4x \left(\partial_\mu (-i\delta \bar{\Psi} \gamma^\mu \Psi) - (-i\delta \bar{\Psi} \gamma^\mu \partial_\mu \Psi) - m \delta \bar{\Psi} \Psi \right) \\ &= \int d^4x \delta \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi, \end{aligned} \quad (10.244)$$

where the boundary integral has been assumed to be zero. This shows that the total variation is

$$\delta S = \int d^4x \left(\delta \bar{\Psi} D \Psi + (\delta \bar{\Psi} D \Psi)^\dagger \right), \quad (10.245)$$

where

$$D = i\gamma^\mu \partial_\mu - m, \quad (10.246)$$

represents the Dirac operator. Requiring that the action variation $\delta S = 0$ is zero for all $\delta \bar{\Psi}$, means that $D\Psi = 0$, which proves eq. (10.69).

Given that the action itself is real, it makes sense for it's variation to be real, as demonstrated above. A nice side effect of demonstrating this is the removal of the redundant variation variable.

Exercise 10.5 Exponential form of $\sqrt{p \cdot \sigma}$, $\sqrt{p \cdot \bar{\sigma}}$.

In [12], Prof Osmond explicitly boosts a $u^s(p_0)$ Dirac spinor from the rest frame with rest frame energy p_0 , and claims

$$\begin{aligned}\sqrt{m}e^{-\frac{1}{2}\eta\sigma^3} &= \sqrt{p \cdot \sigma} \\ \sqrt{m}e^{\frac{1}{2}\eta\sigma^3} &= \sqrt{p \cdot \bar{\sigma}},\end{aligned}\tag{10.247}$$

for the components of $u^s(\Lambda p_0)$.

Validate these identities by squaring both sides.

Answer for Exercise 10.5

First

$$e^{\pm\frac{1}{2}\eta\sigma^3} = \cosh\left(\frac{1}{2}\eta\sigma^3\right) \pm \sinh\left(\frac{1}{2}\eta\sigma^3\right)\sigma^3,\tag{10.248}$$

which squares to ([uvspinors.nb](#))

$$\left(e^{\pm\frac{1}{2}\eta\sigma^3}\right)^2 = \begin{bmatrix} e^{\pm\eta} & 0 \\ 0 & e^{\mp\eta} \end{bmatrix}.\tag{10.249}$$

Explicitly boosting the rest energy p_0 gives

$$\begin{aligned}\begin{bmatrix} p_0 \\ 0 \end{bmatrix} &\rightarrow \begin{bmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{bmatrix} \begin{bmatrix} p_0 \\ 0 \end{bmatrix} \\ &= p_0 \begin{bmatrix} \cosh \eta \\ \sinh \eta \end{bmatrix},\end{aligned}\tag{10.250}$$

so after the boost

$$\begin{aligned}p \cdot \sigma &\rightarrow p_0 (\cosh \eta - \sinh \eta \sigma^3) \\ &= p_0 \begin{bmatrix} \cosh \eta - \sinh \eta & 0 \\ 0 & \cosh \eta + \sinh \eta \end{bmatrix} \\ &= p_0 \begin{bmatrix} e^{-\eta} & 0 \\ 0 & e^{\eta} \end{bmatrix},\end{aligned}\tag{10.251}$$

where $p_0 = m$ is still the rest frame energy. However, according to eq. (10.249) this is exactly

$$\left(\sqrt{m}e^{-\frac{1}{2}\eta\sigma^3}\right)^2\tag{10.252}$$

Since $p \cdot \bar{\sigma}$ flips the signs of the spatial momentum, we have shown that

$$\begin{aligned} \left(\sqrt{m} e^{-\frac{1}{2} \eta \sigma^3} \right)^2 &= p \cdot \sigma \\ \left(\sqrt{m} e^{\frac{1}{2} \eta \sigma^3} \right)^2 &= p \cdot \bar{\sigma}, \end{aligned} \tag{10.253}$$

which isn't a full proof of the claimed result (i.e. the most general orientation isn't considered), but at least validates the claim.

Exercise 10.6 **Verify $v(p)$ solution.**

Prove theorem 10.12.

Answer for Exercise 10.6

Let $D = (i\gamma^\mu \partial_\mu - m)$ represent the Dirac operator. Applying to $e^{ip \cdot x}$ we have

$$\begin{aligned} D e^{ip \cdot x} &= (i\gamma^\mu \partial_\mu - m) e^{ip_\mu x^\mu} \\ &= -(\gamma^\mu p_\mu + m) e^{ip \cdot x} \\ &= -\left(m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + p_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_k \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \right) e^{ip \cdot x} \\ &= -\begin{bmatrix} m & p_0 \sigma^0 + p_k \sigma^k \\ p_0 \sigma^0 - p_k \sigma^k & m \end{bmatrix} e^{ip \cdot x} \\ &= -\begin{bmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{bmatrix} e^{ip \cdot x}. \end{aligned} \tag{10.254}$$

We are now set to apply the Dirac operator to the claimed solution from theorem 10.12.

$$\begin{aligned} D v(p) &= \begin{bmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix} e^{ip \cdot x} \\ &= -\begin{bmatrix} (m \sqrt{p \cdot \sigma} - p \cdot \sigma \sqrt{p \cdot \bar{\sigma}}) \eta \\ (p \cdot \bar{\sigma} \sqrt{p \cdot \sigma} - m \sqrt{p \cdot \bar{\sigma}}) \eta \end{bmatrix} e^{ip \cdot x} \\ &= \begin{bmatrix} \sqrt{p \cdot \sigma} (m - \sqrt{p \cdot \sigma} p \cdot \bar{\sigma}) \eta \\ \sqrt{p \cdot \bar{\sigma}} (\sqrt{p \cdot \bar{\sigma}} p \cdot \sigma - m) \eta \end{bmatrix} e^{ip \cdot x} \\ &= \begin{bmatrix} \sqrt{p \cdot \sigma} (m - \sqrt{m^2}) \eta \\ \sqrt{p \cdot \bar{\sigma}} (\sqrt{m^2} - m) \eta \end{bmatrix} e^{ip \cdot x} \\ &= 0. \end{aligned} \tag{10.255}$$

Exercise 10.7 $v(p)$ normalization.

Prove theorem 10.13.

Answer for Exercise 10.7

Expanding the matrices gives

$$\begin{aligned}
\bar{v}^r v^s &= v^{r\dagger} \gamma^0 v^s \\
&= \begin{bmatrix} \eta^{rT} \sqrt{p \cdot \sigma} & -\eta^{rT} \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix} \\
&= \begin{bmatrix} \eta^{rT} \sqrt{p \cdot \sigma} & -\eta^{rT} \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} -\sqrt{p \cdot \bar{\sigma}} \eta^s \\ \sqrt{p \cdot \sigma} \eta^s \end{bmatrix} \\
&= -\eta^{rT} \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} \eta^s - \eta^{rT} \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \eta^s \\
&= \delta^{rs} 2 \sqrt{m^2} \\
&= 2m \delta^{rs},
\end{aligned} \tag{10.256}$$

and

$$\begin{aligned}
v^{r\dagger} v^s &= \begin{bmatrix} \eta^{rT} \sqrt{p \cdot \sigma} & -\eta^{rT} \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix} \\
&= \eta^{rT} (p \cdot \sigma) \eta^s + \eta^{rT} (p \cdot \bar{\sigma}) \eta^s \\
&= \delta^{rs} (p_0 - \mathbf{p} \cdot \boldsymbol{\sigma} + p_0 + \mathbf{p} \cdot \boldsymbol{\sigma}) \\
&= 2p_0 \delta^{rs}.
\end{aligned} \tag{10.257}$$

Exercise 10.8 $\bar{u}v, \bar{v}u$ relations.

