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An axiomatic introduction of multivectors, vector products, and geometric algebra.

1.1 What's in the pipe.

It's been a while since I did any math or physics writing. This is the first post in a series where I plan to work my way systematically from an introduction of vectors, to the axioms of geometric algebra. I plan to start with an introduction of vectors as directed "arrows", building on that to discuss coordinates, tuples, and column matrix representations, and representation independent ideas. With those basics established, I'll remind the reader about how generalized vector and dot product spaces are defined and give some examples. Finally, with the foundation of vectors and vector spaces in place, I'll introduce the concept of a multivector space, and the geometric product, and start unpacking the implications of the axioms that follow naturally from this train of thought.

The applications that I plan to include in this series will be restricted to Euclidean spaces (i.e. where length is given by the Pythagorean law), primarily those of 2 and 3 dimensions. However, it will be good to also lay the foundations for the non-Euclidean spaces that we encounter in relativistic electromagnetism (there is actually no other kind), and in computer graphics applications of geometric algebra, especially since we can do so nearly for free. I plan to try to introduce the requisite ideas (i.e. the metric, which allows for a generalized dot product) by discussing Euclidean non-orthonormal bases. Such bases have applications in condensed matter physics where there are useful for modelling crystal and lattice structure, and provide a hands conceptual bridge to a set of ideas that might otherwise seem abstract and without "real world" application.

1.1.1 *Motivation.*

Many introductions to geometric algebra start by first introducing the dot product, then bivectors and the wedge product, and eventually define the product of two vectors as the synthetic sum of the dot and wedge

$$\mathbf{x}\mathbf{y} = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y}. \tag{1.1}$$

It takes a fair amount of work to do this well. In the seminal work [3] a few pages are taken for each of the dot and wedge products, showing the similarities and building up ideas, before introducing the geometric product in this fashion. In [2] the authors take a phenomenal five chapters to build up the context required to introduce the geometric product. I am not disparaging the authors for taking that long to build up the ideas, as their introduction of the subject is exceedingly clear and thorough, and they do a lot more than the minimum required to define the geometric product.

The strategy to introduce the geometric product as a sum of dot and wedge can result in considerable confusion, especially since the wedge product is often defined in terms of the geometric product

$$\mathbf{x} \wedge \mathbf{y} = \frac{1}{2} \left(\mathbf{x}\mathbf{y} - \mathbf{y}\mathbf{x} \right). \tag{1.2}$$

The whole subject can appear like a chicken and egg problem. I personally found the subject very confusing initially, and had considerable difficulty understanding which of the many identities of geometric algebra were the most fundamental. For this reason, I found the axiomatic approach of [1] very refreshing. The caveat with that work is that is is exceptionally terse, as they jammed a reformulation of most of physics using geometric algebra into that single book, and it would have been thousands of pages had they tried to make it readable by mere mortals.

When I wrote my own book on the subject, I had the intuition that the way to introduce the subject ought to be like the vector space in abstract linear algebra. The construct of a vector space is a curious and indirect way to define a vector. Vectors are not defined as entities, but simply as members of a vector space, a space that is required to have a set of properties. I thought that the same approach would probably work with multivectors, which could be defined as members of a multivector space, a mathematical construction with a set of properties.

I did try this approach, but was not fully satisfied with what I wrote. I think that dissatisfaction was because I tried to define the multivector first. To define the multivector, I first introduced a whole set of prerequisite ideas (bivector, trivector, blade, k-vector, vector product, ...), but that was also problematic, since the vector multiplication idea required for those concepts wasn't fully defined until the multivector space itself was defined.

My approach shows some mathematical cowardliness. Had I taken the approach of the vector space fully to heart, the multivector could have been defined as a member of a multivector space, and all the other ideas follow from that. In this multi-part series, I'm going to play with this approach anew, and see how it works out.

1.1.2 *Review and background.*

For this discussion, I'm going to assume that the reader is familiar with a wide variety of concepts, including but not limited to

- vectors,
- coordinates,

- matrices,
- basis,
- change of basis,
- dot and cross products,
- real and complex numbers,
- rotations and translations,
- vector spaces, and
- linear transformations.

Despite those assumptions, as mentioned above, I'm going to attempt to build up the basics of vector representation and vector spaces in a systematic fashion, starting from a very elementary level.

My reasons for doing so are mainly to explore the logical sequencing of the ideas required. I've always found well crafted pedagogical sequences rewarding, and will hopefully construct one here that is appreciated by anybody who chooses to follow along.

1.2 Vectors.

Cast yourself back in time, all the way to high school, where the first definition of vector that you would have encountered was probably very similar to the one made famous by the not very villainous *Vector* in "Despicable Me" [4]. His definition was not complete, but it is a good starting point:

Definition 1.1: Vector

A vector is a quantity represented by an arrow with both direction and magnitude.

