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PHY2403H Quantum Field Theory. Lecture 16: Differential cross section, scattering, pair production, transition amplitude, decay rate, S-matrix, connected and amputated diagrams, vacuum fluctuation, symmetry coeffiecient. Taught by Prof. Erich Poppitz

DISCLAIMER: Very rough notes from class, with some additional side notes. These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

1.1 Review

We finished by defining the differential cross section

Definition 1.1: Differential cross section.

$$\frac{d^3\sigma}{dp_x dp_y dp_z} = \frac{\text{number of scattering events with } \mathbf{p}_{\mathrm{f}} \text{ between } (\mathbf{p}_{\mathrm{f}}, \mathbf{p}_{\mathrm{f}} + d\mathbf{p}_{\mathrm{f}})}{\text{flux of incoming particles}}.$$

1.2 Scattering

In QFT we typically study $2 \rightarrow n$ inelastic scattering. Most commonly the nature of the final state particles are different from the nature of the incoming state.

For example, we can collide an electron and anti-electron, and can get muon and anti-muon particles as sketched in fig. 1.1, or pions as sketched in fig. 1.2, or even both as sketched in fig. 1.3.

In the $\lambda\phi^4$ theory we can have scattering events such as $2 \rightarrow 2$ and $2 \rightarrow 2n$ production as sketched in fig. 1.4.

How to calculate in QFT. Initial state of 2 particles A, B with initial state

$$\left|\mathbf{k}_A, \mathbf{k}_B\right\rangle_{\mathrm{in}, T\rightarrow -\infty} \tag{1.1}$$

and final n-particle state

$$\left|\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n\right\rangle_{\mathrm{out}, T\rightarrow +\infty} \tag{1.2}$$

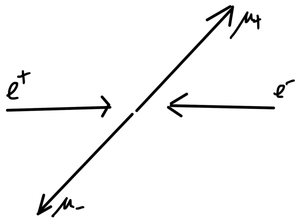


Figure 1.1: Muon pair production.

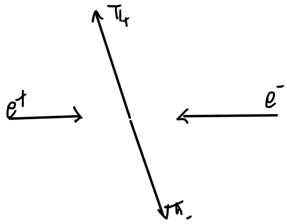


Figure 1.2: Pion pair production.

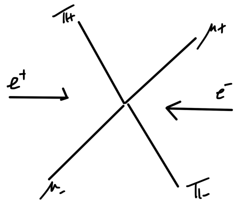
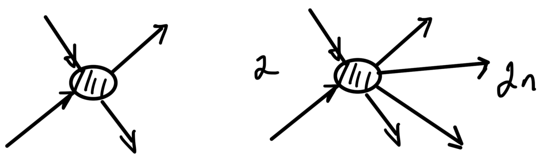


Figure 1.3: Muon and pion pair production.



(a) **(b)**

Figure 1.4: lambda fourth scattering events.

The QM transition amplitude from the initial to the final state is

$$
{}_{\text{out}}\langle \mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n | \, | \mathbf{k}_A, \mathbf{k}_B \rangle_{\text{in}} = \langle \mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n | \, e^{-2iHT} \, | \mathbf{k}_A, \mathbf{k}_B \rangle . \tag{1.3}
$$

This is the amplitude for $AB \to 1 \cdots n$. Ultimately, we want the scattering x-section.

We will also be interested in decay rates, as there are unstable particles in QFT that can decay. This doesn't happen in $\lambda\phi^4$ theory. In a theory with 2 scalar fields Φ, φ with $m_\Phi > 2m_\varphi$. A possible interaction for such a theory is

$$
H_{\text{int}} = \mu \Phi \varphi^2 , \tag{1.4}
$$

which would permit $\Phi \to \varphi\varphi$ decays. HW4 has a coupling like $(h/V)\partial_\mu \phi^a \partial^\mu \phi^a$ for which a $h \to \phi^a \phi^a$ decay is possible.

Definition 1.2: Decay rate.