Prove theorem 10.14.

Answer for Exercise 10.8

We need only expand the matrix products

$$\begin{aligned}
\bar{u}^r v^s &= \begin{bmatrix} \zeta^{rT} \sqrt{p \cdot \bar{\sigma}} & \zeta^{rT} \sqrt{p \cdot \sigma} \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{bmatrix} \\
&= m \zeta^{rT} \eta^s - m \zeta^{rT} \eta^s \\
&= 0,
\end{aligned} \tag{10.258}$$

and

$$\begin{aligned}
 \bar{v}^r u^s &= \begin{bmatrix} -\eta^{rT} \sqrt{p \cdot \bar{\sigma}} & \eta^{rT} \sqrt{p \cdot \sigma} \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \bar{\sigma}} \zeta^s \\ \sqrt{p \cdot \sigma} \bar{\zeta}^s \end{bmatrix} \\
 &= -m \eta^{rT} \zeta^s + m \eta^{rT} \bar{\zeta}^s \\
 &= 0,
 \end{aligned} \tag{10.259}$$

Exercise 10.9 **Dagger orthonormality conditions.**

Prove theorem 10.15.

Answer for Exercise 10.9

$$\begin{aligned}
 u^{r\dagger}(\mathbf{p}) v^s(-\mathbf{p}) &= \begin{bmatrix} \zeta^{rT} \sqrt{p \cdot \sigma} & \zeta^{rT} \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{q \cdot \bar{\sigma}} \eta^s \\ -\sqrt{q \cdot \sigma} \bar{\eta}^s \end{bmatrix} \Big|_{p=(0,\mathbf{p}), q=(0,-\mathbf{p})} \\
 &= \begin{bmatrix} \zeta^{rT} \sqrt{-\mathbf{p} \cdot \sigma} & \zeta^{rT} \sqrt{\mathbf{p} \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{p} \cdot \bar{\sigma}} \eta^s \\ -\sqrt{-\mathbf{p} \cdot \sigma} \bar{\eta}^s \end{bmatrix} \\
 &= \zeta^{rT} \sqrt{-\mathbf{p} \cdot \sigma} \sqrt{\mathbf{p} \cdot \bar{\sigma}} \eta^s - \zeta^{rT} \sqrt{\mathbf{p} \cdot \bar{\sigma}} \sqrt{-\mathbf{p} \cdot \sigma} \bar{\eta}^s \\
 &= 0,
 \end{aligned} \tag{10.260}$$

and

$$\begin{aligned}
 v^{s\dagger}(-\mathbf{p}) u^r(\mathbf{p}) &= \begin{bmatrix} \eta^{s\dagger} \sqrt{q \cdot \sigma} & -\eta^{s\dagger} \sqrt{q \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{p \cdot \bar{\sigma}} \zeta^r \\ \sqrt{p \cdot \sigma} \bar{\zeta}^r \end{bmatrix} \Big|_{p=(0,\mathbf{p}), q=(0,-\mathbf{p})} \\
 &= \begin{bmatrix} \eta^{sT} \sqrt{\mathbf{p} \cdot \sigma} & -\eta^s \sqrt{-\mathbf{p} \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{-\mathbf{p} \cdot \bar{\sigma}} \zeta^r \\ \sqrt{\mathbf{p} \cdot \sigma} \bar{\zeta}^r \end{bmatrix} \\
 &= \eta^{sT} \sqrt{(-\mathbf{p} \cdot \sigma)(\mathbf{p} \cdot \bar{\sigma})} \zeta^r - \eta^s \sqrt{(\mathbf{p} \cdot \sigma)(-\mathbf{p} \cdot \bar{\sigma})} \bar{\zeta}^r \\
 &= 0.
 \end{aligned} \tag{10.261}$$

Exercise 10.10 **Direct product relation for the u 's.**

Prove the u direct product relations of theorem 10.16.

Answer for Exercise 10.10

$$\begin{aligned}
\sum u^s \otimes \bar{u}^s &= \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta^s \\ \sqrt{p \cdot \bar{\sigma}} \zeta^s \end{bmatrix} \otimes \begin{bmatrix} \zeta^{sT} \sqrt{p \cdot \bar{\sigma}} & \zeta^{sT} \sqrt{p \cdot \sigma} \end{bmatrix} \\
&= \sum \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta^s \otimes \zeta^{sT} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \zeta^s \otimes \zeta^{sT} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \zeta^s \otimes \zeta^{sT} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \bar{\sigma}} \zeta^s \otimes \zeta^{sT} \sqrt{p \cdot \sigma} \end{bmatrix} \\
&= \begin{bmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{bmatrix} \\
&= m + p \cdot \gamma.
\end{aligned} \tag{10.262}$$

Exercise 10.11 Derive the Dirac Hamiltonian without using the zero-time field substitution.

Answer for Exercise 10.11

With time left in the mix the fields are

$$\begin{aligned}
\Psi(x) &= \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip \cdot x} u_{\mathbf{p}}^s a_{\mathbf{p}}^s + e^{ip \cdot x} v_{\mathbf{p}}^s b_{\mathbf{p}}^s \right) \\
\Psi^\dagger(x) &= \sum_{r=1}^2 \int \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} \left(e^{iq \cdot x} u_{\mathbf{q}}^{r\dagger} a_{\mathbf{q}}^{r\dagger} + e^{-iq \cdot x} v_{\mathbf{q}}^{r\dagger} b_{\mathbf{q}}^{r\dagger} \right),
\end{aligned} \tag{10.263}$$

and the Hamiltonian is

$$\begin{aligned}
H_{\text{Dirac}} &= \sum_{r,s=1}^2 \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 2 \sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{q}}}} \left(e^{iq \cdot x} u_{\mathbf{q}}^{r\dagger} a_{\mathbf{q}}^{r\dagger} + e^{-iq \cdot x} v_{\mathbf{q}}^{r\dagger} b_{\mathbf{q}}^{r\dagger} \right) \omega_{\mathbf{p}} \left(e^{-ip \cdot x} u_{\mathbf{p}}^s a_{\mathbf{p}}^s - e^{ip \cdot x} v_{\mathbf{p}}^s b_{\mathbf{p}}^s \right) \\
&= \sum_{r,s=1}^2 \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 2 \sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{q}}}} \left(e^{i\omega_{\mathbf{q}} t - i\mathbf{q} \cdot \mathbf{x}} u_{\mathbf{q}}^{r\dagger} a_{\mathbf{q}}^{r\dagger} + e^{-i\omega_{\mathbf{q}} t + i\mathbf{q} \cdot \mathbf{x}} v_{\mathbf{q}}^{r\dagger} b_{\mathbf{q}}^{r\dagger} \right) \omega_{\mathbf{p}} \left(e^{-i\omega_{\mathbf{p}} t + i\mathbf{p} \cdot \mathbf{x}} u_{\mathbf{p}}^s a_{\mathbf{p}}^s \right. \\
&\quad \left. - e^{i\omega_{\mathbf{p}} t - i\mathbf{p} \cdot \mathbf{x}} v_{\mathbf{p}}^s b_{\mathbf{p}}^s \right) \\
&= \sum_{r,s=1}^2 \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 2 \sqrt{\omega_{\mathbf{p}} \omega_{\mathbf{q}}}} \left(e^{i\omega_{\mathbf{q}} t - i\mathbf{q} \cdot \mathbf{x}} u_{\mathbf{q}}^{r\dagger} a_{\mathbf{q}}^{r\dagger} + e^{-i\omega_{\mathbf{q}} t - i\mathbf{q} \cdot \mathbf{x}} v_{-\mathbf{q}}^{r\dagger} b_{-\mathbf{q}}^{r\dagger} \right) \omega_{\mathbf{p}} \left(e^{-i\omega_{\mathbf{p}} t + i\mathbf{p} \cdot \mathbf{x}} u_{\mathbf{p}}^s a_{\mathbf{p}}^s \right. \\
&\quad \left. - e^{i\omega_{\mathbf{p}} t + i\mathbf{p} \cdot \mathbf{x}} v_{-\mathbf{p}}^s b_{-\mathbf{p}}^s \right) \\
&= \sum_{r,s=1}^2 \int \frac{d^3 p}{(2\pi)^3 2} \left(e^{i\omega_{\mathbf{q}} t} u_{\mathbf{q}}^{r\dagger} a_{\mathbf{q}}^{r\dagger} + e^{-i\omega_{\mathbf{q}} t} v_{-\mathbf{q}}^{r\dagger} b_{-\mathbf{q}}^{r\dagger} \right) \left(e^{-i\omega_{\mathbf{p}} t} u_{\mathbf{p}}^s a_{\mathbf{p}}^s - e^{i\omega_{\mathbf{p}} t} v_{-\mathbf{p}}^s b_{-\mathbf{p}}^s \right) \\
&= \sum_{r,s=1}^2 \int \frac{d^3 p}{(2\pi)^3 2} \left(u_{\mathbf{q}}^{r\dagger} u_{\mathbf{p}}^s a_{\mathbf{q}}^{r\dagger} a_{\mathbf{p}}^s - v_{-\mathbf{q}}^{r\dagger} v_{-\mathbf{p}}^s b_{-\mathbf{q}}^{r\dagger} b_{-\mathbf{p}}^s \right),
\end{aligned} \tag{10.264}$$

