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PHY2403H Quantum Field Theory. Lecture 14: Time evolution, Hamiltonian perturbation, ground state. Taught by Prof. Erich Poppitz

DISCLAIMER: Very rough notes from class, with some additional side notes. These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

1.1 Review

Given a field $\phi(t_0, \mathbf{x})$, satisfying the commutation relations

$$\left[\pi(t_0, \mathbf{x}), \phi(t_0, \mathbf{y})\right] = -i\delta(\mathbf{x} - \mathbf{y}) \tag{1.1}$$

we introduced an interaction picture field given by

$$\phi_I(t, x) = e^{iH_0(t-t_0)}\phi(t_0, \mathbf{x})e^{-iH_0(t-t_0)} \tag{1.2}$$

related to the Heisenberg picture representation by

$$\begin{aligned}\phi_H(t, x) &= e^{iH(t-t_0)}\phi(t_0, \mathbf{x})e^{-iH(t-t_0)} \\ &= U^\dagger(t, t_0)\phi_I(t, \mathbf{x})U(t, t_0),\end{aligned} \tag{1.3}$$

where $U(t, t_0)$ is the time evolution operator.

$$U(t, t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \tag{1.4}$$

We argued that

$$i\frac{\partial}{\partial t}U(t, t_0) = H_{\mathrm{I,int}}(t)U(t, t_0) \tag{1.5}$$

We found the glorious expression

$$\boxed{\begin{aligned}U(t, t_0) &= T \exp\left(-i\int_{t_0}^t H_{\mathrm{I,int}}(t')dt'\right) \\ &= \sum_{n=0}^\infty \frac{(-i)^n}{n!}\int_{t_0}^t dt_1 dt_2 \cdots dt_n T\left(H_{\mathrm{I,int}}(t_1)H_{\mathrm{I,int}}(t_2)\cdots H_{\mathrm{I,int}}(t_n)\right)\end{aligned}} \tag{1.6}$$

However, what we are really after is

$$\langle\Omega| T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle \tag{1.7}$$

Such a product has many labels and names, and we'll describe it as "vacuum expectation values of time-ordered products of arbitrary #s of local Heisenberg operators".

1.2 Perturbation

Following §4.2, [1].

$$\begin{aligned} H &= \text{exact Hamiltonian} = H_0 + H_{\text{int}} \\ H_0 &= \text{free Hamiltonian.} \end{aligned} \tag{1.8}$$

We know all about H_0 and assume that it has a lowest (ground state) $|0\rangle$, the "vacuum" state of H_0.
H has eigenstates, in particular H is assumed to have a unique ground state $|\Omega\rangle$ satisfying

$$H|\Omega\rangle = |\Omega\rangle E_0, \tag{1.9}$$

and has states $|n\rangle$, representing excited (non-vacuum states with energies $> E_0$). These states are assumed to be a complete basis

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_n |n\rangle\langle n| + \int dn\,|n\rangle\langle n|\,. \tag{1.10}$$

The latter terms may be written with a superimposed sum-integral notation as

$$\sum_n + \int dn = \sum\!\!\!\!\!\!\int_n, \tag{1.11}$$

so the identity operator takes the more compact form

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum\!\!\!\!\!\!\int_n |n\rangle\langle n|\,. \tag{1.12}$$

For some time T we have

$$e^{-iHT}|0\rangle = e^{-iHT}\left(|\Omega\rangle\langle\Omega|0\rangle + \sum\!\!\!\!\!\!\int_n |n\rangle\langle n|0\rangle\right). \tag{1.13}$$

We now wish to argue that the $\sum\!\!\!\!\!\!\int_n$ term can be ignored.

Argument 1: This is something of a fast one, but one can consider a formal transformation $T \rightarrow T(1-i\epsilon)$, where $\epsilon \rightarrow 0^+$, and consider very large T. This gives

$$\begin{aligned}
\lim_{T \rightarrow \infty, \epsilon \rightarrow 0^+} e^{-iHT(1-i\epsilon)} |0\rangle &= \lim_{T \rightarrow \infty, \epsilon \rightarrow 0^+} e^{-iHT(1-i\epsilon)} \left(|\Omega\rangle \langle \Omega | 0 \rangle + \sum\!\!\!\!\!\!\int_n |n\rangle \langle n | 0 \rangle \right) \\
&= \lim_{T \rightarrow \infty, \epsilon \rightarrow 0^+} e^{-iE_0 T - E_0 \epsilon T} |\Omega\rangle \langle \Omega | 0 \rangle + \sum\!\!\!\!\!\!\int_n e^{-iE_n T - \epsilon E_n T} |n\rangle \langle n | 0 \rangle \\
&= \lim_{T \rightarrow \infty, \epsilon \rightarrow 0^+} e^{-iE_0 T - E_0 \epsilon T} \left(|\Omega\rangle \langle \Omega | 0 \rangle + \sum\!\!\!\!\!\!\int_n e^{-i(E_n - E_0)T - \epsilon T(E_n - E_0)} |n\rangle \langle n | 0 \rangle \right)
\end{aligned} \tag{1.14}$$

The limits are evaluated by first taking T to infinity, then only after that take $\epsilon \rightarrow 0^+$. Doing this, the sum is dominated by the ground state contribution, since each excited state also has a $e^{-\epsilon T(E_n - E_0)}$ suppression factor (in addition to the leading suppression factor).