The decay rate is defined as

$$
\Gamma = \frac{\text{Number of decays } \Phi \to \varphi\varphi \text{ in unit time}}{\text{Number of } \Phi \text{ particles present}}
$$

What is the amplitude for such a decay transition?

$$
\langle \mathbf{k}_\phi |_{\text{in}, T \to -\infty} \to \langle \mathbf{k}_1, \mathbf{k}_2 |_{\text{out}, T \to +\infty} . \tag{1.5}
$$

The amplitude for $\mathbf{k}_\phi \to \mathbf{k}_1, \mathbf{k}_2$.

$$
\langle \mathbf{k}_1, \mathbf{k}_2 | \, e^{-i2HT} \, | \mathbf{k}_\phi \rangle = {}_{\text{out}}\langle \mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_\phi \rangle \tag{1.6}
$$

mysterious seeming statement something like : "The decays are essentially due to interactions with vacuum fluctuations."

1.3 Calculating interactions

We write

$$
\begin{aligned}
{}_{\text{out}}\langle \mathbf{p}_1, \cdots \mathbf{p}_n | \mathbf{k}_A, \mathbf{k}_B \rangle_{\text{in}} &= \lim_{T \to \infty} \langle \mathbf{p}_1, \cdots \mathbf{p}_n | \, e^{-i2HT} \, | \mathbf{k}_A, \mathbf{k}_B \rangle \\
&= \langle \mathbf{p}_1, \cdots \mathbf{p}_n | \, \hat{S} \, | \mathbf{k}_A, \mathbf{k}_B \rangle \\
&= \langle \mathbf{p}_1, \cdots \mathbf{p}_n | \, \mathbf{1} + i\hat{T} \, | \mathbf{k}_A, \mathbf{k}_B \rangle ,
\end{aligned} \tag{1.7}
$$

where $\hat{S}$ is called the S-matrix or scattering matrix, which is decomposed into a unit portion $\mathbf{1}$ which is a convenient way to exclude events with no scattering. $\mathbf{1}$ contributes for $n = 2$ only, but is an n scattering amplitude. We are really interested in the $i\hat{T}$ portion of this amplitude

$$
\langle \mathbf{p}_1, \cdots \mathbf{p}_n | \, i\hat{T} \, | \mathbf{k}_A, \mathbf{k}_B \rangle = (2\pi)^4 \delta^{(4)}(\mathbf{k}_A + \mathbf{k}_B - \sum_{i=1}^{n} \mathbf{p}_i) \times iM(\mathbf{k}_A + \mathbf{k}_B \to \mathbf{p}_1 \cdots \mathbf{p}_n). \tag{1.8}
$$

This amounts to a definition of M. Recall that we found

$$
\begin{aligned}
U(T,-T) &= T\left(e^{-i\int_{-T}^{T} H_I(t')dt'}\right) \\
&= e^{iH_0(T-t_0)} e^{-i2HT} e^{-iH_0(-T-t_0)}.
\end{aligned}
\tag{1.9}
$$

We want to replace the e^{-i2HT} in the matrix element above with U.

In perturbation theory, we assume (conjecture) that

$$
\begin{aligned}
\left|\mathbf{k}_A, \mathbf{k}_B\right\rangle &\sim \left|\mathbf{k}_A, \mathbf{k}_B\right\rangle_{\mathrm{o}} \\
&\sim \mathrm{const}\, a^\dagger_{\mathbf{k}_A} a^\dagger_{\mathbf{k}_B} \left|0\right\rangle
\end{aligned}
\tag{1.10}
$$

Because we'll be squaring the amplitudes, we can assume that the $e^{iH_0(T-t_0)}$ will result in just phase factors that won't survive, so in eq. (1.7) we can insert U

$$
{}_{\mathrm{out}}\left\langle \mathbf{p}_1, \cdots \mathbf{p}_n \middle| \mathbf{k}_A, \mathbf{k}_B \right\rangle_{\mathrm{in}} = \lim_{T\to\infty} \left\langle \mathbf{p}_1, \cdots \mathbf{p}_n \right| U(T,-T) \left|\mathbf{k}_A, \mathbf{k}_B\right\rangle
\tag{1.11}
$$

$$
\left\langle \mathbf{p}_1, \cdots \mathbf{p}_n \right| i\hat{T} \mathbf{k}_A, \mathbf{k}_B = \lim_{T\to\infty(1-i\epsilon)} {}_0\left\langle \mathbf{p}_1, \cdots \mathbf{p}_n \right| T(e^{-i\int_{-T}^{T} H_i(t')dt'}) \left|\mathbf{k}_A, \mathbf{k}_B\right\rangle_0
\tag{1.12}
$$

These are connected and amputated graphs.