where a $\delta^{(3)}(\mathbf{p} - \mathbf{q})$ was factored out and evaluated, and the remaining $v_{-\mathbf{p}}^{r\dagger} u^s, u^{r\dagger} v_{-\mathbf{p}}^s$ terms were killed off. A final use of eq. (10.131a) completes the proof.

Exercise 10.12 **Prove lemma 10.7**

Answer for Exercise 10.12

We will prove only the first, which is representative

$$\begin{aligned} (\bar{v}\gamma^\mu u)^\dagger &= u^\dagger (\gamma^\mu)^\dagger (v^\dagger \gamma^0)^\dagger \\ &= u^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^0 v \\ &= \bar{u} \gamma^\mu v. \end{aligned} \tag{10.265}$$

Exercise 10.13 **Prove lemma 10.8**

Answer for Exercise 10.13

For the two matrix trace, consider

$$\begin{aligned} \text{tr}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) &= 2g_{\mu\nu} \text{tr}(1) \\ &= 8g_{\mu\nu}, \end{aligned} \tag{10.266}$$

but

$$\begin{aligned} \text{tr}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) &= \text{tr}(\gamma_\mu \gamma_\nu) + \text{tr}(\gamma_\nu \gamma_\mu) \\ &= 2 \text{tr}(\gamma_\mu \gamma_\nu), \end{aligned} \tag{10.267}$$

so $\text{tr}(\gamma_\mu \gamma_\nu) = 4g_{\mu\nu}$ as claimed. For the traces of the three matrix products, there are three possible products of interest (for $r \neq s$)

$$\gamma^0 \gamma^r \gamma^s = -i\epsilon^{rst} \begin{bmatrix} 0 & \sigma^t \\ \sigma^t & 0 \end{bmatrix}, \tag{10.268}$$

which is traceless. We also have (for distinct r, s, t)

$$\gamma^r \gamma^s \gamma^t = - \begin{bmatrix} 0 & \sigma^r \sigma^s \sigma^t \\ \sigma^r \sigma^s \sigma^t & 0 \end{bmatrix}, \tag{10.269}$$

which is also traceless. All other three matrix products (except permutations of the two above) are proportional to a single γ^μ , which is traceless. A lazier, brute force proof by Mathematica ([tracesOfDiracMatrixProducts.nb](#)) is also possible. For the four matrix traces, the trace will be zero unless we have two matching pairs of gamma matrices (since $\gamma^0 \gamma^1 \gamma^2 \gamma^3$ or its permutations is traceless.) Assuming such matched pairs, we can reduce the product like so

- $\mu = \nu \implies \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 4g^{\alpha\beta}$
- $\mu = \alpha, \nu \neq \alpha \implies \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = -4g^{\nu\beta}$
- $\mu = \beta (\mu \neq \nu, \mu \neq \alpha) \implies \text{tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta) = 4g^{\nu\alpha}$

It's clear that we can summarize these possibilities as stated in lemma 10.8.

Exercise 10.14

Show that

$$(p^\beta p'^\alpha + p^\alpha p'^\beta - p \cdot p' g^{\alpha\beta}) \times (k'_\beta k_\alpha + k'_\alpha k_\beta - (k \cdot k' + m_\mu^2) g_{\alpha\beta}) = 2(p \cdot k p' \cdot k' + p \cdot k' p' \cdot k + m_\mu^2 p \cdot p')$$

Answer for Exercise 10.14

Proceeding mechanically, but carefully, we have

$$\begin{aligned}
 & p^\beta p'^\alpha k'_\beta k_\alpha + p^\beta p'^\alpha k'_\alpha k_\beta - p^\beta p'^\alpha (k \cdot k' + m_\mu^2) g_{\alpha\beta} \\
 & + p^\alpha p'^\beta k'_\beta k_\alpha + p^\alpha p'^\beta k'_\alpha k_\beta - p^\alpha p'^\beta (k \cdot k' + m_\mu^2) g_{\alpha\beta} \\
 & - p \cdot p' g^{\alpha\beta} k'_\beta k_\alpha - p \cdot p' g^{\alpha\beta} k'_\alpha k_\beta + p \cdot p' g^{\alpha\beta} (k \cdot k' + m_\mu^2) g_{\alpha\beta} \\
 & = p \cdot k' p' \cdot k + p \cdot k p' \cdot k' - p \cdot p' (k \cdot k' + m_\mu^2) \\
 & + p \cdot k p' \cdot k' + p \cdot k' p' \cdot k - p \cdot p' (k \cdot k' + m_\mu^2) \\
 & - p \cdot p' k \cdot k' - p \cdot p' k \cdot k' + 4p \cdot p' (k \cdot k' + m_\mu^2) \\
 & = 2p \cdot k' p' \cdot k + 2p \cdot k p' \cdot k' - 2p \cdot p' k \cdot k' + 2p \cdot p' (k \cdot k' + m_\mu^2) \\
 & = 2p \cdot k' p' \cdot k + 2p \cdot k p' \cdot k' + 2p \cdot p' m_\mu^2 \\
 & = 2(p \cdot k' p' \cdot k + p \cdot k p' \cdot k' + p \cdot p' m_\mu^2).
 \end{aligned} \tag{10.270}$$

Part II

APPENDICES

USEFUL FORMULAS AND REVIEW.

A.1 REVIEW OF OLD MATERIAL.

- Gaussian

$$\int_{-\infty}^{\infty} e^{ax^2} dx = \sqrt{\frac{-\pi}{a}}. \quad (\text{A.1})$$

Here a may be real or imaginary, but must be less than 0 if real.

