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Canonical bivectors in spacetime algebra.

I've been enjoying XylyXylyX's QED Prerequisites Geometric Algebra: Spacetime YouTube series, which is doing a thorough walk through of [1], filling in missing details. The last episode QED Prerequisites Geometric Algebra 15: Complex Structure, left things with a bit of a cliff hanger, mentioning a "canonical" form for STA bivectors that was intriguing.

The idea is that STA bivectors, like spacetime vectors can be spacelike, timelike, or lightlike (i.e.: positive, negative, or zero square), but can also have a complex signature (squaring to a 0,4-multivector.)

The only context that I knew of that one wanted to square an STA bivector is for the electrodynamic field Lagrangian, which has an F^2 term. In no other context, was the signature of F, the electrodynamic field, of interest that I knew of, so I'd never considered this "Canonical form" representation.

Here are some examples:

$$\begin{aligned}
F &= \gamma_{10}, \quad F^2 = 1 \\
F &= \gamma_{23}, \quad F^2 = -1 \\
F &= 4\gamma_{10} + \gamma_{13}, \quad F^2 = 15 \\
F &= \gamma_{10} + \gamma_{13}, \quad F^2 = 0 \\
F &= \gamma_{10} + 4\gamma_{13}, \quad F^2 = -15 \\
F &= \gamma_{10} + \gamma_{23}, \quad F^2 = 2I \\
F &= \gamma_{10} - 2\gamma_{23}, \quad F^2 = -3 + 4I.
\end{aligned} \tag{1.1}$$

You can see in this table that all the F's that are purely electric, have a positive signature, and all the purely magnetic fields have a negative signature, but when there is a mix, anything goes. The idea behind the canonical representation in the paper is to write

$$F = f e^{I\phi}, \tag{1.2}$$

where f^2 is real and positive, assuming that F is not lightlike.

The paper gives a formula for computing f and ϕ, but let's do this by example, putting all the F^2's above into their complex polar form representation, like so

$$
\begin{aligned}
&F = \gamma_{10}, \quad F^2 = 1 \\
&F = \gamma_{23}, \quad F^2 = 1 e^{\pi I} \\
&F = 4\gamma_{10} + \gamma_{13}, \quad F^2 = 15 \\
&F = \gamma_{10} + \gamma_{13}, \quad F^2 = 0 \\
&F = \gamma_{10} + 4\gamma_{13}, \quad F^2 = 15 e^{\pi I} \\
&F = \gamma_{10} + \gamma_{23}, \quad F^2 = 2 e^{(\pi/2) I} \\
&F = \gamma_{10} - 2\gamma_{23}, \quad F^2 = 5 e^{(\pi - \arctan(4/3)) I}
\end{aligned}
\tag{1.3}
$$

Since we can put F^2 in polar form, we can factor out half of that phase angle, so that we are left with a bivector that has a positive square. If we write

$$
F^2 = \left|F^2\right| e^{2\phi I},
\tag{1.4}
$$

we can then form

$$
f = F e^{-\phi I}.
\tag{1.5}
$$

If we want an equation for ϕ, we can just write

$$
2\phi = \operatorname{Arg}(F^2).
\tag{1.6}
$$

This is a bit better (I think) than the form given in the paper, since it will uniformly rotate F^2 toward the positive region of the real axis, whereas the paper's formula sometimes rotates towards the negative reals, which is a strange seeming polar form to use.

Let's compute f for $F = \gamma_{10} - 2\gamma_{23}$, using

$$
2\phi = \pi - \arctan(4/3).
\tag{1.7}
$$

The exponential expands to

$$
e^{-\phi I} = \frac{1}{\sqrt{5}} \left(1 - 2I\right).
\tag{1.8}
$$

Multiplying each of the bivector components by $1 - 2I$, we find

$$
\begin{aligned}
\gamma_{10} \left(1 - 2I\right) &= \gamma_{10} - 2\gamma_{100123} \\
&= \gamma_{10} - 2\gamma_{1123} \\
&= \gamma_{10} + 2\gamma_{23},
\end{aligned}
\tag{1.9}
$$

and

$$
\begin{aligned}
-2\gamma_{23} \left(1 - 2I\right) &= -2\gamma_{23} + 4\gamma_{230123} \\
&= -2\gamma_{23} + 4\gamma_{23}^2 \gamma_{01} \\
&= -2\gamma_{23} + 4\gamma_{10},
\end{aligned}
\tag{1.10}
$$

leaving

$$f = \sqrt{5}\gamma_{10}, \tag{1.11}$$

so the canonical form is

$$F = \gamma_{10} - 2\gamma_{23} = \sqrt{5}\gamma_{10}\frac{1 + 2I}{\sqrt{5}}. \tag{1.12}$$

It's interesting here that f, in this case, is a spatial bivector (i.e.: pure electric field), but that clearly isn't always going to be the case, since we can have a case like,

$$F = 4\gamma_{10} + \gamma_{13} = 4\gamma_{10} + \gamma_{20}I, \tag{1.13}$$

from the table above, that has both electric and magnetic field components, yet is already in the canonical form, with $F^2 = 15$. The canonical f, despite having a positive square, is not necessarily a spatial bivector (as it may have both grades 1,2 in the spatial representation, not just the electric field, spatial grade-1 component.)

Bibliography

[1] Justin Dressel, Konstantin Y Bliokh, and Franco Nori. Spacetime algebra as a powerful tool for electromagnetism. *Physics Reports*, 589:1–71, 2015. 1