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Dirac spinor relations after rest frame boost

In [1], Prof Osmond explicitly boosts a $u^s(p_0)$ Dirac spinor from the rest frame with rest frame energy p_0

$$\begin{aligned}\sqrt{m}e^{-\frac{1}{2}\eta\sigma^3} &= \sqrt{p\cdot\sigma} \\ \sqrt{m}e^{\frac{1}{2}\eta\sigma^3} &= \sqrt{p\cdot\bar{\sigma}},\end{aligned} \tag{1.1}$$

for the components of $u^s(\Lambda p_0)$.

Let's verify this by squaring. First

$$e^{\pm\frac{1}{2}\eta\sigma^3} = \cosh\left(\frac{1}{2}\eta\sigma^3\right) \pm \sinh\left(\frac{1}{2}\eta\sigma^3\right)\sigma^3, \tag{1.2}$$

which squares to (FIXME: link to uvspinor.nb)

$$\left(e^{\pm\frac{1}{2}\eta\sigma^3}\right)^2 = \begin{bmatrix} e^{\pm\eta} & 0 \\ 0 & e^{\mp\eta}\end{bmatrix}. \tag{1.3}$$

Explicitly boosting the rest energy p_0 gives

$$\begin{aligned}\begin{bmatrix} p_0 \\ 0 \end{bmatrix} &\rightarrow \begin{bmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{bmatrix}\begin{bmatrix} p_0 \\ 0 \end{bmatrix} \\ &= p_0 \begin{bmatrix} \cosh\eta \\ \sinh\eta \end{bmatrix},\end{aligned} \tag{1.4}$$

so after the boost

$$\begin{aligned}p\cdot\sigma &\rightarrow p_0\left(\cosh\eta - \sinh\eta\sigma^3\right) \\ &= p_0 \begin{bmatrix} \cosh\eta - \sinh\eta & 0 \\ 0 & \cosh\eta + \sinh\eta \end{bmatrix} \\ &= p_0 \begin{bmatrix} e^{-\eta} & 0 \\ 0 & e^{\eta} \end{bmatrix},\end{aligned} \tag{1.5}$$

where $p_0 = m$ is still the rest frame energy. However, according to eq. (1.3) this is exactly

$$\left(\sqrt{m}e^{-\frac{1}{2}\eta\sigma^3}\right)^2 \tag{1.6}$$

Since $p \cdot \bar{\sigma}$ flips the signs of the spatial momentum, we have shown that

$$\begin{aligned} \left(\sqrt{m}e^{-\frac{1}{2}\eta\sigma^3}\right)^2 &= p \cdot \sigma \\ \left(\sqrt{m}e^{\frac{1}{2}\eta\sigma^3}\right)^2 &= p \cdot \bar{\sigma}, \end{aligned} \tag{1.7}$$

which isn't a full proof of the claimed result (i.e. the most general orientation isn't considered), but at least validates the claim.

Bibliography

[1] Dr. Tobias Osborne. Qft lecture 15, dirac equation, boost from stationary frame. Youtube. URL https://youtu.be/J2lV8uNx0LU?list=PLDfPUNusx1EpRs-wku83aqYSKfR5fFmfS&t=4328. [Online; accessed 18-December-2018]. 1