- Our Fourier transform sign and π placement convention (2π 's with momentum elements and negative exponential sign for the inverse transform)

$$\begin{aligned} f(x) &= \int \frac{d^n k}{(2\pi)^n} f(k) e^{ik \cdot x} \\ f(k) &= \int d^n x f(x) e^{-ik \cdot x}. \end{aligned} \quad (\text{A.2})$$

- Delta function representation.

Setting $f(x) = \delta^n(x)$ implies $f(k) = 1$ and so

$$\delta(x) = \int \frac{d^n k}{(2\pi)^n} e^{ik \cdot x}. \quad (\text{A.3})$$

- Correct sign for the commutator

$$[x_r, p_s] = i\delta_{rs}. \quad (\text{A.4})$$

- Hamilton's equations

$$\begin{aligned} -dH &= d(\mathcal{L} - p\dot{q}) = \frac{\partial \mathcal{L}}{\partial q} dq + \cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}} d\dot{q}} + \frac{\partial \mathcal{L}}{\partial t} dt - dp\dot{q} - p d\dot{q} \\ dH &= \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt, \end{aligned} \quad (\text{A.5})$$

so

$$\frac{\partial H}{\partial q} = -\dot{p}, \quad \frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}. \quad (\text{A.6})$$

- Matrix element for the momentum and position operators

$$\begin{aligned}\langle x|P|x'\rangle &= -i\delta(x-x')\frac{d}{dx} \\ \langle p|X|p'\rangle &= i\delta(p-p')\frac{d}{dp}.\end{aligned}\tag{A.7}$$

- Eigenstates

$$\begin{aligned}p\langle x|p\rangle &= \langle x|P|p\rangle \\ &= \int dx' \langle x|P|x'\rangle \langle x'|p\rangle \\ &= \int dx' (-i)\delta(x-x')\frac{d}{dx} \langle x'|p\rangle \\ &= -i\frac{d}{dx} \langle x|p\rangle,\end{aligned}\tag{A.8}$$

so

$$\langle x|p\rangle \propto e^{ipx}.\tag{A.9}$$

Normalized over all space in d dimensions

$$\langle x|p\rangle = \frac{e^{ip\cdot x}}{(2\pi)^{d/2}}.\tag{A.10}$$

- Time evolution in the Heisenberg picture

$$\frac{dO}{dt} = i[H, O].\tag{A.11}$$

- Commutators of powers of position and momentum operators

$$\begin{aligned}[\hat{q}^n, \hat{p}] &= ni\hat{q}^{n-1} \\ [\hat{p}^n, \hat{q}] &= -ni\hat{p}^{n-1}.\end{aligned}\tag{A.12}$$

More generally, for any function with a power series representation $F(x) = \sum_{k=0}^{\infty} a_k x^k$, we have

$$\begin{aligned}[F(\hat{q}), \hat{p}] &= i\frac{dF}{d\hat{q}} \\ [F(\hat{p}), \hat{q}] &= -i\frac{dF}{d\hat{p}}.\end{aligned}\tag{A.13}$$

- Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (\text{A.14})$$

$$[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c. \quad (\text{A.15})$$

- Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = 0. \quad (\text{A.16})$$

- Lorentz transform identities

$$g_{\nu\rho} = g_{\mu\kappa} \Lambda^\mu{}_\nu \Lambda^\kappa{}_\rho. \quad (\text{A.17})$$

- Noether's first theorem. Given $\delta \mathcal{L} = \partial_\mu J^\mu$,

$$j^\mu = \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \delta \phi - J^\mu, \quad (\text{A.18})$$

satisfies $\partial_\mu j^\mu = 0$.

- Energy momentum tensor

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}. \quad (\text{A.19})$$

- Translation operator

$$\hat{U}(\mathbf{a}) = e^{i\mathbf{a} \cdot \hat{\mathbf{P}}}, \quad (\text{A.20})$$

$$\langle \mathbf{x} | \hat{U}(\mathbf{a}) = \langle \mathbf{x} + \mathbf{a} | \quad (\text{A.21})$$

- Baker-Campbell-Hausdorff theorem theorem 5.2.

$$e^B A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} [B \cdots, [B, A]]. \quad (\text{A.22})$$

A.2 USEFUL RESULTS FROM NEW MATERIAL.

- Hamiltonian for a massive scalar field

$$H = \int d^3x \left(\frac{1}{2} (\pi(\mathbf{x}, t))^2 + \frac{1}{2} (\nabla \phi(\mathbf{x}, t))^2 + \frac{m}{2} (\phi(\mathbf{x}, t))^2 \right) \quad (\text{A.23})$$

$$:H: = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} \quad (\text{A.24})$$

- Canonical commutator

$$[\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}). \quad (\text{A.25})$$

- Creation and annihilation operation

$$[a_{\mathbf{q}}, a_{\mathbf{p}}^{\dagger}] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \quad (\text{A.26})$$

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} (e^{-i\omega_{\mathbf{p}}t} a_{\mathbf{p}} + e^{i\omega_{\mathbf{p}}t} a_{-\mathbf{p}}^{\dagger}) \quad (\text{A.27})$$

$$\pi(\mathbf{x}, t) = \int \frac{d^3q}{(2\pi)^3} \frac{i\omega_{\mathbf{q}}}{\sqrt{2\omega_{\mathbf{q}}}} e^{i\mathbf{q}\cdot\mathbf{x}} (-e^{-i\omega_{\mathbf{q}}t} a_{\mathbf{q}} + e^{i\omega_{\mathbf{q}}t} a_{-\mathbf{q}}^{\dagger}). \quad (\text{A.28})$$

- Relativistic normalization and transformation of momentum state

$$\sqrt{2\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}}^{\dagger} |0\rangle = |\mathbf{p}\rangle, \quad (\text{A.29})$$

$$\hat{U}(\Lambda) |\mathbf{p}\rangle = |\Lambda\mathbf{p}\rangle. \quad (\text{A.30})$$

- Plus and minus operators

$$\begin{aligned} \hat{\phi}_{-}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{-ip\cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}} \\ \hat{\phi}_{+}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{ip\cdot x} \Big|_{p^0=\omega_{\mathbf{p}}} \hat{a}_{\mathbf{p}}^{\dagger}. \end{aligned} \quad (\text{A.31})$$

- Wightman function

$$D(x) = [\hat{\phi}_-(x), \hat{\phi}_+(0)] = \int \frac{d^3 p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip \cdot x} \Big|_{p^0 = \omega_{\mathbf{p}}}. \quad (\text{A.32})$$

- Contractions with momentum states

$$\begin{aligned} \overline{\phi_I(x)} a_{\mathbf{p}}^\dagger &= e^{-ip \cdot x} \\ a_{\mathbf{p}} \overline{\phi_I(x)} &= e^{ip \cdot x}. \end{aligned} \quad (\text{A.33})$$

- Dirac conjugates.

$$\begin{aligned} (\gamma^0)^\dagger &= \gamma^0 \\ (\gamma^k)^\dagger &= -\gamma^k \\ (\gamma^\mu)^\dagger &= \gamma^0 \gamma^\mu \gamma^0. \end{aligned} \quad (\text{A.34})$$

MOMENTUM OF SCALAR FIELD.

B.1 EXPANSION OF THE FIELD MOMENTUM.