What is "connected and amputated"? Explaining by example. $n = 2, \lambda\phi^4/4!$.

$$
\left\langle 0\right| a_{\mathbf{p}_1} a_{\mathbf{p}_2} \left(\cancel{1} - \frac{i\lambda}{4!} \int d^4x \phi_I^4(x) + \frac{1}{2}\left(\frac{i\lambda}{4!}\right)^2 \int d^4x d^4y \phi_I^4(x)\phi_I^4(y) + \cdots \right) a^\dagger_{\mathbf{k}_A} a^\dagger_{\mathbf{k}_B} \left|0\right\rangle
\tag{1.13}
$$

Here time ordering operations are implied, but not written explicitly. Also, the "amputated" indicates that we are going to be dropping the 1 portion of the exponential expansion (as we've also dropped that in eq. (1.12)). We will also be using a relativistic normalization so that the $a^\dagger_{\mathbf{k}_A} a^\dagger_{\mathbf{k}_B}$ terms include $\sqrt{2\omega_{\mathbf{k}_A} 2\omega_{\mathbf{k}_B}}$ contributions and the $a_{\mathbf{p}_1} a_{\mathbf{p}_2}$ include $\sqrt{2\omega_{\mathbf{p}_1} 2\omega_{\mathbf{p}_2}}$ contributions.

$$
T\phi_I(x_1)\phi_I(x_2) = D_F(x_1 - x_2)
\tag{1.14}
$$

When we look at

$$
\begin{aligned}
\phi_I(x_1) a^\dagger_{\mathbf{k}} \sqrt{2\omega_{\mathbf{k}}} &= \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip\cdot x}}{\sqrt{2\omega_{\mathbf{p}}}} a_{\mathbf{p}} a^\dagger_{\mathbf{k}} \sqrt{2\omega_{\mathbf{k}}} \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip\cdot x}}{\sqrt{2\omega_{\mathbf{p}}}} \delta^{(3)}(\mathbf{p} - \mathbf{k}) \sqrt{2\omega_{\mathbf{k}}} \\
&= e^{-ik\cdot x}.
\end{aligned}
\tag{1.15}
$$

Similarly

$$
\begin{aligned}
a_{\mathbf{p}}\phi_I(x_1)\sqrt{2\omega_{\mathbf{p}}} &= \int \frac{d^3k}{(2\pi)^3}\frac{e^{ik\cdot x}}{\sqrt{2\omega_{\mathbf{k}}}} a_{\mathbf{p}} a^\dagger_{\mathbf{k}}\sqrt{2\omega_{\mathbf{k}}} \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{e^{ik\cdot x}}{\sqrt{2\omega_{\mathbf{k}}}} \delta^{(3)}(\mathbf{p}-\mathbf{k})\sqrt{2\omega_{\mathbf{k}}} \\
&= e^{+ip\cdot x}.
\end{aligned}
\tag{1.16}
$$

Summarizing

$$
\begin{aligned}
\phi_I(x_1)a^\dagger_{\mathbf{p}} &= e^{-ip\cdot x} \\
a_{\mathbf{p}}\phi_I(x_1) &= e^{ip\cdot x}.
\end{aligned}
\tag{1.17}
$$

1.4 Example diagrams.

We want to examine the relevant diagrams corresponding to a transition amplitudes for the ϕ^4 theory. Contractions such as

$$
\langle a_{\mathbf{p}_1} a_{\mathbf{p}_2} | a^\dagger_{\mathbf{k}_A} a^\dagger_{\mathbf{k}_B} \rangle_0 . \tag{1.18}
$$

result in diagrams that are <u>not connected</u> as sketched in fig. 1.5.