All the operations that make vectors useful are missing from this definition, such as

- a comparison operator,
- a rescaling operation (i.e. a scalar multiplication operation that changes the length),
- addition and subtraction operators,
- an operator that provides the length of a vector,
- multiplication or multiplication like operations.

The concept of vector, once supplemented with the operations above, will be useful since it models many directed physical quantities that we experience daily.ă These include velocity, acceleration, forces, and electric and magnetic fields.

1.2.1 *Vector comparison.*

In fig. 1.1 (a), we have three vectors, labelled **a**, **b**, **c**, all with different directions and magnitudes, and in fig. 1.1 (b), those vectors have each been translated (moved without rotation or change of length) slightly. Two vectors are considered equal if they have the same direction and magnitude. That is, two vectors are equal if one is the image of the other after translation. In these figures $\mathbf{a} \neq \mathbf{b}, \mathbf{b} \neq \mathbf{c}, \mathbf{c} \neq \mathbf{a}$, whereas any same colored vectors are equal.

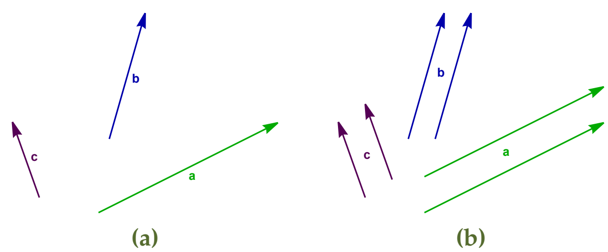

Figure 1.1: Three vectors and example translations.

1.2.2 *Vector (scalar) multiplication.*

We can multiply vectors by scalars by changing their lengths appropriately. In this context a scalar is a real number (this is purposefully vague, as it will be useful to allow scalars to be complex valued later.) Using the example vectors, some rescaled vectors include $2\mathbf{a}, (-1)\mathbf{b}, \pi\mathbf{c}$, as illustrated in fig. 1.2.

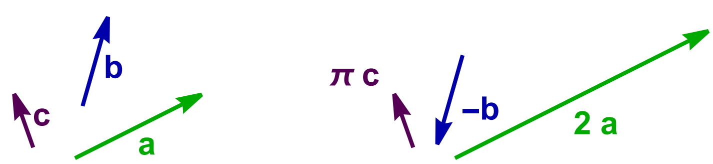

Figure 1.2: Scaled vectors.

1.2.3 *Vector addition.*

Scalar multiplication implicitly provides an algorithm for addition of vectors that have the same direction, as $s\mathbf{x} + t\mathbf{x} = (s + t)\mathbf{x}$ for any scalars s, t. This is illustrated in fig. 1.3 where $2\mathbf{a} = \mathbf{a} + \mathbf{a}$ is formed in two equivalent forms. We see that the addition of two vectors that have the same direction requires lining up those vectors head to tail. The sum of two such vectors is the vector that can be formed from the first tail to the final head. This procecdure is consistent with our experience of directed quantities like forces. Should your buddies pull on your arms with equal forces, your shoulders might object, but you'll still

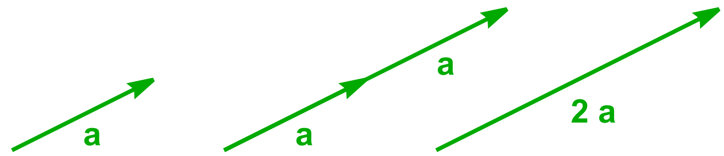

Figure 1.3: Twice a vector.

be in one place, as illustrated in fig. 1.4. However, if one of your friends is stronger, then assuming you

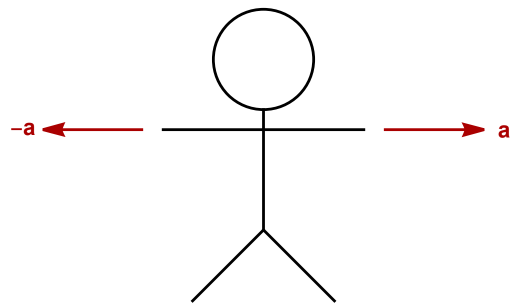

Figure 1.4: Pulled by opposing and equal forces.

haven't planted your feet too firmly to the ground, you'll be moving in the direction of your stronger friend, as illustrated in fig. 1.5.