In eq. (5.64) it was claimed that

$$P^k = \int d^3x \hat{\pi} \partial^k \hat{\phi} = \int \frac{d^3p}{(2\pi)^3} p^k a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \quad (\text{B.1})$$

If I compute this, I get a normal ordered variation of this operator, but also get some time dependent terms. Here's the computation (dropping hats)

$$\begin{aligned} P^k &= \int d^3x \hat{\pi} \partial^k \phi \\ &= \int d^3x \partial_0 \phi \partial^k \phi \\ &= \int d^3x \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_p 2\omega_q}} \partial_0 (a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x}) \partial^k (a_{\mathbf{q}} e^{-iq \cdot x} + a_{\mathbf{q}}^\dagger e^{iq \cdot x}). \end{aligned} \quad (\text{B.2})$$

The exponential derivatives are

$$\begin{aligned} \partial_0 e^{\pm ip \cdot x} &= \partial_0 e^{\pm i p_\mu x^\mu} \\ &= \pm i p_0 \partial_0 e^{\pm ip \cdot x}, \end{aligned} \quad (\text{B.3})$$

and

$$\begin{aligned} \partial^k e^{\pm ip \cdot x} &= \partial^k e^{\pm i p^\mu x_\mu} \\ &= \pm i p^k e^{\pm ip \cdot x}, \end{aligned} \quad (\text{B.4})$$

so

$$\begin{aligned}
P^k &= - \int d^3x \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_p 2\omega_q}} p_0 q^k \left(-a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \left(-a_{\mathbf{q}} e^{-iq \cdot x} + a_{\mathbf{q}}^\dagger e^{iq \cdot x} \right) \\
&= -\frac{1}{2} \int d^3x \frac{d^3p d^3q}{(2\pi)^6} \sqrt{\frac{\omega_p}{\omega_q}} q^k \left(a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q) \cdot x} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q) \cdot x} - a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{i(q-p) \cdot x} - a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q) \cdot x} \right) \\
&= \frac{1}{2} \int \frac{d^3p d^3q}{(2\pi)^3} \sqrt{\frac{\omega_p}{\omega_q}} q^k \left(-a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(\omega_{\mathbf{p}} + \omega_{\mathbf{q}})t} \delta^{(3)}(\mathbf{p} + \mathbf{q}) - a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(\omega_{\mathbf{p}} + \omega_{\mathbf{q}})t} \delta^{(3)}(-\mathbf{p} - \mathbf{q}) \right. \\
&\quad \left. + a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{i(\omega_{\mathbf{q}} - \omega_{\mathbf{p}})t} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(\omega_{\mathbf{p}} - \omega_{\mathbf{q}})t} \delta^{(3)}(\mathbf{q} - \mathbf{p}) \right) \\
&= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger - a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}}t} - a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2i\omega_{\mathbf{p}}t} \right).
\end{aligned} \tag{B.5}$$

What is the rationale for ignoring those time dependent terms? Does normal ordering also implicitly drop any non-paired creation/annihilation operators? If so, why?

B.2 CONSERVATION OF THE FIELD MOMENTUM.

This follows up on unanswered questions related to the apparent time dependent terms in the previous expansion of P^i for a scalar field.

It turns out that examining the reasons that we can say that the field momentum is conserved also sheds some light on the question. P^i is not an a-priori conserved quantity, but we may use the charge conservation argument to justify this despite it not having a four-vector nature (i.e. with zero four divergence.)

The momentum P^i that we have defined is related to the conserved quantity $T^{0\mu}$, the energy-momentum tensor, which satisfies $0 = \partial_\mu T^{0\mu}$ by Noether's theorem (this was the conserved quantity associated with a spacetime translation.)

That tensor was

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}, \tag{B.6}$$

and can be used to define the momenta

$$\begin{aligned}
\int d^3x T^{0k} &= \int d^3x \partial^0 \phi \partial^k \phi \\
&= \int d^3x \pi \partial^k \phi.
\end{aligned} \tag{B.7}$$

Charge $Q^i = \int d^3x j^0$ was conserved with respect to a limiting surface argument, and we can make a similar “beer can integral” argument for P^i , integrating over a large time interval $t \in [-T, T]$ as sketched in fig. 3.1. That is

$$\begin{aligned}
0 &= \partial_\mu \int d^4x T^{0\mu} \\
&= \partial_0 \int d^4x T^{00} + \partial_k \int d^4x T^{0k} \\
&= \partial_0 \int_{-T}^T dt \int d^3x T^{00} + \partial_k \int_{-T}^T dt \int d^3x T^{0k} \\
&= \partial_0 \int_{-T}^T dt \int d^3x T^{00} + \partial_k \int_{-T}^T dt \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger - a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}}t} - a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2i\omega_{\mathbf{p}}t} \right) \\
&= \int d^3x T^{00} \Big|_{-T}^T + T \partial_k \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \right) \\
&\quad - \frac{1}{2} \partial_k \int_{-T}^T dt \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2i\omega_{\mathbf{p}}t} \right).
\end{aligned} \tag{B.8}$$

The first integral can be said to vanish if the field energy goes to zero at the time boundaries, and the last integral reduces to

$$\begin{aligned}
& -\frac{1}{2} \partial_k \int_{-T}^T dt \int \frac{d^3p}{(2\pi)^3} p^k \left(a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}}t} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2i\omega_{\mathbf{p}}t} \right) \\
&= - \int \frac{d^3p}{2(2\pi)^3} p^k \left(a_{\mathbf{p}} a_{-\mathbf{p}} \frac{\sin(-2\omega_{\mathbf{p}}T)}{-2\omega_{\mathbf{p}}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \frac{\sin(2\omega_{\mathbf{p}}T)}{2\omega_{\mathbf{p}}} \right) \\
&= - \int \frac{d^3p}{2(2\pi)^3} p^k \left(a_{\mathbf{p}} a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger \right) \frac{\sin(2\omega_{\mathbf{p}}T)}{2\omega_{\mathbf{p}}}.
\end{aligned} \tag{B.9}$$

The sin term can be interpreted as a sinc like function of $\omega_{\mathbf{p}}$ which vanishes for large \mathbf{p} . It’s not entirely sinc like for a massive field as $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, which never hits zero, as shown in fig. B.1. Vanishing for large \mathbf{p} doesn’t help the whole integral vanish, but we can resort to the Riemann-Lebesgue lemma [18] instead and interpret this integral as one with a plain old high frequency oscillation that is presumed to vanish (i.e. the rest is well behaved enough that it can be labelled as L_1 integrable.)

We see that only the non-time dependent portion of \mathbf{P} matters from a conserved quantity point of view, and having killed off all the time dependent terms, we are left with a conservation relationship for the momenta $\nabla \cdot \mathbf{P} = 0$, where \mathbf{P} in normal order is just

$$:\mathbf{P}: = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{B.10}$$

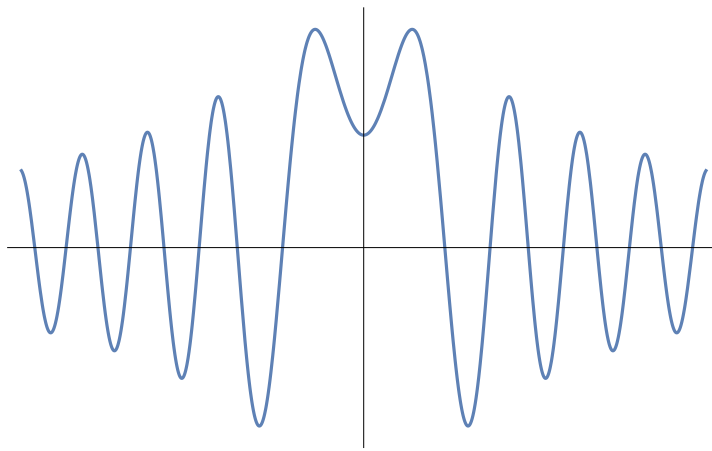


Figure B.1: Angular frequency dependent sinc.

REFLECTION USING PAULI MATRICES.

