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PHY2403H Quantum Field Theory. Lecture 15b: Wick's theorem, vacuum expectation, Feynman diagrams, ϕ^4 interaction, tree level diagrams, scattering, cross section, differential cross section. Taught by Prof. Erich Poppitz

DISCLAIMER: Very rough notes from class, with some additional side notes. These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

1.1 Wick contractions

Here's a double dose of short hand, first an abbreviation for the Feynman propagator

$$D_F(1-2) \equiv D_F(x_1, x_2), \tag{1.1}$$

and second

$$\overline{\phi_i \phi_j} = D_F(i-j), \tag{1.2}$$

which is called a contraction.

Contractions allow time ordered products to be written in a compact form. In HW4 we are set with the task of demonstrating how this is done (i.e. proving Wick's theorem.)

Theorem 1.1: Wick's theorem.

Sounds like stating the theorem is difficult, but the rough idea (from the example below) is that the time ordering of the fields has all the combinations of the pairwise contractions and normal ordered fields.

Illustrating by example for the time ordering of $n = 4$ fields, we have

$$\begin{aligned} T(\phi_1\phi_2\phi_3\phi_4) = {} & {:}\phi_1\phi_2\phi_3\phi_4{:} + \overline{\phi_1\phi_2}{:}\phi_3\phi_4{:} + \overline{\phi_1\phi_3}{:}\phi_2\phi_4{:} + \overline{\phi_1\phi_4}{:}\phi_2\phi_3{:} + \overline{\phi_2\phi_3}{:}\phi_1\phi_4{:} \\ & + \overline{\phi_2\phi_4}{:}\phi_1\phi_3{:} + \overline{\phi_3\phi_4}{:}\phi_1\phi_2{:} + \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \overline{\phi_1\phi_3}\,\overline{\phi_2\phi_4} + \overline{\phi_1\phi_4}\,\overline{\phi_2\phi_3}. \end{aligned} \tag{1.3}$$

Theorem 1.2: Corollary: Vacuum expectation of Wick's theorem expansion

For n even

$$\langle 0| T(\phi_1\phi_2\cdots\phi_n)|0\rangle = \overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4}\overbracket{\phi_5\phi_6}\cdots\overbracket{\phi_{n-1}\phi_n} + \text{all other terms}.$$

For n odd, this vanishes.

1.2 Simplest Feynman diagrams

For $n = 4$ we have

$$\langle 0| T(\phi_1\phi_2\phi_3\phi_4)|0\rangle = \overbracket{\phi_1\phi_2}\overbracket{\phi_3\phi_4} + \overbracket{\phi_1\phi_3}\overbracket{\phi_2\phi_4} + \overbracket{\phi_1\phi_4}\overbracket{\phi_2\phi_3}, \tag{1.4}$$

the set of Wick contractions can be written pictorially fig. 1.1, and are called Feynman diagrams

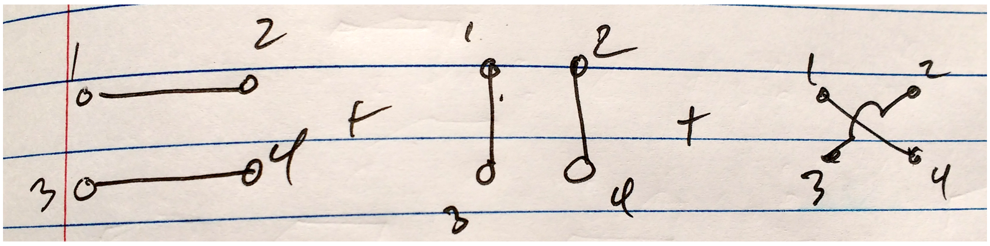


Figure 1.1: Simplest Feynman diagrams.

These are the very simplest Feynman diagrams.

