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PHY2403H Quantum Field Theory. Lecture 15: Perturbation ground state, time evolution operator, time ordered product, interaction. Taught by Prof. Erich Poppitz

DISCLAIMER: Very rough notes from class, with some additional side notes. These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

1.1 Review

We developed the interaction picture representation, which is really the Heisenberg picture with respect to H_0.

Recall that we found

$$U(t, t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}, \tag{1.1}$$

with solution

$$U(t, t') = T \exp\left(-i \int_{t'}^{t} H_{\text{I,int}}(t'') dt'' \right), \tag{1.2}$$

$$\begin{aligned} U(t, t')^\dagger &= T \exp\left(i \int_{t'}^{t} H_{\text{I,int}}(t'') dt'' \right) \\ &= T \exp\left(-i \int_{t}^{t'} H_{\text{I,int}}(t'') dt'' \right) \\ &= U(t', t), \end{aligned} \tag{1.3}$$

and can use this to calculate the time evolution of a field

$$\phi(\mathbf{x}, t) = U^\dagger(t, t_0) \phi_I(\mathbf{x}, t) U(t, t_0) \tag{1.4}$$

and found the ground state ket for H was

$$\left| \Omega \right\rangle = \left. \frac{U(t_0, -T) \left| 0 \right\rangle}{e^{-iE_0(T-t_0)} \left\langle \Omega | 0 \right\rangle} \right|_{T \to \infty(1-i\epsilon)}. \tag{1.5}$$

Question: What's the point of this, since it is self referential?

Answer: We will see, and also see that it goes away. Alternatively, you can write it as

$$|\Omega\rangle \langle\Omega|0\rangle = \left.\frac{U(t_0, -T)\,|0\rangle}{e^{-iE_0(T-t_0)}}\right|_{T\rightarrow\infty(1-i\epsilon)}.$$

We can also show that

$$\langle\Omega| = \left.\frac{\langle 0|\, U(T, t_0)}{e^{-iE_0(T-t_0)}\,\langle 0|\Omega\rangle}\right|_{T\rightarrow\infty(1-i\epsilon)}. \tag{1.6}$$

Our goal is still toe calculate

$$\langle\Omega|\, T\phi(x)\phi(y)\,|\Omega\rangle\,. \tag{1.7}$$

Claim: the "LSZ" theorem (a neat way of writing this) relates this to S matrix elements.

Assuming $x^0 > y^0$

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \frac{\langle 0|\, U(T, t_0)U^\dagger(x^0, t^0)\phi_I(x)U(x^0, t^0)U^\dagger(y^0, t^0)\phi_I(y)U(y^0, t^0)U(t_0, -T)\,|0\rangle}{e^{-i2E_0T}\left|\langle 0|\Omega\rangle\right|^2} \tag{1.8}$$

Normalize $\langle\Omega|\Omega\rangle = 1$, gives

$$\begin{aligned} 1 &= \frac{\langle 0|\, U(T, t_0)U(t_0, -T)\,|0\rangle}{e^{-i2E_0T}\left|\langle 0|\Omega\rangle\right|^2} \\ &= \frac{\langle 0|\, U(T, -T)\,|0\rangle}{e^{-i2E_0T}\left|\langle 0|\Omega\rangle\right|^2}, \end{aligned} \tag{1.9}$$

so that

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \frac{\langle 0|\, U(T, t_0)U^\dagger(x^0, t^0)\phi_I(x)U(x^0, t^0)U^\dagger(y^0, t^0)\phi_I(y)U(y^0, t^0)U(t_0, -T)\,|0\rangle}{\langle 0|\, U(T, -T)\,|0\rangle} \tag{1.10}$$

For $t_1 > t_2 > t_3$

$$\begin{aligned} U(t_1, t_2)U(t_2, t_3) &= Te^{-i\int_{t_2}^{t_1} H_I}\, Te^{-i\int_{t_3}^{t_2} H_I} \\ &= T\left(e^{-i\int_{t_2}^{t_1} H_I} e^{-i\int_{t_3}^{t_2} H_I}\right) \\ &= T(e^{-i\int_{t_3}^{t_1} H_I}), \end{aligned} \tag{1.11}$$

with an end result of

$$U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3). \tag{1.12}$$

(DIY: work through the details – this is a problem in [1])

This gives

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \frac{\langle 0|\, U(T, x^0)\phi_I(x)U(x^0, y^0)\phi_I(y)U(y^0, -T)\,|0\rangle}{\langle 0|\, U(T, -T)\,|0\rangle}. \tag{1.13}$$

If $y^0 > x^0$ we have the same result, but the y's will come first.

