Peeter Joot
peeterjoot@pm.me

Verifying the GA form for the symmetric and antisymmetric components of the different rate of strain.

We found geometric algebra representations for the symmetric and antisymmetric components for a gradient-vector direct product. In particular, given

$$d\mathbf{v} = d\mathbf{x} \cdot (\boldsymbol{\nabla} \otimes \mathbf{v}) \tag{1.1}$$

we found

$$\begin{aligned} d\mathbf{x} \cdot \mathbf{d} &= \frac{1}{2} d\mathbf{x} \cdot \left(\boldsymbol{\nabla} \otimes \mathbf{v} + (\boldsymbol{\nabla} \otimes \mathbf{v})^\dagger \right) \\ &= \frac{1}{2} \left(d\mathbf{x} \left(\boldsymbol{\nabla} \cdot \mathbf{v} \right) + \left\langle \boldsymbol{\nabla} d\mathbf{x} \mathbf{v} \right\rangle_1 \right), \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} d\mathbf{x} \cdot \boldsymbol{\Omega} &= \frac{1}{2} d\mathbf{x} \cdot \left(\boldsymbol{\nabla} \otimes \mathbf{v} - (\boldsymbol{\nabla} \otimes \mathbf{v})^\dagger \right) \\ &= \frac{1}{2} \left(d\mathbf{x} \left(\boldsymbol{\nabla} \cdot \mathbf{v} \right) - \left\langle d\mathbf{x} \mathbf{v} \boldsymbol{\nabla} \right\rangle_1 \right). \end{aligned} \tag{1.3}$$

Let's expand each of these in coordinates to verify that these are correct. For the symmetric component, that is

$$\begin{aligned} d\mathbf{x} \cdot \mathbf{d} &= \frac{1}{2} \left(dx_i \partial_j v_j \mathbf{e}_i + \partial_j dx_i v_k \left\langle \mathbf{e}_j \mathbf{e}_i \mathbf{e}_k \right\rangle_1 \right) \\ &= \frac{1}{2} dx_i \left(\partial_j v_j \mathbf{e}_i + \partial_j v_k \left(\delta_{ji} \mathbf{e}_k + \left(\mathbf{e}_j \wedge \mathbf{e}_i \right) \cdot \mathbf{e}_k \right) \right) \\ &= \frac{1}{2} dx_i \left(\partial_j v_j \mathbf{e}_i + \partial_j v_k \left(\delta_{ji} \mathbf{e}_k + \delta_{ik} \mathbf{e}_j - \delta_{jk} \mathbf{e}_i \right) \right) \\ &= \frac{1}{2} dx_i \left(\partial_j v_j \mathbf{e}_i + \partial_i v_k \mathbf{e}_k + \partial_j v_i \mathbf{e}_j - \partial_j v_j \mathbf{e}_i \right) \\ &= \frac{1}{2} dx_i \left(\partial_i v_k \mathbf{e}_k + \partial_j v_i \mathbf{e}_j \right) \\ &= dx_i \frac{1}{2} \left(\partial_i v_j + \partial_j v_i \right) \mathbf{e}_j. \end{aligned} \tag{1.4}$$

Sure enough, we that the product contains the matrix element of the symmetric component of $\boldsymbol{\nabla} \otimes \mathbf{v}$.

Now let's verify that our GA antisymmetric tensor product representation works out.

$$\begin{aligned} d\mathbf{x}\cdot\mathbf{\Omega} &= \frac{1}{2}\left(dx_i\partial_j v_j\mathbf{e}_i - dx_i\partial_k v_j\left\langle\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\right\rangle_1\right) \\ &= \frac{1}{2}dx_i\left(\partial_j v_j\mathbf{e}_i - \partial_k v_j\left(\delta_{ij}\mathbf{e}_k + \delta_{jk}\mathbf{e}_i - \delta_{ik}\mathbf{e}_j\right)\right) \\ &= \frac{1}{2}dx_i\left(\partial_j v_j\mathbf{e}_i - \partial_k v_i\mathbf{e}_k - \partial_k v_k\mathbf{e}_i + \partial_i v_j\mathbf{e}_j\right) \\ &= \frac{1}{2}dx_i\left(\partial_i v_j\mathbf{e}_j - \partial_k v_i\mathbf{e}_k\right) \\ &= dx_i\frac{1}{2}\left(\partial_i v_j - \partial_j v_i\right)\mathbf{e}_j. \end{aligned} \tag{1.5}$$

As expected, we that this product contains the matrix element of the antisymmetric component of $\boldsymbol{\nabla}\otimes\mathbf{v}$.

We also found previously that $\mathbf{\Omega}$ is just a curl, namely

$$\mathbf{\Omega} = \frac{1}{2}\left(\boldsymbol{\nabla}\wedge\mathbf{v}\right) = \frac{1}{2}\left(\partial_i v_j\right)\mathbf{e}_i\wedge\mathbf{e}_j, \tag{1.6}$$

which directly encodes the antisymmetric component of $\boldsymbol{\nabla}\otimes\mathbf{v}$. We can also see that by fully expanding $d\mathbf{x}\cdot\mathbf{\Omega}$, which gives

$$\begin{aligned} d\mathbf{x}\cdot\mathbf{\Omega} &= dx_i\frac{1}{2}\left(\partial_j v_k\right)\mathbf{e}_i\cdot\left(\mathbf{e}_j\wedge\mathbf{e}_k\right) \\ &= dx_i\frac{1}{2}\left(\partial_j v_k\right)\left(\delta_{ij}\mathbf{e}_k - \delta_{ik}\mathbf{e}_j\right) \\ &= dx_i\frac{1}{2}\left(\left(\partial_i v_k\right)\mathbf{e}_k - \left(\partial_j v_i\right)\mathbf{e}_j\right) \\ &= dx_i\frac{1}{2}\left(\partial_i v_j - \partial_j v_i\right)\mathbf{e}_j, \end{aligned} \tag{1.7}$$

as expected.