In lemma 10.1 we used $\sigma^T = -\sigma_2 \sigma \sigma_2$, which implicitly shows that $(\sigma \cdot \mathbf{x})^T$ is a reflection about the y-axis. This form of reflection will be familiar to a student of geometric algebra (see [4]). I can't recall any mention of the geometrical reflection identity from when I took QM. It's a fun exercise to demonstrate the reflection identity when constrained to the Pauli matrix notation.

Theorem C.1: Reflection about a normal.

Given a unit vector $\hat{\mathbf{n}} \in \mathbb{R}^3$ and a vector $\mathbf{x} \in \mathbb{R}^3$ the reflection of \mathbf{x} about a plane with normal $\hat{\mathbf{n}}$ can be represented in Pauli notation as

$$-\sigma \cdot \hat{\mathbf{n}} \sigma \cdot \mathbf{x} \sigma \cdot \hat{\mathbf{n}}.$$

In standard vector notation, we can decompose a vector into its projective and rejective components

$$\mathbf{x} = (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}). \quad (\text{C.1})$$

A reflection about the plane normal to $\hat{\mathbf{n}}$ just flips the component in the direction of $\hat{\mathbf{n}}$, leaving the rest unchanged. That is

$$-(\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}) = \mathbf{x} - 2(\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}. \quad (\text{C.2})$$

We may write this in σ notation as

$$\sigma \cdot \mathbf{x} - 2\mathbf{x} \cdot \hat{\mathbf{n}} \sigma \cdot \hat{\mathbf{n}}. \quad (\text{C.3})$$

We also know that

$$\begin{aligned} \sigma \cdot \mathbf{a} \sigma \cdot \mathbf{b} &= a \cdot b + i\sigma \cdot (\mathbf{a} \times \mathbf{b}) \\ \sigma \cdot \mathbf{b} \sigma \cdot \mathbf{a} &= a \cdot b - i\sigma \cdot (\mathbf{a} \times \mathbf{b}), \end{aligned} \quad (\text{C.4})$$

or

$$a \cdot b = \frac{1}{2}\{\sigma \cdot \mathbf{a}, \sigma \cdot \mathbf{b}\}, \quad (\text{C.5})$$

where $\{\mathbf{a}, \mathbf{b}\}$ is the anticommutator of \mathbf{a}, \mathbf{b} . Inserting eq. (C.5) into eq. (C.3) we find that the reflection is

$$\begin{aligned}\boldsymbol{\sigma} \cdot \mathbf{x} - \{\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \boldsymbol{\sigma} \cdot \mathbf{x}\} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} &= \boldsymbol{\sigma} \cdot \mathbf{x} - \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} - \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \\ &= \boldsymbol{\sigma} \cdot \mathbf{x} - \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} - \boldsymbol{\sigma} \cdot \mathbf{x} \\ &= -\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}},\end{aligned}\tag{C.6}$$

which completes the proof.

When we expand $(\boldsymbol{\sigma} \cdot \mathbf{x})^T$ and find

$$(\boldsymbol{\sigma} \cdot \mathbf{x})^T = \sigma^1 x^1 - \sigma^2 x^2 + \sigma^3 x^3,\tag{C.7}$$

it is clear that this coordinate expansion is a reflection about the y-axis. Knowing the reflection formula above provides a rationale for why we might want to write this in the compact form $-\sigma^2(\boldsymbol{\sigma} \cdot \mathbf{x})\sigma^2$, which might not be obvious otherwise.

EXPLICIT EXPANSION OF THE DIRAC u, v SPINORS.

We found that the solution of the $u(p), v(p)$ matrices were

$$\begin{aligned} u(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta \\ \sqrt{p \cdot \bar{\sigma}} \zeta \end{bmatrix} \\ v(p) &= \begin{bmatrix} \sqrt{p \cdot \sigma} \eta \\ -\sqrt{p \cdot \bar{\sigma}} \eta \end{bmatrix}, \end{aligned} \tag{D.1}$$

where

$$\begin{aligned} p \cdot \sigma &= p_0 \sigma_0 - \boldsymbol{\sigma} \cdot \mathbf{p} \\ p \cdot \bar{\sigma} &= p_0 \sigma_0 + \boldsymbol{\sigma} \cdot \mathbf{p}. \end{aligned} \tag{D.2}$$

It was pointed out that these square roots can be conceptualized as (in the right basis) as the diagonal matrices of the eigenvalue square roots.

It was also pointed out that we don't tend to need the explicit form of these square roots. We saw that to be the case in all our calculations, where these always showed up in the end in quadratic combinations like $\sqrt{(p \cdot \sigma)^2}, \sqrt{(p \cdot \sigma)(p \cdot \bar{\sigma})}, \dots$, which nicely reduced each time without requiring the matrix roots.

I encountered a case where it would have been nice to have the explicit representation. In particular, I wanted to use Mathematica to symbolically expand $\bar{\Psi} i \gamma^\mu \partial_\mu \Psi$ in terms of $a_{\mathbf{p}}^s, b_{\mathbf{p}}^r, \dots$ representation, to verify that the massless Dirac Lagrangian are in fact the energy and momentum operators (and to compare to the explicit form of the momentum operator found in eq. 3.105 [14]). For that mechanical task, I needed explicit representations of all the $u^s(p), v^r(p)$ matrices to plug in.

It happens that 2×2 matrices can be square-rooted symbolically [squarerootOfFourSigmaDotP.nb](#). In particular, the matrices $p \cdot \sigma, p \cdot \bar{\sigma}$ have nice simple eigenvalues $\pm \|\mathbf{p}\| + \omega_{\mathbf{p}}$. The corresponding unnormalized eigenvectors for $p \cdot \sigma$ are

$$\begin{aligned} e_1 &= \begin{bmatrix} -p^1 + ip^2 \\ p^3 + \|\mathbf{p}\| \end{bmatrix} \\ e_1 &= \begin{bmatrix} -p^1 + ip^2 \\ p^3 - \|\mathbf{p}\| \end{bmatrix}. \end{aligned} \tag{D.3}$$

This means that we can diagonalize $p \cdot \sigma$ as

$$p \cdot \sigma = U \begin{bmatrix} \omega_{\mathbf{p}} + \|\mathbf{p}\| & 0 \\ 0 & \omega_{\mathbf{p}} - \|\mathbf{p}\| \end{bmatrix} U^\dagger, \quad (\text{D.4})$$

where U is the matrix of the normalized eigenvectors

$$\begin{aligned} U &= \begin{bmatrix} e'_1 & e'_2 \end{bmatrix} \\ &= \frac{1}{\sqrt{2\mathbf{p}^2 + 2p^3 \|\mathbf{p}\|}} \begin{bmatrix} -p^1 + ip^2 & -p^1 + ip^2 \\ p^3 + \|\mathbf{p}\| & p^3 - \|\mathbf{p}\| \end{bmatrix}. \end{aligned} \quad (\text{D.5})$$

Letting Mathematica churn through the matrix products eq. (D.4) verifies the diagonalization, and for the roots, we find

$$\sqrt{p \cdot \sigma} = \frac{1}{\sqrt{\omega_{\mathbf{p}} - \|\mathbf{p}\|} + \sqrt{\omega_{\mathbf{p}} + \|\mathbf{p}\|}} \begin{bmatrix} \omega_{\mathbf{p}} - p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} & -p^1 + ip^2 \\ -p^1 - ip^2 & \omega_{\mathbf{p}} + p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \end{bmatrix} \quad (\text{D.6})$$

The matrix $\sqrt{p \cdot \bar{\sigma}}$ has the same form, but we just have to flip the signs on our \mathbf{p} 's.