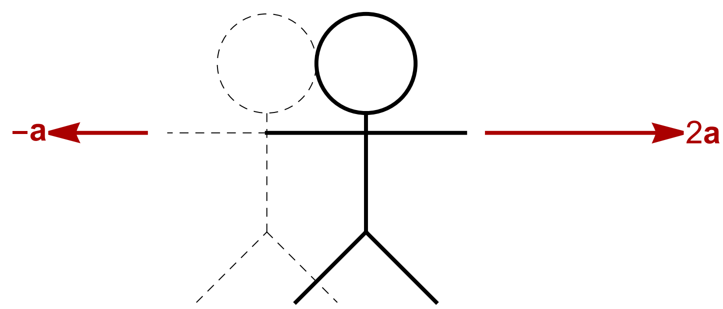

Figure 1.5: Pulled by unequal opposing forces.

It turns out that this arrow daisy chaining procedure is an appropriate way of defining addition for any vectors.

Definition 1.2: Vector addition.

The sum of two vectors can be found by connecting those two vectors head to tail in either order. The sum of the two vectors is the vector that can be formed by drawing an arrow from the initial tail to the final head. This can be generalized by chaining any number of vectors and joining the initial tail to the final head.

This addition procedure is illustrated in fig. 1.6, where $\mathbf{s} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ has been formed. This definition

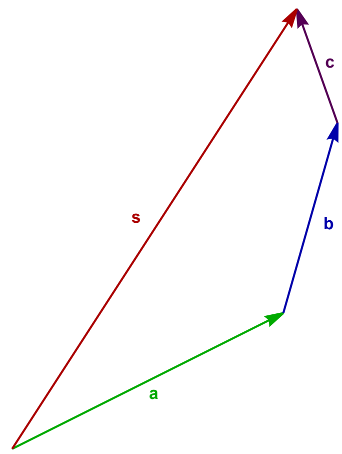

Figure 1.6: Addition of vectors.

of vector addition was inferred from the observation of the rules that must apply to addition of vectors that lay in the same direction (colinear vectors). Is it a cheat to just declare that this rule for addition of colinear vectors also applies to arbitrary vectors? Yes, it probably is, but it's a cheat that works nicely, and one that models physical quantities that we experience daily (velocities, acceleration, force, ...). Illustrating again with a force thought experiment, if you put your arms out at 45 degree angles, and have your friends pull on them with equal forces, you'll move straight ahead. That direction of motion is along the direction of the sum of the forces, if you model those forces as vectors (arrows, with magnitiude proportional to the strength of your friends.) This is crudely illustrated in fig. 1.7. Hopefully, you had a

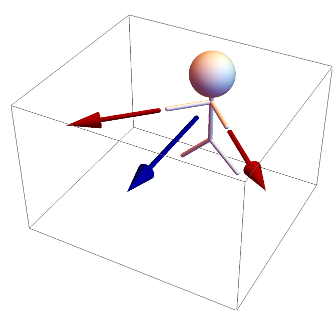

Figure 1.7: Sum of pulls separated by 90 degrees.

high school physics teacher that had you do quantitative experiments of this nature, using springs and force gauges to illustrate the vector nature of force. Such experiments provide a night first hand tangible justification for the vector addition rule above.

1.2.4 *Vector subtraction.*

Since we can scale a vector by -1 and we can add vectors, it is clear how to define vector subtraction

Definition 1.3: Vector subtraction.

The difference of vectors $\mathbf{a}, \mathbf{b}$ is

$$\mathbf{a} - \mathbf{b} \equiv \mathbf{a} + (-1)\mathbf{b}.$$

Graphically, subtracting a vector from another requires flipping the direction of the vector to be subtracted (scaling by -1), , and then adding both head to tail. This is illustrated in fig. 1.8.

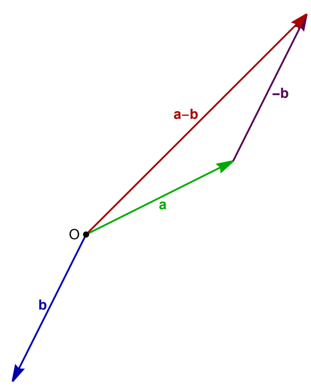

Figure 1.8: Vector subtraction.

1.2.5 *Length and what's to come.*

It is easy to compute the length of a vector that has an arrow representation. One simply lines a ruler of appropriate units along the vector and measures.

We actually want an algebraic way of computing length, but there is some baggage required, including

- Coordinates.
- Bases (plural of basis).
- Linear dependence and independence.

- Dot product.
- Metric.

The next part of this series will cover these topics. Our end goal is geometric algebra, which allows for many coordinate free operations, but we still have to use coordinates, both to read the literature, and in practice. Coordinates and non-orthonormal bases are also a good way to introduce non-Euclidean metrics.

Bibliography

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[4] Vector; supervillain extraordinaire (Despicable Me). A quantity represented by an arrow with direction and magnitude. Youtube. URL https://www.youtube.com/watch?v=bOIe0DIMbI8. [Online; accessed 11-July-2020]. 1.2