Argument 2: With the hand waving required for the argument above, it's worth pointing other (less formal) ways to arrive at the same result. We can write

$$\sum\!\!\!\!\!\!\int_n |n\rangle \langle n| \rightarrow \sum_k \int \frac{d^3 p}{(2\pi)^3} |\mathbf{p}, k\rangle \langle \mathbf{p}, k| \tag{1.15}$$

where k is some unknown quantity that we are summing over. If we have

$$H |\mathbf{p}, k\rangle = E_{\mathbf{p},k} |\mathbf{p}, k\rangle , \tag{1.16}$$

then

$$e^{-iHT} \sum\!\!\!\!\!\!\int |n\rangle \langle n| = \sum_k \int \frac{d^3 p}{(2\pi)^3} |\mathbf{p}, k\rangle e^{-iE_{\mathbf{p},k}} \langle \mathbf{p}, k| . \tag{1.17}$$

If we take matrix elements

$$\begin{aligned}
\langle A| e^{-iHT} \sum\!\!\!\!\!\!\int |n\rangle \langle n| |B\rangle &= \sum_k \int \frac{d^3 p}{(2\pi)^3} \langle A | \mathbf{p}, k\rangle e^{-iE_{\mathbf{p},k}} \langle \mathbf{p}, k | B\rangle \\
&= \sum_k \int \frac{d^3 p}{(2\pi)^3} e^{-iE_{\mathbf{p},k}} f(\mathbf{p}).
\end{aligned} \tag{1.18}$$

If we assume that $f(\mathbf{p})$ is a well behaved smooth function, we have "infinite" frequency oscillation within the envelope provided by the amplitude of that function, as depicted in fig. 1.1. The Riemann-Lebesgue lemma [2] describes such integrals, the result of which is that such an integral goes to zero. This is a different sort of hand waving argument, but either way, we can argue that only the ground state contributes to the sum eq. (1.13) above.

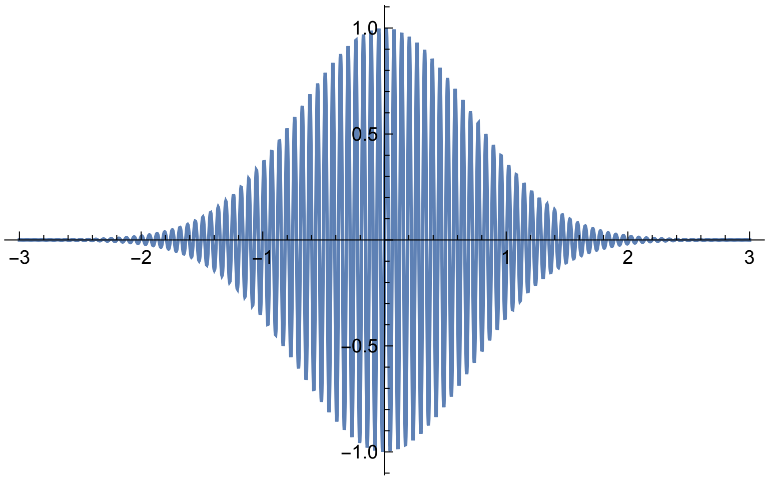

Figure 1.1: High frequency oscillations within envelope of well behaved function.

Ground state of the perturbed Hamiltonian. With the excited states ignored, we are left with

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle \tag{1.19}$$

in the $T \to \infty(1-i\epsilon)$ limit. We can now write the ground state as

$$\begin{aligned}|\Omega\rangle &= \left.\frac{e^{iE_0T-iHT}|0\rangle}{\langle\Omega|0\rangle}\right|_{T\to\infty(1-i\epsilon)}\\ &= \left.\frac{e^{-iHT}|0\rangle}{e^{-iE_0T}\langle\Omega|0\rangle}\right|_{T\to\infty(1-i\epsilon)}.\end{aligned} \tag{1.20}$$

Shifting the very large $T \to T + t_0$ shouldn't change things, so

$$|\Omega\rangle = \left.\frac{e^{-iH(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle}\right|_{T\to\infty(1-i\epsilon)}. \tag{1.21}$$

A bit of manipulation shows that the operator in the numerator has the structure of a time evolution operator.

Claim: (DIY): Equation (1.4), eq. (1.6) may be generalized to

$$U(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)} = T\exp\left(-i\int_{t'}^{t} H_{\mathrm{I,int}}(t'')dt''\right). \tag{1.22}$$

Observe that we recover eq. (1.6) when $t' = t_0$. Using eq. (1.22) we find

$$\begin{aligned}U(t_0,-T)|0\rangle &= e^{iH_0(t_0-t_0)}e^{-iH(t_0+T)}e^{-iH_0(-T-t_0)}|0\rangle\\ &= e^{-iH(t_0+T)}e^{-iH_0(-T-t_0)}|0\rangle\\ &= e^{-iH(t_0+T)}|0\rangle,\end{aligned} \tag{1.23}$$

where we use the fact that $e^{iH_0\tau}|0\rangle = (1 + iH_0\tau + \cdots)|0\rangle = 1|0\rangle$, since $H_0|0\rangle = 0$.

We are left with

$$\boxed{|\Omega\rangle = \frac{U(t_0, -T)|0\rangle}{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle}.} \tag{1.24}$$

We are close to where we want to be. Wednesday we finish off, and then start scattering and Feynman diagrams.

Bibliography

[1] Michael E Peskin and Daniel V Schroeder. *An introduction to Quantum Field Theory*. Westview, 1995. 1.2

[2] Wikipedia contributors. Riemann-lebesgue lemma — Wikipedia, the free encyclopedia, 2018. URL https://en.wikipedia.org/w/index.php?title=Riemann%E2%80%93Lebesgue_lemma&oldid=856778941. [Online; accessed 29-October-2018]. 1.2