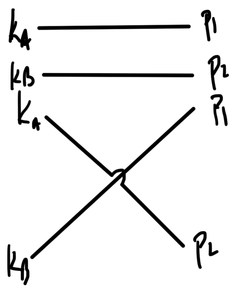


Figure 1.5: Not connected diagrams.

There are no other possibilities for the first order (and these ones are not interesting). For the second order transition amplitudes we want to sum of all the contractions for the expectation

$$
\left\langle a_{\mathbf{p}_1} a_{\mathbf{p}_2} \phi_I^4(x) a^\dagger_{\mathbf{k}_A} a^\dagger_{\mathbf{k}_B} \right\rangle = -i\frac{\lambda}{4!}\sum \text{all contractions.} \tag{1.19}
$$

Our diagrams include fig. 1.6, which are not connected. The figure eight is a vacuum fluctuation that represents virtual processes. Another diagram is fig. 1.7, also not connected.

We want diagrams that we will describe as "connected and amputated". We are clearly discarding non-connected diagrams like those above, but will need to demonstrate what is meant by amputated, and will continue to consider examples to make that clear.

Here's another diagram fig. 1.8 that is also not connected. From the diagrams we can construct the

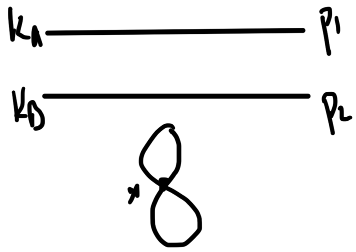


Figure 1.6: Not connected second order interactions, including vacuum fluctuations.

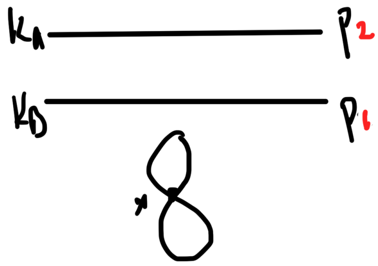


Figure 1.7: Another second order diagram.

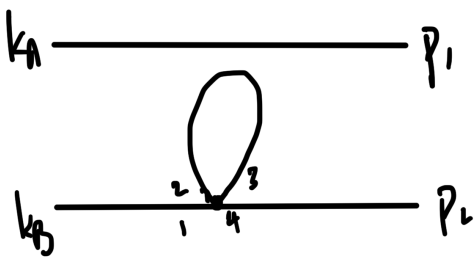


Figure 1.8: Another not-connected diagram.

functionals that they represent. The single line in this one is a $\delta^{(3)}(\mathbf{p}_1 - \mathbf{k}_A)$ whereas the balloon with two strings is

$$\int d^4 x e^{-ik_B \cdot x} D_F(x - x) e^{ip_2 \cdot x}. \tag{1.20}$$

There are similar not-connected variations of the possible diagrams that we will also discard. The connected diagrams all come from contractions such as

$$\langle 0 | a_{\mathbf{p}_1} a_{\mathbf{p}_2} \phi_I^4(x) a^\dagger_{\mathbf{k}_A} a^\dagger_{\mathbf{k}_B} | 0 \rangle \tag{1.21}$$

The diagram for this interaction now has a vertex representing the contractions with $\phi_I^4(x)$ with four edges from that vertex as sketched in fig. 1.9. The algebraic expression for this diagram is

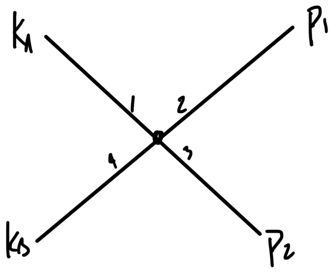


Figure 1.9: Not non-connected diagram.

$$4! \left(\frac{-i\lambda}{4!} \right) \int d^4 x e^{-i(k_A + k_B) \cdot x} e^{i(p_1 + p_2) \cdot x} = -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_A - k_B). \tag{1.22}$$

Such a diagram has the general form

$$(2\pi)^4 \delta^{(4)} \left(\sum \text{in} - \sum \text{final} \right) \times iM(A, B \rightarrow 1, 2), \tag{1.23}$$

so

$$M(A, B \rightarrow 1, 2) = -\lambda. \tag{1.24}$$

Here the "symmetry factor" 4! was added in to count all possible ways of constructing such a diagram.