1.3 ϕ^4 interaction

Introducing another shorthand, we will use an expectation like notation to designate the matrix element for the vacuum state

$$\langle \text{blah} \rangle = \langle 0| \text{blah} |0\rangle . \tag{1.5}$$

For the ϕ^4 theory, this allows us to write the numerator of the perturbed ground state interaction as

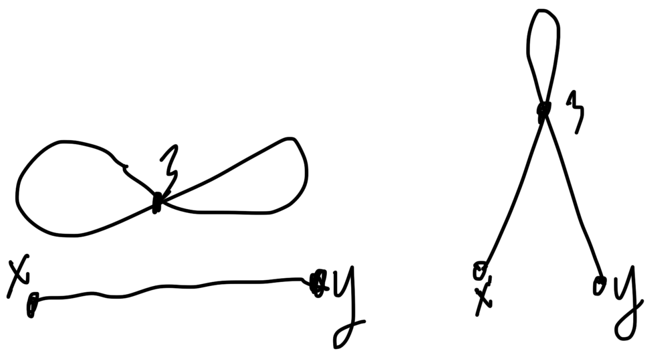
$$\begin{aligned}\langle \Omega| \phi(x)\phi(y)|\Omega\rangle &\sim \langle 0| T\left(\phi_I(x)\phi_I(y)e^{-i\int_{-T}^{T} H_{\mathrm{I,int}}(t')dt'}\right)|0\rangle \\ &= \left\langle \phi_I(x)\phi_I(y)e^{-i\int d^4z\phi^4(z)}\right\rangle .\end{aligned} \tag{1.6}$$

To first order, this is

$$\left\langle T\phi_x\phi_y\right\rangle - i\frac{\lambda}{4}\int d^4z \left\langle T\phi_x\phi_y\phi_z\phi_z\phi_z\phi_z\right\rangle , \tag{1.7}$$

The first braket has the pictorial representation sketched in fig. 1.2. whereas the second has the diagrams sketched in fig. 1.3.

Figure 1.2: First integral diagram.



(a) **(b)**

Figure 1.3: Second integral diagrams.

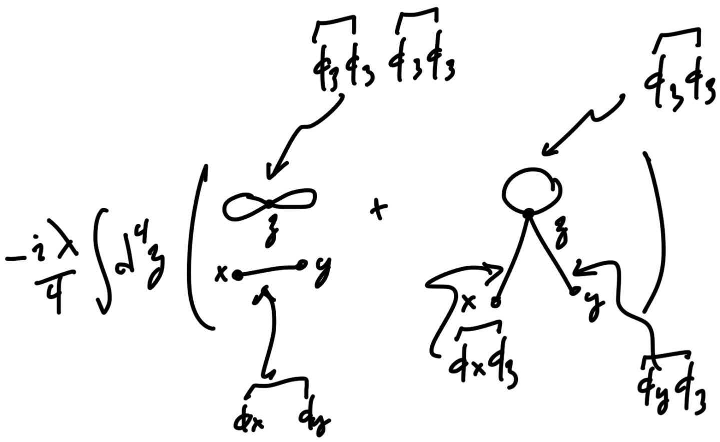


Figure 1.4: Integrals as diagrams.

We can depict the entire second integral in diagrams as sketched in fig. 1.4.

Solving for the perturbed ground state can now be thought of as reduced to drawing pictures. Each line from $x \rightarrow x'$ represents a propagator $D_F(x-x')$, and each vertex $-i\lambda \int d^4 z \times$ symmetry coefficients.[1]

We may also translate back from the diagrams to an algebraic representation. For the first order ϕ^4 interaction, that is

$$\langle T\phi_x \phi_y \rangle - \frac{i\lambda}{4} \int d^4 z D_F(x-y) D_F^2(z-z) + D_F(x-z) D_F(y-z). \tag{1.8}$$

Other diagrams can be similarly translated. For example F5 represents

$$\int d^4 z D^2(z-z) = V_3 T \left(\int \frac{d^4 p}{(2\pi)^4} \frac{1}{\mathbf{p}^2 - m^2 + i\epsilon} \right)^2. \tag{1.9}$$

Clearly, additional interpretation will be required, since this diverges. The resolution of this unfortunately has to be deferred to QFT II, where renormalization is covered.

1.4 Tree level diagrams.

We would like to only discuss tree level diagrams, which exclude diagrams like fig. 1.5 [2].

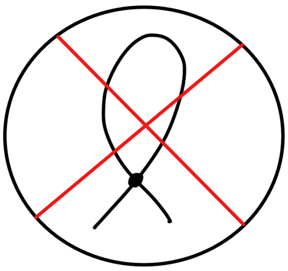


Figure 1.5: Not a tree level diagram.