We are now ready to plug in $\zeta^{1T} = (1, 0), \zeta^{2T} = (0, 1), \eta^{1T} = (1, 0), \eta^{2T} = (0, 1)$ to find the explicit form of our u 's and v 's

$$\begin{aligned}
 u^1(p) &= \frac{1}{\sqrt{\omega_{\mathbf{p}} - \|\mathbf{p}\|} + \sqrt{\omega_{\mathbf{p}} + \|\mathbf{p}\|}} \begin{bmatrix} \omega_{\mathbf{p}} - p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \\ -p^1 - ip^2 \\ \omega_{\mathbf{p}} + p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \\ p^1 + ip^2 \end{bmatrix} \\
 u^2(p) &= \frac{1}{\sqrt{\omega_{\mathbf{p}} - \|\mathbf{p}\|} + \sqrt{\omega_{\mathbf{p}} + \|\mathbf{p}\|}} \begin{bmatrix} -p^1 + ip^2 \\ \omega_{\mathbf{p}} + p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \\ p^1 - ip^2 \\ \omega_{\mathbf{p}} - p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \end{bmatrix} \\
 v^1(p) &= \frac{1}{\sqrt{\omega_{\mathbf{p}} - \|\mathbf{p}\|} + \sqrt{\omega_{\mathbf{p}} + \|\mathbf{p}\|}} \begin{bmatrix} \omega_{\mathbf{p}} - p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \\ -p^1 - ip^2 \\ -\omega_{\mathbf{p}} - p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \\ -p^1 - ip^2 \end{bmatrix} \\
 v^2(p) &= \frac{1}{\sqrt{\omega_{\mathbf{p}} - \|\mathbf{p}\|} + \sqrt{\omega_{\mathbf{p}} + \|\mathbf{p}\|}} \begin{bmatrix} -p^1 + ip^2 \\ \omega_{\mathbf{p}} + p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \\ -p^1 + ip^2 \\ -\omega_{\mathbf{p}} + p^3 + \sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \end{bmatrix}.
 \end{aligned} \tag{D.7}$$

This is now a convenient form to try the next symbolic manipulation task. If nothing else this takes some of the mystery out of the original compact notation, since we see that the u, v 's are just 4 element column vectors, and we know their explicit form should we want them.

Also note that in class we made a note that we should take the positive roots of the eigenvalue diagonal matrix. It doesn't look like that is really required. We need not even use the same sign for each root. Squaring the resulting matrix root in the end will recover the original $p \cdot \sigma$ matrix.

D.1 COMPACT REPRESENTATION OF $\sqrt{p \cdot \sigma}$, $\sqrt{p \cdot \bar{\sigma}}$.

With the help of Mathematica, eq. (D.6) was found, a compact representation of the root of $p \cdot \sigma$. A bit of examination shows that we can do much better. The leading scalar term can be simplified by squaring it

$$\begin{aligned} \left(\sqrt{\omega_{\mathbf{p}} - \|\mathbf{p}\|} + \sqrt{\omega_{\mathbf{p}} + \|\mathbf{p}\|} \right)^2 &= \omega_{\mathbf{p}} - \|\mathbf{p}\| + \omega_{\mathbf{p}} + \|\mathbf{p}\| + 2\sqrt{\omega_{\mathbf{p}}^2 - \mathbf{p}^2} \\ &= 2\omega_{\mathbf{p}} + 2m, \end{aligned} \quad (\text{D.8})$$

where the on-shell value of the energy $\omega_{\mathbf{p}}^2 = m^2 + \mathbf{p}^2$ has been inserted. Using that again in the matrix, we have

$$\begin{aligned} \sqrt{p \cdot \sigma} &= \frac{1}{\sqrt{2\omega_{\mathbf{p}} + 2m}} \begin{bmatrix} \omega_{\mathbf{p}} - p^3 + m & -p^1 + ip^2 \\ -p^1 - ip^2 & \omega_{\mathbf{p}} + p^3 + m \end{bmatrix} \\ &= \frac{1}{\sqrt{2\omega_{\mathbf{p}} + 2m}} \left((\omega_{\mathbf{p}} + m)\sigma^0 - p^1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - p^2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - p^3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \\ &= \frac{1}{\sqrt{2\omega_{\mathbf{p}} + 2m}} \left((\omega_{\mathbf{p}} + m)\sigma^0 - p^1 \sigma^1 - p^2 \sigma^2 - p^3 \sigma^3 \right) \\ &= \frac{1}{\sqrt{2\omega_{\mathbf{p}} + 2m}} \left((\omega_{\mathbf{p}} + m)\sigma^0 - \boldsymbol{\sigma} \cdot \mathbf{p} \right). \end{aligned} \quad (\text{D.9})$$

We've now found a nice algebraic form for these matrix roots

$$\begin{aligned} \sqrt{p \cdot \sigma} &= \frac{1}{\sqrt{2\omega_{\mathbf{p}} + 2m}} (m + p \cdot \sigma) \\ \sqrt{p \cdot \bar{\sigma}} &= \frac{1}{\sqrt{2\omega_{\mathbf{p}} + 2m}} (m + p \cdot \bar{\sigma}). \end{aligned} \quad (\text{D.10})$$

As a check, let's square one of these explicitly

$$\begin{aligned}
 (\sqrt{p \cdot \sigma})^2 &= \frac{1}{2\omega_{\mathbf{p}} + 2m} (m^2 + (p \cdot \sigma)^2 + 2m(p \cdot \sigma)) \\
 &= \frac{1}{2\omega_{\mathbf{p}} + 2m} (m^2 + (\omega_{\mathbf{p}}^2 - 2\omega_{\mathbf{p}}\sigma \cdot \mathbf{p} + \mathbf{p}^2) + 2m(p \cdot \sigma)) \\
 &= \frac{1}{2\omega_{\mathbf{p}} + 2m} (2\omega_{\mathbf{p}}^2 - 2\omega_{\mathbf{p}}\sigma \cdot \mathbf{p} + 2m(\omega_{\mathbf{p}} - \sigma \cdot \mathbf{p})) \\
 &= \frac{1}{2\omega_{\mathbf{p}} + 2m} (2\omega_{\mathbf{p}}(\omega_{\mathbf{p}} + m) - (2\omega_{\mathbf{p}} + 2m)\sigma \cdot \mathbf{p}) \\
 &= \omega_{\mathbf{p}} - \sigma \cdot \mathbf{p} \\
 &= p \cdot \sigma,
 \end{aligned} \tag{D.11}$$

which validates the result.

We can also put the spinor solutions u, v in a nice compact square-root-free format

$$\begin{aligned}
 u(p) &= \frac{1}{\sqrt{2m + 2\omega_{\mathbf{p}}}} \begin{bmatrix} (m + p \cdot \sigma)\zeta \\ (m + p \cdot \bar{\sigma})\zeta \end{bmatrix} \\
 v(p) &= \frac{1}{\sqrt{2m + 2\omega_{\mathbf{p}}}} \begin{bmatrix} (m + p \cdot \sigma)\eta \\ -(m + p \cdot \bar{\sigma})\eta \end{bmatrix}.
 \end{aligned} \tag{D.12}$$

?? is probably a much nicer starting point for evaluating the various u, v, \bar{u}, \bar{v} product relationships. In particular, again using [Mathematica uvspinors.nb](#), this is a nice representation for showing that

$$\begin{aligned}
 \bar{u}^r(p)\gamma^k u^s(p) &= 2\delta^{rs}p^k \\
 \bar{v}^r(p)\gamma^k v^s(p) &= 2\delta^{rs}p^k,
 \end{aligned} \tag{D.13}$$

which is roughly the form of the relationship that I suspected existed, but had some trouble deriving manually.