Next order How about an amplitude like

$$\langle 0 | a_{\mathbf{p}_1} a_{\mathbf{p}_2} \frac{1}{2} \left(\frac{-i\lambda}{4!} \right)^2 \int d^4 x \int d^4 y \phi_I^4(x) \phi_I^4(y) a^\dagger_{\mathbf{k}_A} a^\dagger_{\mathbf{k}_B} | 0 \rangle \tag{1.25}$$

Disconnected diagrams include fig. 1.10. However, we have connected diagrams like fig. 1.11. The loop in this diagram represents an interaction with "vacuum fluctuation". Such an interaction is

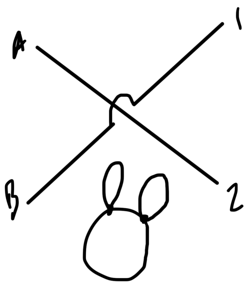


Figure 1.10: Disconnected third order interaction.

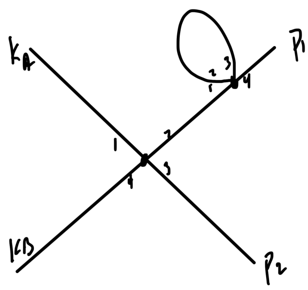


Figure 1.11: Connected diagram.

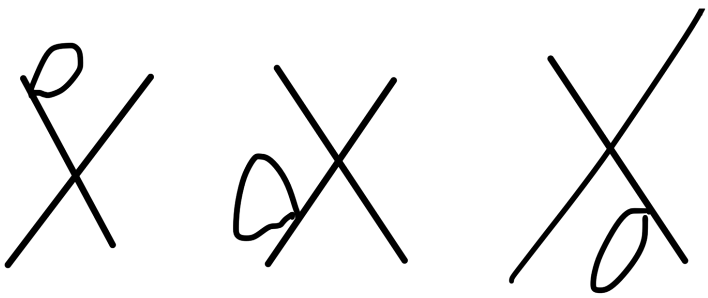


Figure 1.12: Other amputatable diagrams.

not relevant to scattering, and we may consider just the portion of the diagram that leaves off this vacuum fluctuation. This is what is meant by amputated. Amputated diagrams do not include such factors. Other example interactions that may also be amputated include fig. 1.12.

At the next order we can have fun interactions like that of fig. 1.13, which is not amputatable (it connects branches), and must be considered.

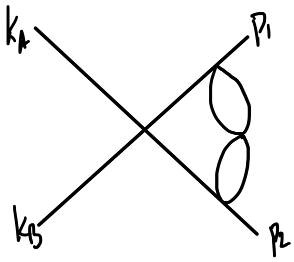
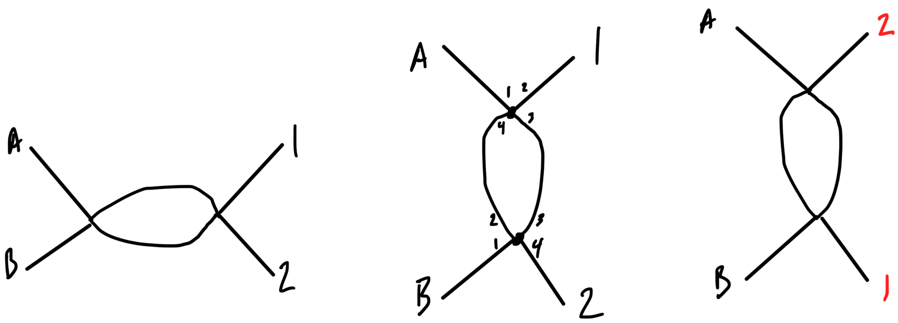


Figure 1.13: Fun interaction.

At the λ^2 order, the relevant diagrams are sketched in fig. 1.14 At this order $\phi^4(x), \phi^4(y)$ each contribute a vertex with 4 edges.



(a) (b) (c)

Figure 1.14: Second order connected amputated diagrams.

> **Definition 1.3: Amputated**
>
> Omit anything that only effects input or output lines.