For the braket [3]

$$\left\langle \int d^4 z \phi_1 \phi_2 \phi_3 \phi_4 \phi_z \phi_z \phi_z \phi_z \right\rangle \tag{1.10}$$

we draw diagrams like those of fig. 1.6, the first of which is a tree level diagram.

[1] Symmetry coefficients weren't discussed until the next lecture. This means making combinatorial arguments to count the number of equivalent diagrams.

[2] I think this is what is referred to as connected, amputated graphs in the next lecture. Such diagrams are the ones of interest for scattering and decay problems.

[3] I'd written: $\langle \int \phi_1 \phi_2 \phi_3 \phi_4 \lambda \int d^4 z \phi_z \phi_z \phi_z \phi_z \rangle$. Is this two fold integral what was intended, or my correction in eq. (1.10)?

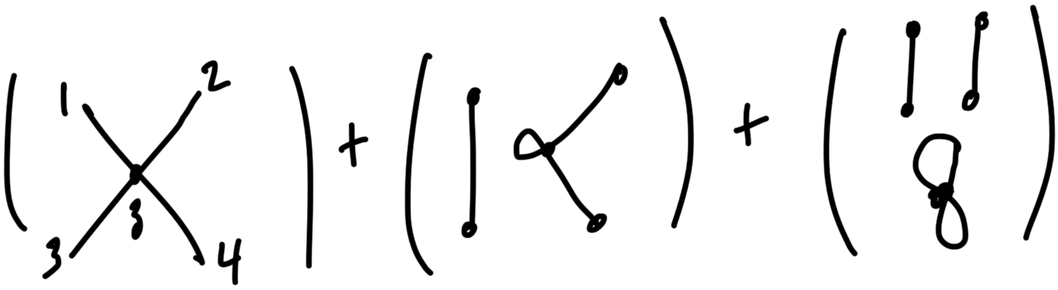


Figure 1.6: First order interaction diagrams.

1.5 Scattering.

In QM we did lots of scattering problems as sketched in fig. 1.7, and were able to compute the reflected and transmitted wave functions and quantities such as the reflection and transmission coefficients

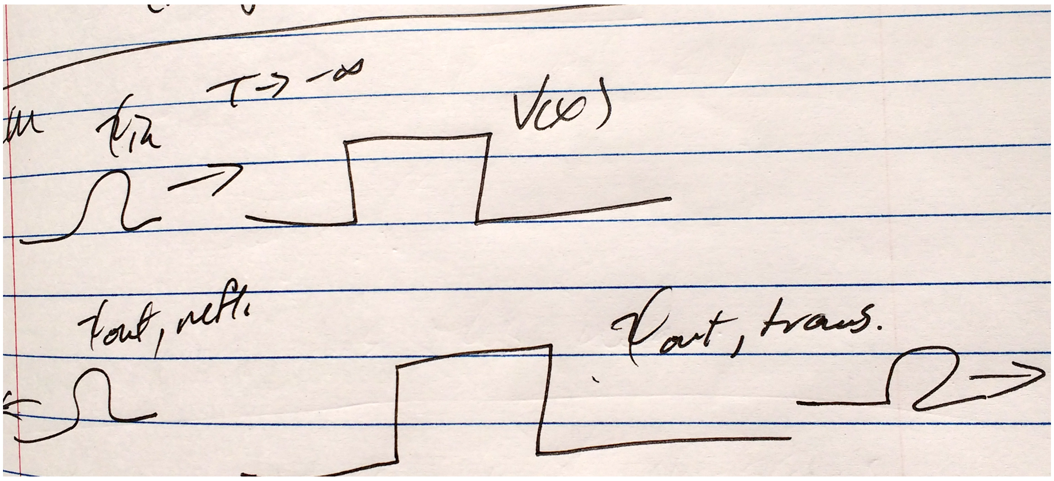


Figure 1.7: Reflection and transmission of wave packets.

$$
\begin{aligned}
R &= \frac{\left|\Psi_{\text{ref}}\right|^2}{\left|\Psi_{\text{in}}\right|^2} \\
T &= \frac{\left|\Psi_{\text{trans}}\right|^2}{\left|\Psi_{\text{in}}\right|^2}.
\end{aligned}
\tag{1.11}
$$

We'd like to consider scattering in some region of space with a non-zero potential, such as the scattering of a plane wave with known electron flux rate as sketched in fig. 1.8. We can imagine that we have a detector capable of measuring the number of electrons with momentum $\mathbf{p}_{\text{out}}$ per unit time.

