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# Reflection using Pauli matrices.

In class yesterday (lecture 19, notes not yet posted) we used $\boldsymbol{\sigma}^{\mathrm{T}} = -\sigma_2 \boldsymbol{\sigma} \sigma_2$, which implicitly shows that $(\boldsymbol{\sigma} \cdot \mathbf{x})^{\mathrm{T}}$ is a reflection about the y-axis. This form of reflection will be familiar to a student of geometric algebra (see [1]). I can't recall any mention of the geometrical reflection identity from when I took QM. It's a fun exersize to demonstrate the reflection identity when constrained to the Pauli matrix notation.

**Theorem 1.1: Reflection about a normal.**

Given a unit vector $\hat{\mathbf{n}} \in \mathbb{R}^3$ and a vector $\mathbf{x} \in \mathbb{R}^3$ the reflection of $\mathbf{x}$ about a plane with normal $\hat{\mathbf{n}}$ can be represented in Pauli notation as

$$-\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}.$$

In standard vector notation, we can decompose a vector into its projective and rejective components

$$\mathbf{x} = (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}) . \tag{1.1}$$

A reflection about the plane normal to $\hat{\mathbf{n}}$ just flips the component in the direction of $\hat{\mathbf{n}}$, leaving the rest unchanged. That is

$$-(\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}) = \mathbf{x} - 2(\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}. \tag{1.2}$$

We may write this in $\boldsymbol{\sigma}$ notation as

$$\boldsymbol{\sigma} \cdot \mathbf{x} - 2\mathbf{x} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}. \tag{1.3}$$

We also know that

$$\begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{a} \boldsymbol{\sigma} \cdot \mathbf{b} &= a \cdot b + i \boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}) \\ \boldsymbol{\sigma} \cdot \mathbf{b} \boldsymbol{\sigma} \cdot \mathbf{a} &= a \cdot b - i \boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}), \end{aligned} \tag{1.4}$$

or

$$a \cdot b = \frac{1}{2} \left\{ \boldsymbol{\sigma} \cdot \mathbf{a}, \boldsymbol{\sigma} \cdot \mathbf{b} \right\}, \tag{1.5}$$

where $\{\mathbf{a}, \mathbf{b}\}$ is the anticommutator of $\mathbf{a}, \mathbf{b}$. Inserting eq. (1.5) into eq. (1.3) we find that the reflection is

$$\begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{x} - \left\{ \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \boldsymbol{\sigma} \cdot \mathbf{x} \right\} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} &= \boldsymbol{\sigma} \cdot \mathbf{x} - \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} - \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \\ &= \boldsymbol{\sigma} \cdot \mathbf{x} - \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} - \boldsymbol{\sigma} \cdot \mathbf{x} \\ &= -\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{x} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \end{aligned} \tag{1.6}$$

which completes the proof.

When we expand $(\boldsymbol{\sigma} \cdot \mathbf{x})^{\mathrm{T}}$ and find

$$(\boldsymbol{\sigma} \cdot \mathbf{x})^{\mathrm{T}} = \sigma^1 x^1 - \sigma^2 x^2 + \sigma^3 x^3, \tag{1.7}$$

it is clear that this coordinate expansion is a reflection about the y-axis. Knowing the reflection formula above provides a rationale for why we might want to write this in the compact form $-\sigma^2(\boldsymbol{\sigma} \cdot \mathbf{x})\sigma^2$, which might not be obvious otherwise.

# Bibliography

[1] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, Cambridge, UK, 1st edition, 2003. 1