We now also know that we can return to eq. (D.7) and put the explicit (root-free) representation of the u, v spinors in a slightly tidier form

$$\begin{aligned}
 u^1(p) &= \frac{1}{\sqrt{2m + 2\omega_{\mathbf{p}}}} \begin{bmatrix} \omega_{\mathbf{p}} - p^3 + m \\ -p^1 - ip^2 \\ \omega_{\mathbf{p}} + p^3 + m \\ p^1 + ip^2 \end{bmatrix} \\
 u^2(p) &= \frac{1}{\sqrt{2m + 2\omega_{\mathbf{p}}}} \begin{bmatrix} -p^1 + ip^2 \\ \omega_{\mathbf{p}} + p^3 + m \\ p^1 - ip^2 \\ \omega_{\mathbf{p}} - p^3 + m \end{bmatrix} \\
 v^1(p) &= \frac{1}{\sqrt{2m + 2\omega_{\mathbf{p}}}} \begin{bmatrix} \omega_{\mathbf{p}} - p^3 + m \\ -p^1 - ip^2 \\ -\omega_{\mathbf{p}} - p^3 + m \\ -p^1 - ip^2 \end{bmatrix} \\
 v^2(p) &= \frac{1}{\sqrt{2m + 2\omega_{\mathbf{p}}}} \begin{bmatrix} -p^1 + ip^2 \\ \omega_{\mathbf{p}} + p^3 + m \\ -p^1 + ip^2 \\ -\omega_{\mathbf{p}} + p^3 + m \end{bmatrix}.
 \end{aligned} \tag{D.14}$$

MATHEMATICA NOTEBOOKS

These Mathematica notebooks, some just trivial ones used to generate figures, others more elaborate, and perhaps some even polished, can be found in

<https://github.com/peeterjoot/mathematica/tree/master/>.

The free **Wolfram CDF player**, is capable of read-only viewing these notebooks to some extent.

- Oct 4, 2018 [phy2403-quantum-field-theory/phononEnergyDensity.nb](#)
phy2403 Hw1 problem 2(d). Phonon energy density. Ratio of the phonon energy density to the rest-mass energy density (in hardwood).
- Oct 9, 2018 [phy2403-quantum-field-theory/ps1Problem5MassiveFieldDerivatives.nb](#)
This is a check (up to the 7th derivative) that shows that the massive field wipes out the odd derivatives that contribute to the Euler-Maclaurin sum used to calculate the magnitude of the Casimir effect. Hw1 problem V.
- Oct 9, 2018 [phy2403-quantum-field-theory/ps1Problem5numeric.nb](#)
Hw1 problem V. part 7 (g). comparative force: $1\mu F$ capacitor with 1V charge.
- Oct 11, 2018 [phy2403-quantum-field-theory/mexicanHatPotentialManipulate.nb](#)
Manipulate and figures for the Mexican hat potential.
- Oct 15, 2018 [phy2403-quantum-field-theory/lecture10PlotMomentumParaboloid.nb](#)
This is plots for Lecture 10: constantMomentumSurfaceFig2, constantMomentumSurfaceFig1 (paraboloids vs. light cone).
- Oct 15, 2018 [phy2403-quantum-field-theory/pauliMatrix.nb](#)
Hw2 Problem II.1, explicitly write our the U_L matrix, and related expansions.
- Oct 17, 2018 [phy2403-quantum-field-theory/hyperboliodAndLightCone.nb](#)
This is a plot for Lecture 11. spaceLikeAndLightConeFig1
- Oct 29, 2018 [phy2403-quantum-field-theory/traceOfPauliMatrixProducts.nb](#)
Calculate the product of $(\mathbf{p} \cdot \boldsymbol{\sigma})(\mathbf{p} \cdot \boldsymbol{\sigma})$, and then take the trace of that.

- Oct 29, 2018 [phy2403-quantum-field-theory/RiemannLebesgueFig1.nb](#)
Plot: RiemannLebesgueFig1. A gaussian envelope high frequency oscillation.
- Oct 29, 2018 [phy2403-quantum-field-theory/traceOfHermitianTimesPauliVector.nb](#)
Is the trace of the product of a Hermitian and a Pauli vector zero? No. Ends up kind of dot product like.
- Oct 30, 2018 [phy2403-quantum-field-theory/ps2Problem1_2_evaluateBesselIntegral.nb](#)
Hw2 Problem I.2. Bessel function result for the final integral after the branch cut manipulation.
- Oct 30, 2018 [phy2403-quantum-field-theory/spacelikeWightmanFunction.nb](#)
Attempt to explicitly evaluate the Wightman function. Did not converge.
- Oct 31, 2018 [phy2403-quantum-field-theory/noncommutativeMatrixMultiply.nb](#)
An attempt to use the NCAAlgebra package (non-commutative algebra) to calculate the commutators of the charge operators for the Left and Right currents of Hw2 problem II.6. Did not work.
- Oct 31, 2018 [phy2403-quantum-field-theory/howToSimplifyNonCommutativeMultiplyQuestion.nb](#)
Post to Mathematica SE. How to get FullSimplify to act on scalars (-1, 0, 1, ...) that are multiplied by potentially non-commutative symbols (operators, block matrices, ...)?
- Nov 8, 2018 [phy2403-quantum-field-theory/sincVsSineLargeT.nb](#)
Figure for momentum.tex : sortOfSincFunctionFig1
- Nov 20, 2018 [phy2403-quantum-field-theory/ps3p3Integrals.nb](#)
Appendix for Hw3, III.2. Relevant integrals.
- Nov 20, 2018 [phy2403-quantum-field-theory/problemSet3YukawaIntegral.nb](#)
Hw3 problem I. Yukawa integral.
- Nov 20, 2018 [phy2403-quantum-field-theory/ps3p1TRFig1.nb](#)
Figure: ps3p1TRFig1. Plots $T e^{-R}/R$ for $T \sim R, T \gg R$
- Nov 22, 2018 [phy2403-quantum-field-theory/gammaAndMorePauliCalculations.nb](#)
Eigenvalues of $i\sigma \cdot \mathbf{x}$

- Nov 25, 2018 [phy2403-quantum-field-theory/diracWeylMatrixRepresentationAndIdentities.nb](#)
 Explicit 4x4 expansion of the Dirac matrices using the Weyl representation, and verification of some identities.
- Dec 4, 2018 [phy2403-quantum-field-theory/hw4p2numbers.nb](#)
 Numeric calculations for Hw4 Problem 2. hw4p2MathematicaFig1
- Dec 11, 2018 [phy2403-quantum-field-theory/squarerootOfFourSigmaDotP.nb](#)
 Define Dirac Matrices (in the Weyl representation). Calculate $\sqrt{p \cdot \sigma}$, $\sqrt{p \cdot \bar{\sigma}}$, square roots of 2x2 matrices, and representations of u, v spinors.
- Dec 18, 2018 [phy2403-quantum-field-theory/uvspinors.nb](#)
 A less experimental version of squarerootOfFourSigmaDotP.nb, that uses the results obtained from that code without actually taking the roots using the Eigensystem solution. Defines: DiracMatrix (Weyl representation); complex – allow for less general complex numbers, instead of: $a + I b$, complex[a, b] always formed from real a, b.; diracMatrix (using complex instead of Complex); pauliMatrix (using complex instead of Complex); u, v, and square roots of $p \cdot \sigma$'s
- Dec 25, 2018 [phy2403-quantum-field-theory/tracesOfDiracMatrixProducts.nb](#)
 Traces of two, three, four Dirac matrix products.

Part III

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