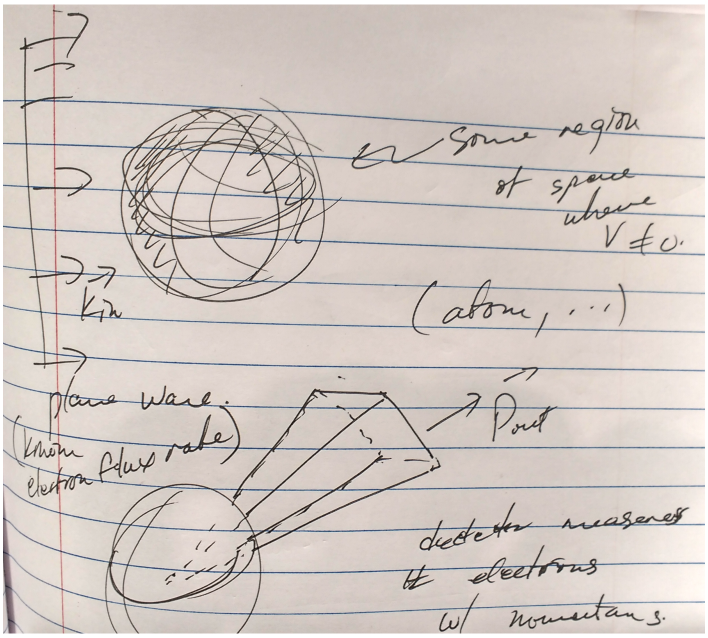


Figure 1.8: Plane wave scattering off a potential.

Definition 1.1: Total cross section (X-section).

$$\sigma_{\text{total}} = \frac{\text{number of scattering events with } \mathbf{p}_{\text{out}} \neq \mathbf{k}_{\text{in}} \text{ per unit time}}{\text{Flux of incoming particles}},$$

where the flux is the number of particles crossing a unit area in unit time.

Units of the x-section are (with $\hbar = c = 1$)

$$[\sigma] = \text{area} = \frac{1}{M^2}. \tag{1.12}$$

The concept of scattering cross section may not be new, as it can even be encountered in classical mechanics. One such scenario is sketched in fig. 1.9 where the cross section is just the area

$$\sigma = \pi R^2. \tag{1.13}$$

Other classical fields where cross section is encountered includes antenna theory (radar scattering profiles, ...).

Definition 1.2: Differential cross section.

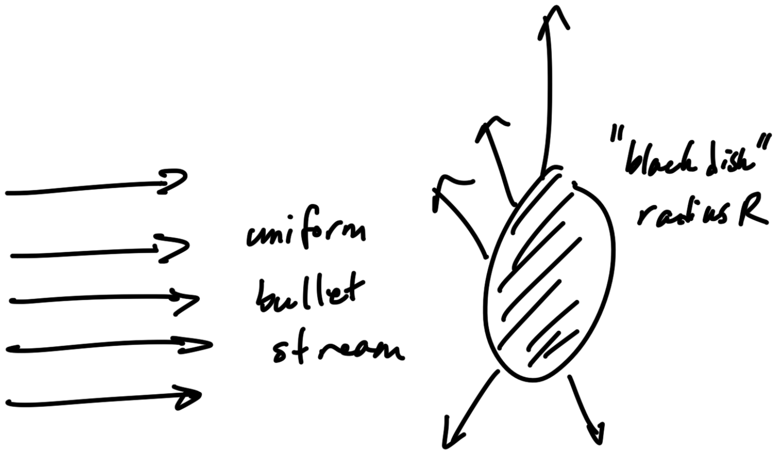


Figure 1.9: Classical scattering.

$$\frac{d^3\sigma}{dp_x dp_y dp_z} = \frac{\text{number of scattering events with } \mathbf{p}_{\text{out}} \text{ between } (\mathbf{p}, \mathbf{p} + \Delta\mathbf{p})}{\text{flux}}.$$