1.5 The recipe.

The general transition amplitude for a $2 \to n$ event has the form

$$\langle p_1 \cdots p_n | k_A k_B \rangle = (2\pi)^4 \delta^{(4)}(\sum k_{\text{in}} - \sum p_{\text{out}}) iM(A, B \to 1, \cdots n). \tag{1.26}$$

Our recipe is

1. $iM = \sum$ of all connected amputated diagrams, lines and vertices.
2. to every internal line (not connected to input or final particle)
3. associate a propagator

$$\frac{i}{p^2 - m^2 - i\epsilon}, \tag{1.27}$$

   where p is the 4-momentum of the line. External lines are $\equiv 1$.

4. Impose non-conservation with every vertex.
5. integrate $\int d^4p/(2\pi)^4$ over all momenta not fixed.
6. symmetry factors
7. vertex: $(-i\lambda)$.

1.6 Back to our scalar theory

Applying these rules to the diagram fig. 1.15, we get

$$-i\lambda = iM, \tag{1.28}$$

or

$$M = -\lambda. \tag{1.29}$$

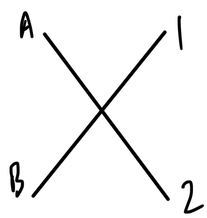


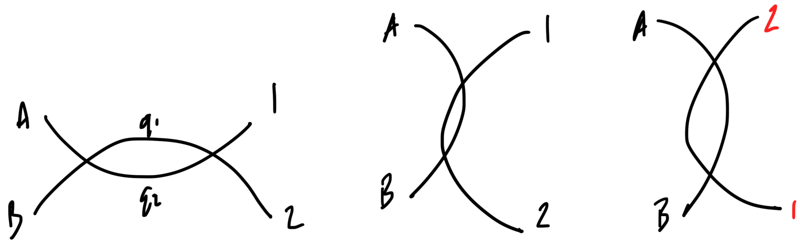
Figure 1.15: First order interaction.

For the second order diagrams The first diagram gives

$$(-i\lambda)^2 \frac{i}{q_1^2 - m^2 - i\epsilon} \frac{i}{q_2^2 - m^2 - i\epsilon}, \tag{1.30}$$

where $q_1 + q_2 = k_A + k_B$, so we can let $q_2 = k_A + k_B - q_1$, which gives

$$\int \frac{d^4q_1}{(2\pi)^4} (-i\lambda)^2 \frac{i}{q_1^2 - m^2 - i\epsilon} \frac{i}{\left(k_A + k_B - q_1\right)^2 - m^2 - i\epsilon} \tag{1.31}$$



(a) (b) (c)

Figure 1.16: Second order diagrams.

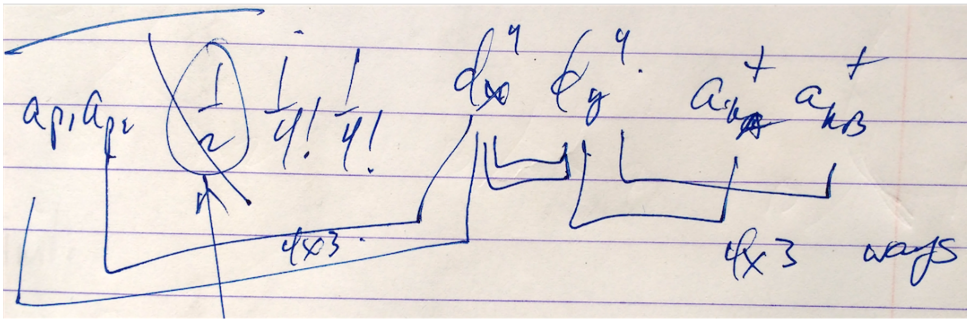


Figure 1.17: Symmetry coefficient counting.

Calculating the symmetry coefficients is a counting game, illustrated roughly in fig. 1.17, where the 1/2 factor was eliminated by the two choices, and the rest by factorial counting (4 ways to pick first, leaving 3 ways for the next choice, two for the next, until the last.) In the end we have a symmetry factor of $(4 \times 3) \times 2 \times (4 \times 3)$.