Claim:

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \frac{\langle 0|\,T\left(\phi_I(x)\phi_I(y)e^{-i\int_{-T}^{T}H_{\mathrm{I,int}}(t')dt'}\right)|0\rangle}{\langle 0|\,T(e^{-i\int_{-T}^{T}H_{\mathrm{I,int}}(t')dt'})\,|0\rangle}. \tag{1.14}$$

More generally

$$\langle\Omega|\,\phi_I(x_1)\cdots\phi_I(x_n)\,|\Omega\rangle = \frac{\langle 0|\,T\left(\phi_I(x_1)\cdots\phi_I(x_n)e^{-i\int_{-T}^{T}H_{\mathrm{I,int}}(t')dt'}\right)|0\rangle}{\langle 0|\,T(e^{-i\int_{-T}^{T}H_{\mathrm{I,int}}(t')dt'})\,|0\rangle}. \tag{1.15}$$

This is the holy grail of perturbation theory.

In QFT II you will see this written in a path integral representation

$$\langle\Omega|\,\phi_I(x_1)\cdots\phi_I(x_n)\,|\Omega\rangle = \frac{\int[\mathcal{D}\phi]\phi(x_1)\phi(x_2)\cdots\phi(x_n)e^{-iS[\phi]}}{\int[\mathcal{D}\phi]e^{-iS[\phi]}}. \tag{1.16}$$

1.2 Unpacking it.

$$\begin{aligned}\int_{-T}^{T}H_{\mathrm{I,int}}(t) &= \int_{-T}^{T}\int d^3\mathbf{x}\frac{\lambda}{4}\left(\phi_I(\mathbf{x},t)\right)^4 \\ &= \int d^4x\frac{\lambda}{4}\left(\phi_I\right)^4\end{aligned} \tag{1.17}$$

so we have

$$\frac{\langle 0|\,T\left(\phi_I(x_1)\cdots\phi_I(x_n)e^{-i\frac{\lambda}{4}\int d^4x\phi_I^4(x)}\right)|0\rangle}{\langle 0|\,Te^{-i\frac{\lambda}{4}\int d^4x\phi_I^4(x)}\,|0\rangle}. \tag{1.18}$$

The numerator expands as

$$\begin{aligned}&\langle 0|\,T\left(\phi_I(x_1)\cdots\phi_I(x_n)\right)|0\rangle - i\frac{\lambda}{4}\int d^4x\,\langle 0|\,T\left(\phi_I(x_1)\cdots\phi_I(x_n)\phi_I^4(x)\right) \\ &+\frac{1}{2}\left(-i\frac{\lambda}{4}\right)^2\int d^4xd^4y\,\langle 0|\,T\left(\phi_I(x_1)\cdots\phi_I(x_n)\phi_I^4(x)\phi_I^4(y)\right)|0\rangle + \cdots\end{aligned} \tag{1.19}$$

so we see that the problem ends up being the calculation of time ordered products.

1.3 Calculating perturbation

Let's simplify notation, dropping interaction picture suffixes, writing $\phi(x_i) = \phi_i$.

Let's calculate $\langle 0|\,T\left(\phi_1\cdots\phi_n\right)|0\rangle$. For $n = 2$ we have

$$\begin{aligned}\langle 0|\,T\left(\phi_1\cdots\phi_n\right)|0\rangle &= D_F(x_1 - x_2) \\ &\equiv D_F(1-2)\end{aligned} \tag{1.20}$$

TO BE CONTINUED. The rest of the lecture was very visual, and hard to type up. I'll do so later.

1.4 Problems

Exercise 1 $U(T, t_0)U(t_0, -T)$

Show that

$$U(T, t_0)U(t_0, -T) = U(T, -T)$$

Answer of exercise 1

We can see that from

$$\begin{aligned} U(T, t_0) &= e^{iH_0(T-t_0)}e^{-iH(T-t_0)}\cancel{e^{-iH_0(t_0-t_0)}} \\ U(t_0, -T) &= \cancel{e^{iH_0(t_0-t_0)}}e^{-iH(t_0--T)}e^{-iH_0(-T-t_0)}, \end{aligned} \tag{1.21}$$

so

$$\begin{aligned} U(T, t_0)U(t_0, -T) &= e^{iH_0(T-t_0)}e^{-iH(T-t_0)}e^{-iH(t_0+T)}e^{-iH_0(-T-t_0)} \\ &= e^{iH_0(T-t_0)}e^{-iH2T}e^{-iH_0(-T-t_0)}, \end{aligned} \tag{1.22}$$

whereas

$$\begin{aligned} U(T, -T) &= e^{iH_0(T-t_0)}e^{-iH(T--T)}e^{-iH_0(-T-t_0)} \\ &= e^{iH_0(T-t_0)}e^{-iH2T}e^{-iH_0(-T-t_0)} \end{aligned} \tag{1.23}$$

Bibliography

[1] Michael E Peskin and Daniel V Schroeder. *An introduction to Quantum Field Theory*. Westview, 1995. 1.1